Conservative Binary Dynamics at Order α^5 in Electrodynamics

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We compute the potential-photon contributions to the classical relativistic scattering angle of two charged nonspinning bodies in electrodynamics through fifth order in the coupling. We use the scattering amplitudes framework, effective field theory, and multiloop integration techniques based on integration by parts and differential equations. At fifth order, the result is expressed in terms of cyclotomic polylogarithms. Our calculation demonstrates the feasibility of the corresponding calculations in general relativity, including the evaluation of the encountered four-loop integrals.

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Introduction.—The spectacular detection of gravitational waves [1] has opened a new window on the universe. The expected increase in precision of up to 2 orders of magnitude for the next generation gravitational wave detectors [2] requires commensurate advances in theoretical predictions. Analytic perturbative approaches based on post-Newtonian (PN) [3] and post-Minkowskian (PM) [4] expansions have seen major advances in recent years—see the reviews, e.g., Ref. [5] for further details and references. The connection of these approaches to quantum field theory (QFT) scattering processes has been long understood [6,7]. This has recently invigorated the PM approach by leveraging advances in QFT scattering, including generalized unitarity, the double copy [8], and advanced integration techniques. The double copy [8] expresses gravitational scattering amplitudes in terms of simpler gauge-theory amplitudes, while generalized unitarity [9] builds loop integrands from simpler tree-level amplitudes. Integration by parts (IBP) methods [10] allow the reduction of integrands to a set of independent master integrals whose values can usually be determined from differential equations [11,12]. The extraction of classical physics from quantum scattering is greatly simplified by basic concepts from effective field theories (EFTs), systematized for the gravitational-wave problem in Ref. [13] and applied to the PM framework in Ref. [14]. These methods have pushed the state of the art to 4PM $\mathcal{O}(G^4)$ [15–23]. By combining perturbative, numerical relativity [24] and self force [25] results, the effective one-body (EOB) approach [26] can yield high-precision waveform templates for both bound and unbound motion [7]. Despite some technical questions regarding the inclusion of radiation reaction, the strikingly good agreement between EOB-improved 4PM scattering predictions and numerical relativity [27] provides strong motivation for pursuing PM calculations to ever higher orders to help match the precision of future measurements.

Amplitude methods efficiently solve the problem of constructing integrands for the gravitational two-body problem to high PM orders. These methods use tree-level amplitudes as building blocks for loop-level integrands and their efficiency derives from the physical nature of the former. The primary difficulty for high-order predictions is often the evaluation of the resulting loop integrals. In particular, the integrals encountered in the classical gravitational two-body problem at 5PM (four-loop) order are overwhelmingly more involved than those encountered at lower orders.

To explore possible solutions to such difficulties, we turn to the simpler theory of electrodynamics (QED). It is a useful toy model for general relativity (GR) [28–33] because it retains certain essential features while having far fewer integral topologies due to the absence of photon self-interactions. Moreover, while in the quantum theory the two-derivative nature of gravitational interactions leads to far more complicated integrals, the classical limit restricts the number of loop momenta in each diagram's numerator so that the gravitational integrals are of a similar complexity as the QED ones. Our QED example demonstrates the necessary performance of the IBP reduction for carrying out four-loop calculations. This gives us confidence that our integration setup is applicable in GR. Here we use FIRE [34] and LITERED [35]. We also have written a special purpose IBP code for cross-checking results.

In this Letter, we compute potential photon contributions to the conservative QED scattering angle of two charged nonspinning objects to fifth order in the fine structure constant α, leaving aside conservative contributions from radiation photons. In the following, we will refer to this expansion as the "post-Lorentzian" (PL) expansion, which is in direct correspondence to the PM expansion in GR. Similarly, we refer to the analog of the PN expansion as the post-Coulombian (PC) expansion.

Unlike GR, this calculation does not exhibit divergences associated with the separation of potential and radiation modes [36]. The scattering angles including radiation effects were previously found through 3PL in Refs. [28,32,33].

