Near-Optimal Performance of Quantum Error Correction Codes

Guo Zheng,[1,*,†] Wenhao He,[2,3,*] Gideon Lee,[1] and Liang Jiang[1,‡]
[1]Pritzker School of Molecular Engineering, The University of Chicago, Chicago, Illinois 60637, USA
[2]School of Physics, Peking University, Beijing 100871, China
[3]Center for Computational Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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The Knill-Laflamme conditions distinguish exact quantum error correction codes, and they have played a critical role in the discovery of state-of-the-art codes. However, the family of exact codes is a very restrictive one and does not necessarily contain the best-performing codes. Therefore, it is desirable to develop a generalized and quantitative performance metric. In this Letter, we derive the near-optimal channel fidelity, a concise and optimization-free metric for arbitrary codes and noise. The metric provides a narrow two-sided bound to the optimal code performance, and it can be evaluated with exactly the same input required by the Knill-Laflamme conditions. We demonstrate the numerical advantage of the near-optimal channel fidelity through multiple qubit code and oscillator code examples. Compared to conventional optimization-based approaches, the reduced computational cost enables us to simulate systems with previously inaccessible sizes, such as oscillators encoding hundreds of average excitations. Moreover, we analytically derive the near-optimal performance for the thermodynamic code and the Gottesman-Kitaev-Preskill code. In particular, the Gottesman-Kitaev-Preskill code's performance under excitation loss improves monotonically with its energy and converges to an asymptotic limit at infinite energy, which is distinct from other oscillator codes.

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Introduction.—Quantum error correction (QEC) has central importance in scaling up quantum devices. The seminal work [1] by Knill and Laflamme (KL) outlines the necessary and sufficient conditions for exact QEC. The KL conditions are celebrated for their conciseness and computational efficiency. Practically, they have guided the discoveries and analysis of many state-of-the-art qubit codes [2–4] and oscillator codes [5–9].

The KL conditions deal with *exact* error correction, in that they tell us whether or not any given error is exactly correctable by a given code. However, there are two issues with this: first, the set of correctable errors may not correspond exactly to the errors that occur in real devices—one often needs to approximate practical noise sources by considering a truncated set of Kraus operators and/or through techniques such as Pauli Twirling [10,11]. Furthermore, codes that *exactly* correct leading order errors do not necessarily outperform codes that “approximately” correct errors at all orders [4,12]. Therefore, it is critical to develop a concise and efficiently computable metric (similar to the KL conditions) that quantifies the capabilities of general codes. Such an extension uncovers the fundamental limit set by the encoding and noise, which is a critical benchmark for code designs.

One such widely accepted benchmark is the channel (or process) fidelity [13–17]. Under this metric, the optimal recovery can be found through convex optimizations, which motivated works in optimization-based QEC [12–14,18–25]. The optimal recovery fidelity can also serve as a guide for encoding designs [12,15,21,26]. However, these methods have two drawbacks. First, although convex optimization algorithms are decently optimized, they remain computationally expensive compared to optimization-free methods. Consequently, past numerical works only work with small Hilbert space sizes, such as systems with fewer than five qubits [13,27,28] or oscillators containing at most ten average excitations [12,20,29]. Second, these optimization techniques are inherently numerical. While powerful, they do not yield analytical forms for many of the quantities we are interested in, e.g., the parametric dependence of the code performance.

Near-optimal recoveries [25,30–35] have the potential to circumvent these limitations. These channels have constructive forms and solve a relaxed optimization problem: their performances provide two-sided bounds on the optimal fidelity. These channels have led to attempts to generalize the KL conditions [25,31,36–38], further leading to the development of codes like the thermodynamic code [39–41]. However, the generalized conditions still involve optimizations and/or the Bures metric, which are generally challenging to analyze. The common solutions were to derive bounds on the scaling of the near-optimal fidelity assuming large system sizes, which no longer provides a two-sided bound on the optimal performance.

Moreover, the parametric dependence on other system parameters remains unknown.

In this Letter, we derive a concise and optimization-free performance metric, the near-optimal channel fidelity. The metric is achievable by the transpose channel [30], also known as the Petz channel [32,33,42], and provides a narrow two-sided bound on the optimal fidelity. Crucially, the only input to our expression is the QEC matrix, which is exactly what is required to verify the KL conditions. Therefore, our result is a quantitative generalization of the KL conditions for arbitrary codes and noise processes. We also develop a perturbative expression, providing intuition on how codes' performances are connected to the structures of their error subspaces. More importantly, the perturbative form well approximates the near-optimal fidelity and can be computed analytically. Taking multiple qubit codes as examples, we numerically validate the proximity of the closed-form near-optimal expression and the optimal fidelity obtained from convex optimizations. Furthermore, we analytically compute the near-optimal fidelity for the thermodynamic code in the thermodynamic limit, for which only a scaling with system size was known in past works. After examining a few representative oscillator codes, we provide rigorous insights on why certain codes' performances under excitation loss improve monotonically with increased energy, such as the Gottesman-Kitaev-Preskill (GKP) code [5]. While past numerical simulations of the GKP code were limited to a few average excitations, we extend our result to hundreds of excitations. We also obtain GKP's performance analytically, parametrized by system parameters and loss rates. With the analytical expression, we find the GKP's performance admits an asymptotic limit at infinite energy.

