Theory of Compression Channels for Postselected Quantum Metrology

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The postselected quantum metrological scheme is especially advantageous when the final measurements are either very noisy or expensive in practical experiments. In this Letter, we put forward a general theory on the compression channels in postselected quantum metrology. We define the basic notions characterizing the compression quality and illuminate the underlying structure of lossless compression channels. Previous experiments on postselected optical phase estimation and weak-value amplification are shown to be particular cases of this general theory. Furthermore, for two categories of bipartite systems, we show that the compression loss can be made arbitrarily small even when the compression channel acts only on one subsystem. These findings can be employed to distribute quantum measurements so that the measurement noise and cost are dramatically reduced.

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Quantum metrology utilizes quantum coherence and entanglement to boost the measurement precision quantified by the quantum Fisher information (QFI) [1–4]. In standard quantum metrology, given an ensemble of metrological samples, quantum mechanics allows one to optimize the quantum measurements so that the information about the signal is maximally extracted. Yet, another metrological scheme, called postselected quantum metrology arises in the context of weak-value amplification (WVA) [5–11], where a postselection measurement is performed to select a subensemble of the samples before the information-extracting measurement. Comparing to standard metrology, though the QFI encoded in the subensemble averaged over its postselection probability cannot be larger than the QFI in standard metrology [12–17], there are several advantages due to postselection when the cost of the postselection measurement becomes cheap: (i) WVA outperforms the standard metrology in the presence of certain types of technical noise [18]. (ii) WVA can be viewed as a filter to reduce the number of detected samples in standard metrology without losing the precision significantly. As such, in Hamiltonian learning [19,20], postselection can be employed to reduce the sample complexity [21], i.e., the number of samples to achieve a given precision. Practically speaking, when the final information-extracting measurements are subjected to various kinds of imperfections, such as detector saturation, limited memory and computational power, etc., postselected quantum metrology is provably outperforms the standard one [22–26].

Recently, postselection has been applied in a broad context in quantum metrology beyond the setup of WVA [27]. The advantage of postselection as a filter or compression channel persists in this broad context as demonstrated in the experiment of optical phase estimation [28]. While postselection can be also applied to classical metrology, previous works [27,29] show that the nonclassicality can further boost the precision. However, despite these advances, a comprehensive theory for designing the lossless postselection measurement channels in the most general setups beyond WVA remains uncharted. In standard quantum metrology, for arbitrary parameter-dependent quantum states, the optimal measurements saturating the quantum Cramér-Rao bound were studied by Braunstein and Caves [30] and recently applied to the case of noisy detection [31]. Analogously, in postselected metrology a similar fundamental question has not been addressed.

In this work, we answer this question by proposing a theory that unifies weak-value metrology, postselected metrology, as well as standard metrology. The crucial observation is that standard metrology only makes use of the measurement statistics and discards the postmeasurement states completely while postselected metrology utilizes a specific set of postmeasurement states and discards the rest. As such, we generalize the optimal measurement condition from standard metrology to postselected metrology. By keeping track of these conditions, we identify the generic structure of the lossless postselection channel for pure states. Previous setup on postselected metrology [21,27,28,32] and WVA [17] are special cases of this

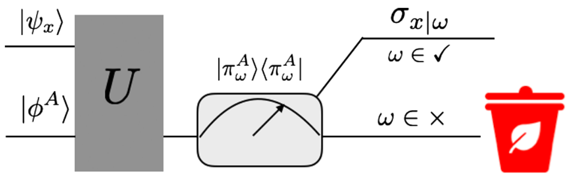

FIG. 1. The protocol of postselected quantum metrology. "A" denotes the ancilla. The unitary operation combined with the following projective measurement on the ancilla implement the postselection channel K_ω. After postselection, measurements can be further performed (not shown), as in standard metrology, on these postselected states to extract information about the estimation parameter.

general theory. Finally, for bipartite entangled states, when the compression channel is restricted to one subsystem, we construct two categories of examples, which can be compressed substantially with only a negligible amount of loss.