It is worth noting that QED amplitudes at small momentum transfer, i.e., QED amplitudes in the classical regime, are an integral part of the analysis of ultraperipheral collisions at particle colliders and enter the relevant cross sections through interference with nuclear S-matrix elements. This so-called Coulomb-nuclear interference is of current theoretical and experimental interest in light of, e.g., the TOTEM experiment [37] probing physics down to momentum-squared transfer $|t| = 8\times10^{-4}\ \mathrm{GeV}^2$ and center-of-mass energy $\sqrt{s} = 13\ \mathrm{TeV}$. It was first studied by Bethe in Ref. [38] and reanalyzed from different perspectives in [39]; see Ref. [40] for a summary of various approaches and of the relevant cross sections. Recent theoretical improvements include effects of excited nuclei [41], an all-order-in α analysis of the leading eikonal [42], and an interpretation [43] of the data of Ref. [37] in light of a new treatment of IR divergences [44]. Interestingly, the momentum transfer of 2PL and 3PL terms dominates (at low t) or is of the same order, respectively, as the contribution of excited nuclei [41]. While analyzing them from this perspective is beyond the scope of this Letter, it would be interesting to explore the phenomenological consequences of higher-order terms.

Basic setup.—The starting point for our calculation are quantum scattering amplitudes, from which classical observables can be extracted via several approaches [14–16,31,45]. Here we use the realization that the classical elastic four-point scattering amplitude is an appropriately defined exponential [17]

$$\mathrm{i}\mathcal{M}(q) = \int_J (e^{\mathrm{i}I_r(J)} - 1), \tag{1}$$

of the classical radial action [46], $I_r(J) = \int p_r dr$, defined as an integral of the radial momentum, p_r, over the scattering trajectory. The radial action is a function of the total angular momentum $J = pb$ of the $2\to2$ scattering process of two massive particles with center-of-mass momentum p and impact parameter b; its Fourier conjugate variable is the momentum transfer q. The radial action (and therefore the classical limit of the amplitude) determines the scattering angle,

$$\chi = -\frac{\partial I_r(J)}{\partial J}. \tag{2}$$

The classical limit corresponds to large angular momentum $J \gg 1$ in $\hbar = 1$ units, see, e.g., Refs. [14,16], and translates to the hierarchy of scales $s, m_1^2, m_2^2 \sim J^2|t| \gg |t| = |q|^2$. The phase space splits into two regions

$$\text{hard: } \ell \gg |q|, \qquad \text{soft: } \ell \sim |q|. \tag{3}$$

Classical physics is captured by the soft region, whereas the hard region contributes only to quantum effects [15]. At L loops, it is contained in the terms proportional to $|q|^{L-2}\ln|q|$ and $|q|^{L-2}$ for even and odd L, respectively. To refine these contributions, we identify the potential and radiation sub-regions [47], characterized by a small velocity v:

$$\text{potential: } \ell \sim (v,1)|q|, \qquad \text{radiation: } \ell \sim (v,v)|q|. \tag{4}$$

Here we focus on the contribution where all photon loop momenta ℓ_i are in the potential region. As in GR, the potential region does not account for all conservative effects, which, starting at 4PM or 4PL [18], also require the inclusion of radiation modes. At 5PL, we leave such contributions to future work. However, in contrast to the gravitational case, the potential-region contribution in QED gives rise to a well-defined local classical potential due to the absence of nonlinearities of the field equations and the associated absence of tail effects [36]. Nonetheless, as is standard in quantum field theory computations (see, e.g., Ref. [48]), we use dimensional regularization, setting $D = 4 - 2\epsilon$, to handle divergences in intermediate expressions.

QED and loop integrands.—To describe two electrically charged classical spinless compact objects of mass m_1 and m_2 at scales much larger than their size, we use the minimally coupled scalar QED Lagrangian in R_ξ gauge,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \sum_{i=1}^{2}[|D_\mu\phi_i|^2 - m_i^2|\phi_i|^2], \tag{5}$$

where the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eQ_iA_\mu$ contains the photon field $A_\mu(x)$ and the electric charge Q_i for each object measured in terms of the elementary charge e, related to the fine structure constant $\alpha = e^2/(4\pi) \simeq 1/137$. Below it will be convenient to define an effective small coupling $\alpha_{\mathrm{eff}} = \alpha Q_1Q_2/J$ in terms of the charges of the scattering objects. The Maxwell field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Feynman vertices of scalar electrodynamics used in the calculation are derived from the Lagrangian in Eq. (5) using standard procedures [49]. This allows us to construct QED amplitudes via Feynman diagrams to higher orders in perturbation theory. At leading order in α only a single diagram contributes. At order α^5 there are a total of 1536