Background.—The Knill-Laflamme (KL) conditions [1] are the necessary and sufficient conditions for exact QEC codes. For completeness, we briefly review the conditions here. In QEC settings, the logical information is encoded through a code with d_L logical codewords $\{|\mu_L\rangle\}$, which subsequently passes through a noise channel $\mathcal{N}$ with Kraus form $\{\hat{N}_i\}$. The QEC matrix is defined as

$$M_{[\mu l],[\nu k]} = \langle \mu_L | \hat{N}_l^\dagger \hat{N}_k | \nu_L \rangle \tag{1}$$

in index notation. The KL conditions state that a code is a exact error-correcting code if and only if the QEC matrix can be written as $M = I_L \otimes A$, with I_L denoting a *logical* d_L-dimensional identity matrix.

The KL conditions assess codes with the assumption that any physical recovery is allowed. Such an idea can be extended to general codes: the performance of an encoding, $\mathcal{E}$, against certain noise, $\mathcal{N}$, is determined by the performance of the optimal recovery, $\mathcal{R}^{\text{opt}}$. To define optimality, in this Letter, we adopt the metric of channel fidelity. For any quantum channel $\mathcal{Q}$, the channel fidelity is defined as [14,16]

$$F(\mathcal{Q}) \coloneqq \langle \Phi | \mathcal{Q} \otimes \mathcal{I}_R(|\Phi\rangle\langle\Phi|)|\Phi\rangle, \tag{2}$$

where $|\Phi\rangle$ is the purified maximally mixed state, and $\mathcal{I}_R$ is the identity channel acting on the *reference* ancillary system. The optimal recovery, $\mathcal{R}^{\text{opt}}$, is defined as a recovery that achieves the optimal fidelity

$$F^{\text{opt}} \coloneqq \max_{\mathcal{R}} F(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E}) = F(\mathcal{R}^{\text{opt}}\circ\mathcal{N}\circ\mathcal{E}), \tag{3}$$

where $\circ$ indicates channel compositions. For discussions below, we refer to the channel fidelity as fidelity for simplicity. Our choice of metric is well-motivated by two important properties of the channel fidelity. First, the channel fidelity is directly connected to other widely adopted metrics such as the average input-output fidelity [43,44]. Second, the metric is linear in the Choi matrix of the recovery, which causes the optimization to fall in the category of semidefinite programming (SDP) [45], a subfield of convex optimization.

Main result.—Here, we propose the near-optimal fidelity as a quantitative metric that can be evaluated without optimization.

Theorem 1 (the near-optimal fidelity). For a d_L-dimensional encoding, $\mathcal{E}$, and a noise channel, $\mathcal{N}$, the near-optimal fidelity is

$$\tilde{F}^{\text{opt}} = \frac{1}{d_L^2} \left\| \text{Tr}_L \sqrt{M} \right\|_F^2, \tag{4}$$

where M is the QEC matrix, $(\text{Tr}_L B)_{l,k} = \sum_\mu B_{[\mu l],[\mu k]}$ denotes the partial trace over the code space indices, and $\|\cdot\|_F$ is the Frobenius norm. The near-optimal fidelity gives a two-sided bound on the optimal fidelity as

$$\frac{1}{2}\left(1 - \tilde{F}^{\text{opt}}\right) \leq 1 - F^{\text{opt}} \leq 1 - \tilde{F}^{\text{opt}}. \tag{5}$$

Worth noticing, the gap between the two-sided bound is proportional to the optimal infidelity. Therefore, when the code performs well against the noise channel, the near-optimal fidelity is a close approximation to the optimal fidelity. Equally importantly, it is remarkable that Eq. (4) is only dependent on the QEC matrix. Therefore, our metric requires exactly the same resources as the KL conditions, but it provides a quantitative performance metric beyond a binary Yes-or-No output. Moreover, our result can be further extended to general channel reversals [60,61], subsystem codes [62], and mixed-state codes [63]. Our result implies that the QEC matrix contains richer information about the code and noise structure beyond the KL conditions. For example, one can greatly reduce the complexity of optimization-based methods via adopting the error subspaces as an efficient basis to describe the action of the noise channel [45,64,65].

The near-optimal fidelity is achievable by the transpose channel [30,31] by construction, which is exactly where our expression inherits the two-sided bound. While there are other channels possessing similar near-optimal properties [25,34], the transpose channel has the most concise fidelity expression. The derivation of Eq. (4) is based on the observation that the QEC matrix is the Gram matrix of the error subspaces. When the QEC matrix is invertible, the error subspaces can be orthonormalized by the Gram matrix. In such an orthonormal basis, the transpose channel is equivalent to a measure-and-recover operation, which leads to the expression of Eq. (4). The derivation can be generalized to scenarios of degenerate QEC matrix [45].