Generalized optimal measurement condition.—We consider a pure quantum state of a quantum sensor described by $\rho_x = |\psi_x\rangle\langle\psi_x|$, where x is the estimation parameter. The QFI associated with this state is $I(\rho_x) = 4g(\rho_x) = \sum_{\omega\in\Omega} I_\omega(\rho_x)$, where $g(\rho_x) \equiv \langle\partial_x^\perp\psi_x|\partial_x^\perp\psi_x\rangle$, $|\partial_x^\perp\psi_x\rangle \equiv |\partial_x\psi_x\rangle - \langle\psi_x|\partial_x\psi_x\rangle|\psi_x\rangle$ [30,33,34], $I_\omega(\rho_x) \equiv 4\langle\partial_x^\perp\psi_x|E_\omega|\partial_x^\perp\psi_x\rangle$, and E_ω is a positive-operator-valued measure (POVM) operator, satisfying $E_\omega \geq 0$ and $\sum_{\omega\in\Omega} E_\omega = \mathbb{I}$ [35]. In postselected quantum metrology, a postselection measurement channel denoted as $\{K_\omega\}$ is performed on the system, where $\omega\in\Omega$ and Ω is the set of all measurement outcomes. As shown in Fig. 1, such a generalized measurement can be implemented by a unitary operation entangling the system and ancilla, followed by a projective measurement on the ancilla. After performing the postselection channel, but before postselection is made, the joint state of the system and the ancilla becomes $\sigma_x^{\mathrm{SA}} = \sum_{\omega\in\Omega} p(\omega|x)\sigma_{x|\omega} \otimes |\pi_\omega^A\rangle\langle\pi_\omega^A|$, where $p(\omega|x) = \langle\psi_x|E_\omega|\psi_x\rangle$ and $\sigma_{x|\omega} = K_\omega|\psi_x\rangle\langle\psi_x|K_\omega^\dagger/p(\omega|x)$ [36]. Throughout this work, states and operators that do not act on the system only will be specified through the superscript, "A", "SA" etc. The QFI corresponding to σ_x^{SA} is [13,37], $I(\sigma_x^{\mathrm{SA}}) = \sum_{\omega\in\Omega} I_\omega(\sigma_x^{\mathrm{SA}})$, where

$$I_\omega(\sigma_x^{\mathrm{SA}}) \equiv I_\omega^{\mathrm{cl}}(p(\omega|x)) + p(\omega|x)I(\sigma_{x|\omega}), \tag{1}$$

and $I_\omega^{\mathrm{cl}}(p(\omega|x)) \equiv [\partial_x p(\omega|x)]^2/p(\omega|x)$. The physical meaning of Eq. (1) is clear: The QFI for each measurement outcome ω consists of two parts, the classical QFI and the average QFI for the postmeasurement states. Clearly, the postselection channel cannot increase the QFI, i.e., $I(\sigma_x^{\mathrm{SA}}) \leq I(\rho_x)$. A more refined statement is the following [37]:

$$I_\omega(\sigma_x^{\mathrm{SA}}) \leq I_\omega(\rho_x). \tag{2}$$

If Eq. (2) is saturated for all measurement outcomes, then $I(\sigma_x^{\mathrm{SA}}) = I(\rho_x)$.

TABLE I. The necessary and sufficient conditions for the saturation of the bounds of various QFIs corresponding to a regular POVM operator E_ω with $\sqrt{E_\omega}|\psi_x\rangle \neq 0$. See Supplemental Material for details [37].

Saturation	Necessary and sufficient condition	
$I_\omega(\sigma_x^{\mathrm{SA}}) = I_\omega(\rho_x)$,	$\mathrm{Im}\langle\partial_x^\perp\psi_x|E_\omega|\psi_x\rangle = 0.$	(T1)
$I(\sigma_{x|\omega}) = 0$,	$\sqrt{E_\omega}|\partial_x^\perp\psi_x\rangle = c\sqrt{E_\omega}|\psi_x\rangle$, $c\in\mathbb{C}$.	(T2)
$I_\omega^{\mathrm{cl}}(p(\omega|x)) = 0$,	$\mathrm{Re}\langle\partial_x^\perp\psi_x|E_\omega|\psi_x\rangle = 0.$	(T3)
$p(\omega|x)I(\sigma_{x|\omega}) = I_\omega(\rho_x)$,	$\sqrt{E_\omega}|\partial_x^\perp\psi_x\rangle \perp \sqrt{E_\omega}|\psi_x\rangle$.	(T4)
$I_\omega^{\mathrm{cl}}(p(\omega|x)) = I_\omega(\rho_x)$,	$\sqrt{E_\omega}|\partial_x^\perp\psi_x\rangle = c\sqrt{E_\omega}|\psi_x\rangle$, $c\in\mathbb{R}$.	(T5)

To compress the number of samples without sacrificing the precision, we demand the discarded set contains no information. Therefore, a precondition to reaching this goal is that Eq. (2) must be saturated even before the selection process is made. For a regular POVM measurement, where $\langle\psi_x|E_\omega|\psi_x\rangle \neq 0$ (see [37] and Ref. [33] for an elaborated definition), the necessary and sufficient condition to saturate the inequality (2) is given by Eq. (T1) in Table I. For a null POVM measurement where $\langle\psi_x|E_\omega|\psi_x\rangle = 0$, $I_\omega^{\mathrm{cl}}(p(\omega|x)) = I_\omega(\rho_x)$ and $I(\sigma_{x|\omega}) = 0$, see Ref. [33] for details. Thus no information is left in the postmeasurement state $\sigma_{x|\omega}$. As a consequence, if $\sigma_{x|\omega}$ is the state one would like to retain, one should avoid designing E_ω as a null POVM measurement operator.