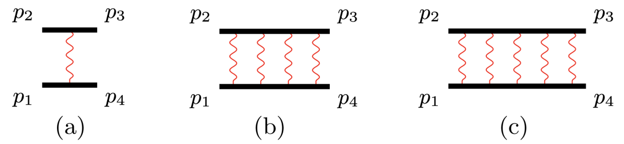

FIG. 1. Sample Feynman diagrams describing the scattering of two (macroscopic) charges at (a) leading, (b) fourth, and (c) fifth order in α in an all-outgoing momentum convention.

relevant Feynman diagrams, which we organize into 23 graphs containing only cubic vertices. Sample diagrams at various orders in perturbation theory are shown in Fig. 1. The primary calculation is performed in Feynman gauge $\xi = 1$, but we verified the ξ independence of the amplitudes at one numerical kinematic point.

Soft expansion, integral reduction, and integration.— With the L-loop integrand at hand we extract terms that can contribute to the classical limit by expanding in the soft region Eq. (3) to L^{th} order. Starting from the p_i in Fig. 1, special variables [51],

$$\bar{p}_1 = -p_1 + \frac{q}{2} = p_4 - \frac{q}{2}, \quad \bar{p}_2 = -p_2 - \frac{q}{2} = p_3 + \frac{q}{2}, \tag{6}$$

which satisfy $\bar{p}_1 \cdot q = \bar{p}_2 \cdot q = 0$, $\bar{p}_i^2 = \bar{m}_i^2$ simplify the analysis. After the soft expansion, the dependence on q^2 and $\bar{m}_i$ is trivially fixed by scaling arguments. Therefore all integrals are functions of the single kinematic variable,

$$y \coloneqq \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = \frac{1 + x^2}{2x} = \sigma + \mathcal{O}(q^2), \tag{7}$$

where the x parametrization is useful to simply the resulting expressions, and $\sigma = p_1 \cdot p_2/(m_1 m_2) = (1 - v^2)^{-1/2}$.

The soft expansion yields many terms, some with higher-rank tensor integrals and higher-power matter propagators. Yet, the key factor characterizing the difficulty of the calculation is not the number of integrals but rather their "IBP complexity," defined for each of them as the number of irreducible numerators plus the number of "dots" corresponding to doubled and higher powers of propagators. For QED at L loops, one obtains integrals with up to rank L numerators and up to L additional dots for a maximum IBP complexity of $2L$. Remarkably, similar counting in the classical limit shows that the maximum IBP complexity in GR is identical to that of QED. We should mention, however, that besides the appearance of many more diagram families in GR, they require more effort to evaluate at the same complexity ranking. In contrast, the ladder diagrams of GR can be evaluated using precisely the same setup as for QED.

The integral of the soft-expanded integrand is subsequently reduced to a combination of master integrals via IBP. These integrals are naturally organized in terms of families of diagrams with only cubic vertices and their contractions (contact diagrams). Different families share many common contact integrals so that we find it convenient to organize the computation in terms of a single set of "global" master integrals shared across all families. The global master integrals are evaluated via differential equations [11], following the steps described in Ref. [51]. In particular, we use an implementation of Lee's algorithm [52], together with various software packages [53] to find a basis of master integrals $\vec{\mathcal{I}}(x, \epsilon)$ that brings the corresponding differential equation to canonical form [12],

$$\partial_x \vec{\mathcal{I}}(x, \epsilon) = \epsilon \sum_{w \in \mathbb{W}} w(x) A_w \vec{\mathcal{I}}(x, \epsilon), \tag{8}$$

with rational matrices A_w and logarithmic kernels $\mathbb{W}$.

To fully specify the solutions we supply boundary conditions in the near-static limit $x \to 1$ (equivalent to $v \to 0$) by expanding the master integrals in velocity as explained in Ref. [51]. Another constraint is analyticity at $v \to 0$. The resulting solutions are a power series in ϵ whose coefficients are generically generalized polylogarithms [54] in x.

Potential-region scattering angles through $\mathcal{O}(\alpha^4)$.— Equations (1) and (2) give the potential-region contributions to the scattering angles through fourth order as

$$\chi_{\text{pot}}^{\text{1PL}} = \alpha_{\text{eff}} \frac{-2\sigma}{\sqrt{\sigma^2 - 1}},$$

$$\chi_{\text{pot}}^{\text{2PL}} = \alpha_{\text{eff}}^2 \frac{\pi}{2\sqrt{1 + 2\nu(\sigma - 1)}},$$

$$\chi_{\text{pot}}^{\text{3PL}} = \alpha_{\text{eff}}^3 \frac{-2\sigma(2\sigma^2 - 3) + 4\nu(\sigma - 1)(\sigma^3 + 3\sigma^2 - 3)}{3[1 + 2\nu(\sigma - 1)](\sigma^2 - 1)^{3/2}},$$

$$\begin{aligned}\chi_{\text{pot}}^{\text{4PL}} = \alpha_{\text{eff}}^4 & \frac{3\pi}{8(1 + 2\nu(\sigma - 1))^{3/2}} \\ & \times \left[1 + \frac{\nu}{2(\sigma^2 - 1)} \left\{ 3\sigma^4 - 11\sigma^3 + 3\sigma^2 + \sigma + 14 \right.\right. \\ & - \frac{7\sigma^2 - 1}{\sigma^3} + 2(3\sigma^3 - 4\sigma^2 + 9\sigma - 4) \frac{\log(x)}{\sqrt{\sigma^2 - 1}} \\ & \left.\left. + (3\sigma^2 + 1) \left[\frac{\log(x)}{\sqrt{\sigma^2 - 1}}\right]^2 \right\}\right],\end{aligned} \tag{9}$$

where $\nu = m_1 m_2/(m_1 + m_2)^2$ is the symmetric mass ratio. In Refs. [32,33], the scattering angles, including radiative effects, were presented through α^3. The α^4 potential-region contribution is new. For completeness, we have also computed the conservative radiation contribution to the scattering angle using the methods described in Ref. [18] and provide the result in the Supplemental Material [55]. Interestingly, $\chi_{\text{pot}}^{\text{4PL}}$ is related by a derivative with respect to $\log(x)$ (holding other appearances of σ fixed) to the energy-loss at 3PL [32,33] in the $Q_1^3 Q_2^3$ charge sector

$$\Delta E_{\text{c.m.}}^{Q_1^3 Q_2^3} \sim (3\sigma^3 - 4\sigma^2 + 9\sigma - 4) + (3\sigma^2 + 1) \frac{\log(x)}{\sqrt{\sigma^2 - 1}}. \tag{10}$$

In GR, the 3PM energy loss is related to the divergent part of the 4PM tail contribution [56] which in the full expression cancels against the divergent part of the potential-region scattering angle. In contrast, the relation appears to hold for the finite part in QED but not in GR. It would be interesting to investigate this further.

Scattering angles at $\mathcal{O}(\alpha^5)$.—Compared to lower orders, where analogous GR calculations are available, our four-loop result is at a previously unexplored order in the PM/PL expansion and warrants some further details.

The complete 5PL amplitude is composed of 1536 Feynman diagrams of which we have to evaluate 213, while the rest can be obtained by crossing. For the purpose of the integrand reduction via integration by parts, we organize the integrand into 23 ladder-type diagrams with cubic vertices by multiplying and dividing by any missing propagators. The soft expansion is straightforward. After applying IBP reduction to the soft expanded four-loop integrand we obtain a set of master integrals for each family. Removing redundant master integrals between the 23 families leaves 1107 global master integrals. Here, we again bring the master integral differential equations into canonical form, see Eq. (8).

The logarithmic kernels for the differential equation at four loops are explicitly given by

$$\mathbb{W} = \left\{\frac{1}{x}, \frac{1}{1+x}, \frac{1}{x-1}, \frac{2x}{1+x^2}, \frac{1+2x}{1+x+x^2}, \frac{2x-1}{1-x+x^2}\right\} \coloneqq \{f_0^0, f_2^0, f_1^0, 2f_4^1, 2f_3^1+f_3^0, 2f_6^1-f_6^0\}, \tag{11}$$

and reexpressed via cyclotomic kernels $f_j^i \coloneqq x^i/\Phi(j,x)$; $\Phi(j,x)$ being the jth cyclotomic polynomial. Therefore, the solutions of the differential equations are naturally written in terms of cyclotomic harmonic polylogarithms (CPLs), originally introduced in Ref. [57], and have appeared in different contexts, see, e.g., Ref. [58],

$$C_{a_1,\ldots,a_n}^{b_1,\ldots,b_n}(x) = \int_0^x \mathrm{d}z\, f_{a_1}^{b_1}(z) C_{a_2,\ldots,a_n}^{b_2,\ldots,b_n}(z), \tag{12}$$

with $C_0^0(x) \coloneqq \log(x)$.