In standard quantum metrology, the postmeasurement states are all discarded and only the measurement statistics is retained. In this case, one would like to saturate the inequality, $I_\omega^{\mathrm{cl}}(p(\omega|x)) \leq I_\omega(\rho_x)$, which was studied by the classic work of Braunstein and Caves [30] and the recent work Ref. [31], see also Eq. (T5). In postselected metrology, we require the average QFI of the retained postmeasurement state saturates the quantum limit, i.e., $p(\omega|x)I(\sigma_{x|\omega}) \leq I_\omega(\rho_x)$. Finally, for the measurement outcome or the corresponding postmeasurement state to be discarded, we would like them to carry no information, i.e., $I_\omega^{\mathrm{cl}}(p(\omega|x)) = 0$ or $I(\sigma_{x|\omega}) = 0$. The saturation conditions of these bounds are given in Table I. We shall call Eqs. (T1)–(T5) as the *generalized* optimal measurement conditions, as they generalize the results by Braunstein and Caves [30] to account for the case where the information about the parameter is losslessly encoded either in the measurement statistics or the postmeasurement states. As we will see, they play a fundamental role in the theory of compression channels.

Lossless compression channel.—In the standard metrology, the discarded set, as indicated by the red trash bin in Fig. 1, is $\{\sigma_{x|\omega}\}_{\omega\in\Omega}$ and no QFI is left in the postmeasurement states. In postselected metrology, we require all the QFI to be transferred to the desired postmeasurement states and the discarded set contains no information. We use $\omega\in ✓$ and $\omega\in\times$ to indicate the desired and undesired outcomes, respectively. Throughout this work, we consider a minimum retained set $\{\sigma_{x|\omega}\}_{\omega\in\checkmark}$, where the discarded set is $\Omega\cup\{\sigma_{x|\omega}\}_{\omega\in\times}$ [38]. When $\sum_{\omega\in\checkmark}p(\omega|x)<1$, we can view the postselection as a compression channel. It is worth noting that even if $\sum_{\omega\in\checkmark}p(\omega|x)$ is small, resulting in a small number of metrological samples in the postselected ensemble in each round, the experiment is assumed to be repeated for a sufficiently large number of rounds so that the classical Cramér-Rao bound is saturated.

Let us first introduce the essential notions for the theory of compression channels. The loss of the QFI per input sample can be expressed as $\gamma\equiv 1-\sum_{\omega\in\checkmark}p(\omega|x) I_\omega(\sigma_{x|\omega})/I(\rho_x)$. We define $c\equiv 1/\sum_{\omega\in\checkmark}p(\omega|x)$ as the *compression capacity* for a postselection channel, characterizing the ability of a designed postselection measurement to reduce the number of samples. If $\gamma=0$ and $c>1$ then a postselection measurement is called a lossless compression channel (LCC). We shall restrict our attention to *efficient* postselection channels where $c\in(1,\infty)$ [39]. We further define the *compression gain*, $\eta\equiv\sum_{\omega\in\checkmark}I_\omega(\sigma_{x|\omega})/I(\rho_x)$, as the ratio between the postselected QFI and the one standard metrology, characterizing information gain per detected sample. This characterizes the advantage of the former over the latter when the cost of final detection dominates over the cost of postselection [27].

For generic quantum systems, we find LCC must satisfy the following theorem:

Theorem 1.—For a pure state $|\psi_x\rangle$, the POVM operators in an efficient LCC must satisfy

$$\langle\psi_x^\perp|\sum_{\omega\in\checkmark}E_\omega|\psi_x^\perp\rangle=1, \tag{3}$$

$$\langle\partial_x^\perp\psi_x|E_\omega|\psi_x\rangle=0, \tag{4}$$

with $p(\omega|x)>0$ for $\omega\in\checkmark$ and $\sum_{\omega\in\checkmark}p(\omega|x)<1$, where $|\psi_x^\perp\rangle\equiv|\partial_x^\perp\psi_x\rangle/\sqrt{g(\rho_x)}$ is the normalized vector along $|\partial_x^\perp\psi_x\rangle$ direction. The proof is straightforward with the following intuition: Eq. (3) guarantees that in the undesired outcome, measurement statistics and postmeasurement states contain no QFI, i.e., $I_\omega(\rho_x)=0$ for $\omega\in\times$; Eq. (4) ensures that for the desired outcome where $\omega\in\checkmark$, the measurement statistics again contains no QFI and retained states reach the quantum limit given by Eq. (T4). As such, the QFI is fully preserved after the postselection. Alternatively, the following theorem illuminates the underlying structure of an LCC:

Theorem 2.—For a pure state $|\psi_x\rangle$, the POVM operators in the retained set of an LCC can be expressed as follows:

$$E_\omega=q_\omega\rho_x^\perp+\Lambda_\omega, \tag{5}$$

where $q_\omega\in(0,1]$ satisfying $\sum_{\omega\in\checkmark}q_\omega=1$, $\rho_x^\perp\equiv|\psi_x^\perp\rangle\langle\psi_x^\perp|$ and Λ_ω is the gauge operator, which does not contribute to the QFI and can be chosen in many ways as long as it satisfies

$$\langle\psi_x^\perp|\Lambda_\omega|\psi_x^\perp\rangle=\langle\psi_x^\perp|\Lambda_\omega|\psi_x\rangle=0, \tag{6}$$

and $\langle\psi_x|\Lambda_\omega|\psi_x\rangle=\lambda_\omega\in(0,1)$. The compression capacity and gain are $c=1/\sum_{\omega\in\checkmark}\lambda_\omega$ and $\eta=\sum_{\omega\in\checkmark}q_\omega/\lambda_\omega$, respectively.

Theorems 1 and 2 are our second main results. Practically, the LCC (5) depends on the true value of x, so in general adaptive estimation is required [40–46]. If we assume full accessibility of the postselection measurements on the whole Hilbert space, i.e., Eq. (5) is always implementable regardless of the choice of Λ_ω, then λ_ω can be tuned arbitrarily small, say $\lambda_\omega=\varepsilon$. Then we find $\eta=Lc=1/\varepsilon$, where L is the number of desired outcomes.

When $\eta>1$, we know the QFI per detected sample in an LCC is amplified, though it will be counterbalanced when the postselection probability is accounted for [12,13,27]. While Ref. [27] relates such an amplification to the noncommutativity of observables, here thanks to Theorems 1 and 2, we can directly compute the enhancement of the parametric sensitivity of the postselected state to the estimation parameter. For example, apart from an irrelevant unitary rotation, one can take $K_\omega=\sqrt{q_\omega}\rho_x^\perp+\sqrt{\lambda_\omega}\rho_x$, corresponding to $\Lambda_\omega=\lambda_\omega\rho_x$. While the postmeasurement state is still $|\psi_x\rangle$, the parameter derivative of the postselected state becomes [37]

$$\partial_x\Big(K_\omega|\psi_x\rangle/\sqrt{p(\omega|x)}\Big)=|\partial_x\psi_x\rangle+\Big(\sqrt{q_\omega/\lambda_\omega}-1\Big)\times\sqrt{g(\rho_x)}|\psi_x^\perp\rangle. \tag{7}$$

Finally, it is worthwhile to note that if $\lambda_\omega=0$, the LCC (5) degenerates into a null measurement operator $q_\omega\rho_x^\perp$. Using our prior knowledge about the estimation parameter denoted as x_*, the projector $\rho_{x_*}^\perp$ can approach to quantum limit asymptotically as $x_*\to x$ [33].

Binary postselection.—For a binary postselection, i.e., the desired set ✓ contains only one single outcome. In this case, for simplicity we use ✓ as a shorthand notation for $\omega\in\checkmark$. Apparently $q_\checkmark=1$ and $c=\eta=1/\lambda_\checkmark$, but there are many choices of choosing $\Lambda_\checkmark$. If we take the gauge operator $\Lambda_\checkmark=\lambda_\checkmark\rho_x+\mathcal{P}_0$, where $\mathcal{P}_0$ is the projector to the orthogonal complement to the subspace spanned by $|\psi_x\rangle$ and $|\psi_x^\perp\rangle$, then Eq. (5) becomes $E_\checkmark=(\lambda_\checkmark-1)\rho_x+\mathbb{I}$, which is the LCC proposed in Refs. [21,32].

In two-level systems, the gauge operator $\Lambda_\checkmark$ is forced to take the form $\Lambda_\checkmark=\lambda_\checkmark\rho_x$. Consider a parameter-dependent state $|\psi_x\rangle=\cos(x\Delta/2)|0\rangle+\mathrm{i}\sin(x\Delta/2)|1\rangle$, where Δ is a known constant and $\{|0\rangle,|1\rangle\}$ is a parameter-independent basis. This is the example investigated in the experiment in Ref. [28]. Our theory predicts that the postselected POVM measurement operator in the basis of $\{|0\rangle,|1\rangle\}$ is

$$E_{\checkmark}=\begin{bmatrix}\lambda_{\checkmark}\cos^2(\frac{x\Delta}{2})+\sin^2(\frac{x\Delta}{2}) & \mathrm{i}(1-\lambda_{\checkmark})\sin(x\Delta)\\ -\mathrm{i}(1-\lambda_{\checkmark})\sin(x\Delta) & \cos^2(\frac{x\Delta}{2})+\lambda_{\checkmark}\sin^2(\frac{x\Delta}{2})\end{bmatrix}. \quad (8)$$