The boundary conditions are evaluated in the near-static limit $x \to 1$. Analyticity in this limit provides 814 boundary conditions. The remaining 293 conditions correspond to appropriately normalized scalar integrals which are fixed by explicitly expanding in the near-static limit, as explained in Ref. [51].

Inserting the master-integral values into the IBP-reduced integrand yields the potential part of the classical amplitude for the scattering of two charged particles,

$$\mathcal{M}_5 = -J^5 \alpha_{\mathrm{eff}}^5 |q|^2 \log(|q|^2)\left[-4\sigma(15-20\sigma^2+8\sigma^4) + \sum_{k=1}^{12}\left(\nu r_k^{(1)} + \nu^2 r_k^{(2)}\right) f_k\right] + \text{iteration}, \tag{13}$$

where the result is organized in terms of the symmetric mass ration ν. The "iterations" are dictated by the amplitude-action relation in Eq. (1) [17] and we verified their connection with lower-loop amplitudes. We also introduced a basis of transcendental functions f_k and algebraic coefficients $r_k^{(i)}$,

$$\begin{aligned}
r_1^{(1)} &= \frac{15}{\sigma^2} - 208\sigma^6 + 128\sigma^5 - 625\sigma^4 - 320\sigma^3 + 705\sigma^2 \\
&\quad + 240\sigma + 65, \\
r_2^{(1)} &= \sqrt{\sigma^2-1}\left[-\frac{60(5\sigma^2-1)}{\sigma^3} - 80\sigma(16\sigma^2+23)\right], \\
r_3^{(1)} &= \frac{90(6\sigma^2-1)}{\sigma^4} - 10(350\sigma^2+319), \\
r_4^{(1)} &= -\frac{5760\sigma}{\sqrt{\sigma^2-1}}, \\
r_5^{(1)} &= 120(\sigma^2-1)^{3/2}(2\sigma^2-1), \\
r_8^{(1)} &= 120(\sigma^2-1)(\sigma^2+\sigma-1), \\
r_9^{(1)} &= r_{12}^{(1)} = 240(\sigma^2-1)^2, \\
r_{11}^{(1)} &= 120(\sigma^2-1)(\sigma^2+2\sigma-1), \\
r_6^{(1)} &= r_7^{(1)} = r_{10}^{(1)} = 0, \\
r_1^{(2)} &= \frac{405\sigma(15-44\sigma^2)}{16(1-4\sigma^2)^2} - \frac{15(10\sigma^2+2\sigma-3)}{\sigma^3} \\
&\quad + \frac{-2048\sigma^7 + 6656\sigma^6 + 17872\sigma^5 + 20000\sigma^4}{16} \\
&\quad + \frac{-7740\sigma^3 - 22560\sigma^2 - 6635\sigma - 2080}{16}, \\
r_2^{(2)} &= \sqrt{\sigma^2-1}\left[\frac{45(1232\sigma^4 - 1168\sigma^2 + 287)}{16(4\sigma^2-1)^3}\right. \\
&\quad + \frac{30(20\sigma^3 - 9\sigma^2 - 4\sigma + 3)}{\sigma^4} \\
&\quad \left. + \frac{5}{16}(1776\sigma^4 + 8192\sigma^3 + 10820\sigma^2 + 11776\sigma + 3223)\right], \\
r_3^{(2)} &= -\frac{30(16\sigma^4 + 36\sigma^3 - 11\sigma^2 - 6\sigma + 3)}{\sigma^5} \\
&\quad + 20(212\sigma^3 + 350\sigma^2 + 328\sigma + 319), \\
r_4^{(2)} &= \frac{2880(\sigma+1)(3\sigma+1)}{\sqrt{\sigma^2-1}}, \\
r_6^{(2)} &= 480(\sigma^2-1)^{3/2}(2\sigma^2-1), \\
r_7^{(2)} &= 45\sigma(\sigma^2-1)^{5/2}, \\
r_9^{(2)} &= -480(\sigma^2-1)(\sigma^2-\sigma-1), \\
r_{10}^{(2)} &= -135(\sigma^2-1)^2, \\
r_{12}^{(2)} &= -480(\sigma^2-1)(\sigma^2-2\sigma-1), \\
r_5^{(2)} &= r_8^{(2)} = r_{11}^{(2)} = 0.
\end{aligned} \tag{14}$$

The transcendental functions have compact representations in terms of CPLs in the variable x

$$
\begin{aligned}
&f_1=1, \quad f_2=C_0^0(x), \quad f_3=C_{0,0}^{0,0}(x), \quad f_4=C_{0,0,0}^{0,0,0}(x),\\
&f_5=-C_{1,0}^{0,0}(x)+C_{2,0}^{0,0}(x)+\frac{\pi^2}{4},\\
&f_6=-C_{2,0}^{0,0}(x)+C_{4,0}^{1,0}(x)-\frac{\pi^2}{16},\\
&f_7=C_{3,0}^{0,0}(x)+2C_{3,0}^{1,0}(x)+C_{6,0}^{0,0}(x)-2C_{6,0}^{1,0}(x)+\frac{\pi^2}{6},\\
&f_8=-C_{0,1,0}^{0,0,0}(x)+C_{0,2,0}^{0,0,0}(x)+\frac{\pi^2}{4}C_0^0(x)+\frac{7\zeta_3}{2},\\
&f_9=-C_{0,2,0}^{0,0,0}(x)+C_{0,4,0}^{0,1,0}(x)-\frac{\pi^2}{16}C_0^0(x)-\frac{21\zeta_3}{16},\\
&f_{10}=C_{0,3,0}^{0,0,0}(x)+2C_{0,3,0}^{0,1,0}(x)+C_{0,6,0}^{0,0,0}(x)-2C_{0,6,0}^{0,1,0}(x)\\
&\qquad+\frac{1}{6}\pi^2C_0^0(x)+\frac{28\zeta_3}{9},\\
&f_{11}=-C_{1,0,0}^{0,0,0}(x)+C_{2,0,0}^{0,0,0}(x)-\frac{7\zeta_3}{4},\\
&f_{12}=-C_{2,0,0}^{0,0,0}(x)+C_{4,0,0}^{1,0,0}(x)+\frac{21\zeta_3}{32}.
\end{aligned} \tag{15}
$$

Remarkably, the amplitude (13) does not depend on the full cyclotomic alphabet, but only on the subset

$$\mathbb{W}'=\left\{\frac{1}{x},\frac{x}{1-x^2},\frac{x-1}{(x+1)(1+x^2)},\frac{1-x^2}{1+x^2+x^4}\right\}. \tag{16}$$

Note that the last letter appears only in the ν^2 sector of the amplitude, while the second letter does not appear there. The transcendental constants in Eq. (15) are chosen such that the expansion of the functions around the static point $x=1$ only involves rational numbers. Therefore the post-Coulombian expansion of the amplitude is manifestly free of these constants and it is natural to conjecture that this property holds to all orders. This is in contrast to GR where π^2 is present at 4PM and is closely tied to the appearance of elliptic integrals. For the particular combination of CPL's in the classical amplitude we find that there exists an expression in terms of classical polylogarithms with *real* arguments. The expression can be found in the Supplemental Material to this Letter [55].

The $\mathcal{O}(\alpha^5)$ angle follows from Eqs. (13), (1) and (2):

$$
\begin{aligned}
\chi_{\rm pot}^{\rm 5PL}=&\,\alpha_{\rm eff}^5\frac{1}{30[1+2\nu(\sigma-1)]^2(\sigma^2-1)^{5/2}}\\
&\times\left[-4\sigma(15-20\sigma^2+8\sigma^4)+\sum_{k=1}^{12}\left(\nu r_k^{(1)}+\nu^2r_k^{(2)}\right)f_k\right].
\end{aligned} \tag{17}
$$

For both the 4PL and 5PL potential-region scattering angles we performed a number of nontrivial checks. We verified the independence of the result on the gauge parameter ξ after IBP reduction to a master-integral basis, checking much of the integrand and integral tables.