It should be noted that unlike Ref. [28] which assumes small values of $x\Delta$, Eq. (8) is the exact LCC for any values of x. Of course, when $x\Delta \ll 1$, to the zeroth order of $x\Delta$, it recovers the LCC in Ref. [28], i.e., $E_{\checkmark}=\lambda_{\checkmark}|0\rangle\langle 0|+|1\rangle\langle 1|$.

Restricted postselections.—When $|\psi_x\rangle$ is a bipartite entangled state between two subsystems A and B, the postselection measurement on only A generically leads to loss, i.e., $\gamma>0$. This is because the LCC, according to Theorem 2, in general acts globally on both the system and the environment. Nevertheless, we demonstrate the existence of approximate LCC for two classes of examples, where the loss is very tiny. To proceed, let us define $\mathcal{C}_x^{\mathrm{AB}}\equiv|\partial_x^{\perp}\psi_x^{\mathrm{AB}}\rangle\langle\psi_x^{\mathrm{AB}}|$, $\mathcal{C}_x^{\mathrm{A}}=\mathrm{Tr}_{\mathrm{B}}\mathcal{C}_x^{\mathrm{AB}}$, $\varrho_x^{\perp\mathrm{A}}=\mathrm{Tr}_{\mathrm{B}}\rho_x^{\perp\mathrm{AB}}$, $\varrho_x^{\mathrm{A}}=\mathrm{Tr}_{\mathrm{B}}\rho_x^{\mathrm{AB}}$. Consider a postselection channel on the subsystem A only, i.e., $E_\omega^{(\mathrm{A})}\otimes\mathbb{I}^{(\mathrm{B})}$. Then Theorem 1 becomes $\mathrm{Tr}(\varrho_x^{\perp\mathrm{A}}\sum_{\omega\in\checkmark}E_\omega^{\mathrm{A}})=1$, and $\mathrm{Tr}(\mathcal{C}_x^{\mathrm{A}}E_\omega^{\mathrm{A}})=0$, $\omega\in\checkmark$, with $\mathrm{Tr}(\varrho_x^{\mathrm{A}}\sum_{\omega\in\checkmark}E_\omega^{\mathrm{A}})<1$.

The first category of examples is the weak-entanglement limit, which includes WVA as a special case. We consider a separable pure initial state $|\psi_0^{\mathrm{AB}}\rangle\equiv|\phi_0^{\mathrm{A}}\rangle\otimes|\varphi_0^{\mathrm{B}}\rangle$. The Hamiltonian is $H_{\mathrm{AB}}=x(H_{\mathrm{A}}\otimes H_{\mathrm{B}})$. By a judicious choice of the initial states, one can always make $\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle=0$ or $\langle\varphi_0^{\mathrm{B}}|H_{\mathrm{B}}|\varphi_0^{\mathrm{B}}\rangle=0$ such that $\langle\psi_0^{\mathrm{AB}}|H_{\mathrm{AB}}|\psi_0^{\mathrm{AB}}\rangle=0$. The QFI is $I(\rho_x)=4(\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}^2|\phi_0^{\mathrm{A}}\rangle\langle\varphi_0^{\mathrm{B}}|H_{\mathrm{B}}^2|\varphi_0^{\mathrm{B}}\rangle)$. In local estimation theory, x is usually considered to be very small. In the limit $x\to 0$, we find $|\psi_x\rangle=|\phi_0^{\mathrm{A}}\rangle\otimes|\varphi_0^{\mathrm{B}}\rangle$ and $|\partial_x^{\perp}\psi_x\rangle=-\mathrm{i}H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle\otimes H_{\mathrm{B}}|\varphi_0^{\mathrm{B}}\rangle$ are disentangled. Then, it can be readily calculated that $\varrho_x^{\perp\mathrm{A}}=H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}/\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}^2|\phi_0^{\mathrm{A}}\rangle$, $\varrho_x^{\mathrm{A}}=|\phi_0^{\mathrm{A}}\rangle\langle\phi_0^{\mathrm{A}}|$, and $\mathcal{C}_x^{\mathrm{A}}=H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle\langle\phi_0^{\mathrm{A}}|$.

Now if $\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle=0$, similar with Eq. (5), one can construct

$$E_\omega^{\mathrm{A}}=q_\omega\varrho_x^{\perp\mathrm{A}}+\varepsilon|\phi_0^{\mathrm{A}}\rangle\langle\phi_0^{\mathrm{A}}|, \quad (9)$$

where $\sum_{\omega\in\checkmark}q_\omega=1$, ε is arbitrarily small. In this case $\eta=Lc=1/\varepsilon$ as before, i.e., we can achieve arbitrarily large compression capacity and gain without loss. On the other hand, if $\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle\neq 0$ but $\langle\varphi_0^{\mathrm{B}}|H_{\mathrm{B}}|\varphi_0^{\mathrm{B}}\rangle=0$. One can simply take

$$E_\omega^{\mathrm{A}}=q_\omega\varrho_x^{\perp\mathrm{A}}. \quad (10)$$

In this case, the compression capacity is $c=\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}^2|\phi_0^{\mathrm{A}}\rangle/\langle\phi_0^{\mathrm{A}}|H_{\mathrm{A}}|\phi_0^{\mathrm{A}}\rangle^2$, which is the ratio between the second and first order moments of the energy of the subsystem A. The compression gain is $\eta=Lc$.

It is worth noting that WVA [5,17] falls into this category. Considering the von Neumann measurement model [47], where the system consists of a two-level subsystem and the continuous-variable meter. The Hamiltonian of the system is $H=x\sigma_z\otimes P_u$, where $P_u=-\mathrm{i}\partial/\partial u$ and u is the position of the meter. The initial state is $|\phi_\theta\rangle\otimes|\varphi_0\rangle$,where $|\phi_\theta\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$ and $|\varphi_n\rangle=\int du\varphi_n(u)|u\rangle$ with $\varphi_n(u)$ is the nth order normalized Hermite-Gaussian function defined as $\varphi_n(u)=(1/\sqrt{2^n n!})(1/2\pi\sigma^2)^{1/4}e^{-u^2/(4\sigma^2)}H_n(u/\sqrt{2}\sigma)$, where $H_n(x)$ is well-known Hermite polynomial [48]. The parameter-dependent state is $|\psi_x\rangle=e^{-\mathrm{i}x\sigma_z\otimes\hat{P}}|\phi\rangle\otimes|\varphi_0\rangle$. The QFI before postselection is $I(\rho_x)=1/\sigma^2$. One can readily show $\langle\varphi_0|P_u|\varphi_0\rangle=0$. In the limit $x\to 0$, one can find $|\psi_{x=0}\rangle=|\phi_\theta\rangle\otimes|\varphi_0\rangle$ and $|\psi_{x=0}^{\perp}\rangle=|\phi_{-\theta}\rangle\otimes|\varphi_1\rangle$. The WVA employs a binary postselection channel $E_{\checkmark}^{\mathrm{WVA}}=|\phi_{\theta_*}\rangle\langle\phi_{\theta_*}|\otimes\mathbb{I}$, where $|\phi_{\theta_*}\rangle$ is almost orthogonal to the initial spin state $|\phi_\theta\rangle$, i.e., $\theta_*=\theta-\pi+2\varepsilon$ and ε is an arbitrarily small but nonzero real number. Apparently, the WVA postselection channel satisfies Eq. (4), meaning the measurement statistics associated with the retained set contains no QFI. However, the loss $\gamma=\cos^2(\theta+\varepsilon)$, indicating there is information loss in the undesired measurement statistics and undesired postmeasurement states, unless $\theta=\pi/2$, where the loss is $\sin^2\varepsilon$.

On the other hand, Eqs. (9) and (10) predict that the LCC acting on the two-level system is of the same form as the WVA postselection channel, but with a different choice of θ_*. Here, $\theta_*=-\theta$ if $\theta\neq\pi/2$ and $\theta_*=-\theta+2\varepsilon$ if $\theta=\pi/2$. The compression gain is $\eta=c=1/\cos^2\theta$ if $\theta\neq\pi/2$ and $1/\sin^2\varepsilon$ if $\theta=\pi/2$, which is consistent with previous analysis. Furthermore, the same LCC can be alternatively constructed by a judicious choice of the gauge operator $\Lambda_{\checkmark}$ so that Eq. (5) becomes a one-body operator that only acts on the two-level systems [37]. If we consider postselecting on the meter, Eq. (9) predicts that $E_{\checkmark}=\mathbb{I}\otimes(|\varphi_1\rangle\langle\varphi_1|+\varepsilon|\varphi_0\rangle\langle\varphi_0|)$ is also an LCC with $\eta=c=1/\varepsilon^2$. Interestingly, measurement of this type with $\varepsilon=0$ was previously explored to reach the superresolution of incoherent imaging [49,50]. The performance of these LCCs is numerically calculated in Fig. 2.