We have also computed the scattering angle to fourth post-Coulombian order using the Fokker-type Lagrangian of Wheeler-Feynman electrodynamics [50] from Eq. (4.1) of Ref. [29], and applied a variant of their proposed order-reduction procedure to eliminate higher time derivatives [59]. For the overlapping terms, we find complete agreement with the velocity expansion of our PL expressions in the near-static limit. In particular we match the expansion of the 5PL angle,

$$
\begin{aligned}
\chi_{\rm pot}^{\rm 5PL}=&\,\alpha_{\rm eff}^5\left[-\frac{2}{5v^5}+\frac{4}{3v^3}+\frac{2(8\nu-3)}{3v}+\frac{8}{9}\nu(5-18\nu)v\right.\\
&\left.+\nu\left(\frac{80\nu^2}{3}-\frac{532\nu}{27}+\frac{226}{45}\right)v^3+\mathcal{O}(v^5)\right].
\end{aligned} \tag{18}
$$

Notice that the $\mathcal{O}(\alpha^5)$ potential starts contributing only at $\mathcal{O}(v^3)$, while $\mathcal{O}(v^{n\leq 1})$ are fixed by lower PL orders.

The probe limit, $\nu\to 0$, in which one mass is much smaller than the other, provides another important check. In this limit the angle has a simple expression [46,60] to all orders in α:

$$\chi^{(0)}=-\pi+\frac{2}{\sqrt{1-\alpha_{\rm eff}^2}}\arctan\left[\frac{\sqrt{\sigma^2-1}}{\sigma}\frac{\sqrt{1-\alpha_{\rm eff}^2}}{\alpha_{\rm eff}}\right]. \tag{19}$$

Expanding in small $\alpha_{\rm eff}\ll 1$, we obtain

$$
\begin{aligned}
\chi^{(0)}=&-\alpha_{\rm eff}\frac{2\sigma}{\sqrt{\sigma^2-1}}+\alpha_{\rm eff}^2\frac{\pi}{2}+\alpha_{\rm eff}^3\frac{2\sigma(2\sigma^2-3)}{3(\sigma^2-1)^{3/2}}\\
&+\alpha_{\rm eff}^4\frac{3\pi}{8}-\alpha_{\rm eff}^5\frac{2\sigma(8\sigma^4-20\sigma^2+15)}{15(\sigma^2-1)^{5/2}}+\mathcal{O}(\alpha_{\rm eff}^6),
\end{aligned} \tag{20}
$$

in agreement with the probe-limit of the explicit PL results in Eqs. (9) and (17).

We note that $\chi_{\rm pot}^{\rm 5PL}$ exhibits singularities at $\sigma=0,\pm 1/2$ which lie outside of the physical scattering region $1<\sigma$. Similar poles are present in the complete 4PM results in GR [17–23]. It would be interesting to investigate their fate after the analytic continuation to the bound regime [61]. QED offers a cleaner environment to study this issue due to the absence of the tail effect.

Conclusions.—In this Letter we studied the scattering of two classical charges through fifth order in the fine structure constant α. While our primary objective was to connect to the rapid progress in the post-Minkowskian approach to gravitational-wave physics and explore the feasibility of analogous calculations in GR, we point out the possible phenomenological relevance of our 5PL QED amplitude as well as of amplitudes at lower PL orders [33] to ultraperipheral scattering as probed, e.g., at the TOTEM experiment. These aspects deserve further study.

Amplitude-based approaches efficiently solve the problem of constructing integrands for the foreseeable future, even in GR. The critical issue addressed here is whether the overwhelming increase in complexity of integrals encountered at 5PM and beyond impedes further progress. Electrodynamics is an especially useful test case because its diagram topologies are a subset of those appearing in GR. In the overlap, the integrals have identical IBP complexity in the classical limit and share the same set of master integrals. Moreover, most of the master integrals are also shared by more involved diagram topologies that appear in the gravitational calculation. Our results indicate that the powerful field-theory integration methods are sufficient to meet the 5PM challenge. We anticipate much more progress to follow in the near future.

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