Another category of examples with negligible loss is when the energy fluctuation of the postselected subsystems dominates over the other if they are noninteracting but the initial state is entangled. We consider the Hamiltonian of the two systems is $H_{\mathrm{AB}}=x(H_{\mathrm{A}}+H_{\mathrm{B}})$. Denote the eigenstates of H_{A} as $H_{\mathrm{A}}|E_n^{\mathrm{A}}\rangle=E_n^{\mathrm{A}}|E_n^{\mathrm{A}}\rangle$, where $n=1,2\ldots,d_{\mathrm{A}}\equiv\dim\mathcal{H}_{\mathrm{A}}$ and $E_1^{\mathrm{A}}\leq E_2^{\mathrm{A}}\leq\cdots\leq E_{d_{\mathrm{A}}}^{\mathrm{A}}$. We then split the Hilbert space into several orthogonal and disjoint subspaces spanned by the energy eigenstates, i.e., $\mathcal{H}_{\mathrm{A}}=\oplus_k\mathcal{V}_k^{\mathrm{A}}$ with $\mathcal{V}_k^{\mathrm{A}}\cap\mathcal{V}_l^{\mathrm{A}}=\{0\}$ for $k\neq l$ so that one can construct a set of mutually orthogonal states with the same average energy, i.e., $\langle\phi_k^{\mathrm{A}}|H_{\mathrm{A}}|\phi_k^{\mathrm{A}}\rangle=\mathcal{E}$ with $|\phi_k^{\mathrm{A}}\rangle\in\mathcal{V}_k^{\mathrm{A}}$ is a superposition of energy eigenstates to ensure nonvanishing QFI.

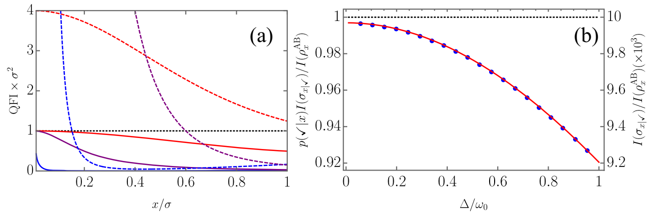

FIG. 2. In both figures, $K_{✓} = \sqrt{E_{✓}}$. (a) The solid lines correspond to $p(✓|x)I(\sigma_{x|✓})/I(\rho_x)$ with $I(\rho_x) = 1/\sigma^2$, where the deviation from 1 represents the loss γ, while the dashed lines correspond to the gain $\eta = I(\sigma_{x|✓})/I(\rho_x)$. Each color represents one postselection scheme—red: LCC on the two-level system with $\theta_* = -\theta = -\pi/3$; blue: WVA with $\theta_* = -2\pi/3 + 10^{-2}$; purple: LCC on the meter discussed in the main text with $\varepsilon = 10^{-4}$. (b) Postselection on the three-qubit entangled state. The LCC is given by Eq. (11) with $x = 10^{-5}$, $\theta = \pi/3$, $p_1 = 1 - p_2 = 2/3$, and $\varepsilon = 10^{-4}$. The red solid line corresponds to the plot of $1 - \gamma$ (left frame ticks) while the blue round dots correspond to the gain η (right frame ticks).

We consider entangled initial state $|\psi_0^{\mathrm{AB}}\rangle = \sum_k \sqrt{p_k}|\phi_k^{\mathrm{A}}\rangle \otimes |\varphi_k^{\mathrm{B}}\rangle$, where $p_k > 0$ satisfying $\sum_k p_k = 1$, $\{|\varphi_k^{\mathrm{B}}\rangle\}$ is a set of orthonormal basis on the subsystems B. The QFI before postselection is $I(\rho_x^{\mathrm{AB}}) = 4(\delta h_{\mathrm{A}}^2 + \delta h_{\mathrm{B}}^2) \approx 4\delta h_{\mathrm{A}}^2$, where $\delta h_{\mathrm{A,B}} \equiv \sqrt{\mathrm{Var}(H_{\mathrm{A,B}})_{\varrho_0^{\mathrm{A,B}}}}$ and $\delta h_{\mathrm{B}} \ll \delta h_{\mathrm{A}}$ is assumed. While an LCC exists for arbitrary value of x [37], as before, we focus on the local estimation for small x and consider the following LCC:

$$E_\omega^{\mathrm{A}} = \sum_k r_{\omega k}|\phi_k^{\perp \mathrm{A}}\rangle\langle\phi_k^{\perp \mathrm{A}}| + \varepsilon \mathcal{P}_{\mathrm{supp}(\varrho_0^{\mathrm{A}})}, \tag{11}$$

where ε is an arbitrarily small positive number, $\sum_{\omega \in ✓} r_{\omega k} = 1$, $|\phi_k^{\perp \mathrm{A}}\rangle \equiv (H_{\mathrm{A}} - \mathcal{E})|\phi_k^{\mathrm{A}}\rangle / \sqrt{\mathrm{Var}(H_{\mathrm{A}})_{|\phi_k^{\mathrm{A}}\rangle}}$ satisfying $\langle\phi_k^{\perp \mathrm{A}}|\phi_l^{\mathrm{A}}\rangle = 0$ and $\langle\phi_k^{\perp \mathrm{A}}|\phi_l^{\perp \mathrm{A}}\rangle = \delta_{kl}$ and $\mathcal{P}_{\mathrm{supp}(\varrho^{\mathrm{A}})}$ is any projector to the support of the reduced density matrix $\varrho_0^{\mathrm{A}} = \sum_k p_k|\phi_k^{\mathrm{A}}\rangle\langle\phi_k^{\mathrm{A}}|$. Equation (11) generalizes Eq. (5) beautifully while preserving the similar structure. The scaling of loss, capacity, and gain in this case are $\gamma \sim O[(\delta h_{\mathrm{B}}/\delta h_{\mathrm{A}})^2] + O(L\varepsilon)$, $c \sim 1/O(L\varepsilon)$, and $\eta \sim \{1 - O[(\delta h_{\mathrm{B}}/\delta h_{\mathrm{A}})^2]\}/O(\varepsilon)$.

For example, consider A and B consists of two qubits and one qubit respectively with the Hamiltonian $H_{\mathrm{A}} = \omega_0(\sigma_z^{(1)} + \sigma_z^{(2)})$ and $H_{\mathrm{B}} = \Delta\sigma_z^{(3)}$. The initial state is $|\psi_0^{\mathrm{AB}}\rangle = \sqrt{p_1}|\phi_1^{\mathrm{A}}\rangle \otimes |\varphi_1^{\mathrm{B}}\rangle + \sqrt{p_2}|\phi_2^{\mathrm{A}}\rangle \otimes |\varphi_2^{\mathrm{B}}\rangle$, where $|\phi_1^{\mathrm{A}}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\phi_2^{\mathrm{A}}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, $|\varphi_1^{\mathrm{B}}\rangle = |\phi_\theta\rangle$ and $|\varphi_1^{\mathrm{B}}\rangle = |\phi_{\theta-\pi}\rangle$, where $|\phi_\theta\rangle$ is defined previously. We consider binary postselection and employ the LCC $E_{✓}^{\mathrm{A}} = |\phi_1^{\perp \mathrm{A}}\rangle\langle\phi_1^{\perp \mathrm{A}}| + \varepsilon|\phi_2^{\mathrm{A}}\rangle\langle\phi_2^{\mathrm{A}}|$, where $|\phi_1^{\perp \mathrm{A}}\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$ is orthogonal to $|\phi_1^{\mathrm{A}}\rangle$ and $|\phi_2^{\perp \mathrm{A}}\rangle = 0$ so it does not appear. The performance of this compression channel is numerically calculated in Fig. 2.

Conclusion.—We propose a unified theory, which implies that quantum measurements can be viewed as either information-extracting apparatus as in the standard quantum metrology, or information filters as in the postselected quantum metrology. It can be employed to distribute the optimal measurements through postselections so that the cost of the final detections are dramatically reduced, in synergy with recent efforts on distributed quantum sensing (see, e.g., [51–54]). As a result, we anticipate our results will find applications in quantum sensing technologies, such as optical imaging and interferometry [49,50,55], magnetometry [56], frequency estimation [57], etc. Many problems are open for future exploration, including the compression of mixed states, multiparameter states [4,58], multipartite-entangled states, etc.

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[36] We are only concerned with fundamental advantages of postselections and therefore do not consider mixed initial states of the ancilla, which leads to inefficiency in reading the postselection outcomes, see Fig. 1.

[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.250802 for the expression of the quantum Fisher information (QFI) after the postselection measurement, the saturation conditions for the various bounds of the QFI, proof of Theorem 2 in the main text, calculations on the sensitivity of the postselected state, weak value amplifications and the example of restricted compression, and expressions of $|\partial_x^\perp\psi_x\rangle\langle\partial_x^\perp\psi_x|$ and $|\partial_x^\perp\psi_x\rangle\langle\psi_x|$ in terms of ρ_x.

[38] Alternatively, one may also wish to retain the desired outcomes, where the discarded set is $\{\sigma_{x|\omega}\}_{\omega\in\times}$, which will be discussed elsewhere. With Table I, the corresponding theory for the latter can be developed analogously.

[39] As mentioned previously, for $\omega\in\checkmark$, E_ω should not be null POVM operator and therefore $p(\omega|x)$ must be strictly positive and c cannot blow up. One may think in the limit $c=\infty$ and $p(\omega\in\checkmark|x)=0$, the compression is too strong so that the sample "crushes" and we will not collect any desired samples.

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