Computationally, our approach has a drastically reduced cost compared to optimization-based methods. In many cases, the QEC matrix can be analytically computed. Otherwise, it is possible to efficiently compute the matrix depending on the code and noise of interest. In such cases, suppose we are considering N_K number of noise Kraus operators, the cost of evaluating $\tilde{F}^{\mathrm{opt}}$ is $\mathcal{O}((d_L N_K)^3)$. As a reference, the SDP for optimal recovery costs $\tilde{\mathcal{O}}((d_L N)^{5.246})$ [66,67], where N is the physical Hilbert space dimension. For example, if we consider qubit codes, N scales exponentially with the number of qubits, n. However, it is sufficient numerically to truncate the number of noise Kraus operators to a polynomial scaling, $N_K \propto n^r$. In many cases, r only depends on the target precision and physical error rates.

While the exact form of the near-optimal fidelity, Eq. (4), is an elegant expression, its advantage lies in its numerical complexity. The component of matrix square root makes it cumbersome to obtain an analytical expression for the near-optimal fidelity. Therefore, we develop the following corollary based on a perturbative decomposition of the QEC matrix.

Corollary 1. The noise channel's Kraus representation can be chosen such that $(1/d_L)\mathrm{Tr}_L M = D$, with M being the QEC matrix and D being a diagonal matrix. With the residual matrix $\Delta M := M - I_L \otimes D$, the near-optimal infidelity has a perturbative expansion through

$$1 - \tilde{F}^{\mathrm{opt}} = \frac{1}{d_L}\left\| f(D) \odot \Delta M \right\|_F^2 + \mathcal{O}\left(\frac{1}{d_L}\left\| f(D) \odot \Delta M \right\|_F^3\right), \tag{6}$$

where $f(D)_{[\mu l],[\nu k]} = [1/(\sqrt{D_{ll}} + \sqrt{D_{kk}})]$ and the Hadamard product $(A \odot B)_{ij} = A_{ij} B_{ij}$.

This corollary is proved with the Daleckii-Krein theorem of matrix square root expansions [45,68,69]. Equation (6) conveniently expresses the infidelity as a function of ΔM and D instead of the square root of M, thus making it tangible to obtain analytical expressions. Physically, $(1/d_L) I_L \otimes \mathrm{Tr}_L M$ and ΔM correspond to the correctable and uncorrectable QEC matrix, respectively. In Corollary 1, we applied a unitary to diagonalize the correctable matrix. However, it is generally nontrivial to analytically express the unitary. As a compromise, D can be instead defined as the diagonal entries of the correctable matrix, $\mathrm{diag}[(1/d_L)\mathrm{Tr}_L M] = \mathrm{diag}(D)$. Such a truncation overestimates the error, but its effect is negligible as long as the off-diagonal yet correctable elements are sufficiently small [45].

It is important to note that the infidelity includes contributions from the uncorrectable matrix modulated by a function of the correctable matrix, $f(D)$. Intuitively, while the overlaps between error subspaces lead to uncorrectable errors, their effects are weighted by the probabilities of their respective quantum trajectories, which are contained in D. Past works [12,21,70–73] have proposed code performance estimators based on the QEC matrix to perform efficient code optimizations. Nonetheless, they mostly only took into account the uncorrectable matrix, ΔM, or attempted to consider the effects of D in heuristic ways. As a comparison, Eq. (6) suggests a combination of effects from ΔM and D with guaranteed performance.

A few observations can be drawn from Eq. (6). For example, for the near-optimal and optimal fidelity, the error caused by the uncorrectable matrix is suppressed quadratically. While the quadratic scaling was also observed in Ref. [35], our result is not limited to one-parameter family of channels and instead presents the full expression. Moreover, Eq. (6) has a noteworthy property: if ΔM is traceless, the code is an exact QEC code when the perturbative form vanishes [45]. This is useful for applications that require vanishing error probability, such as computing the achievable rates [74].

Examples.—In Fig. 1, we numerically validate the two-sided bound presented in Eq. (5) for qubit codes under amplitude damping noise [45], which is a practical but non-Pauli noise channel. We adopt the convention that $[\![n,k,d]\!]$ represents encoding k logical qubits in n physical qubits with distance d, while for $((n,k,d))$, k represents the logical dimension instead. Even for well-studied codes like

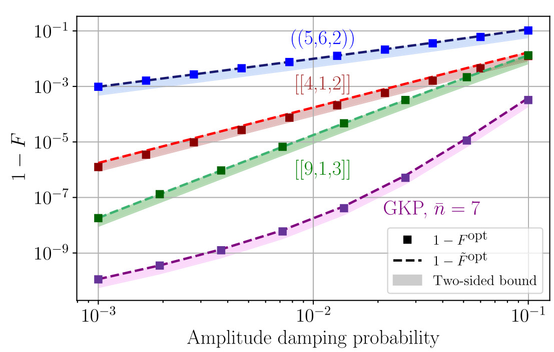

FIG. 1. Optimal infidelity for qubit codes $[\![4,1,2]\!]$, ((5,6,2)), and $[\![9,1,3]\!]$ and the GKP code with $\bar{n} = 7$. The noise channel is amplitude damping noise, i.e., loss for oscillators. The shaded regions represent the optimal infidelity intervals bounded by the two-sided bound given by the near-optimal infidelity in Eq. (5).

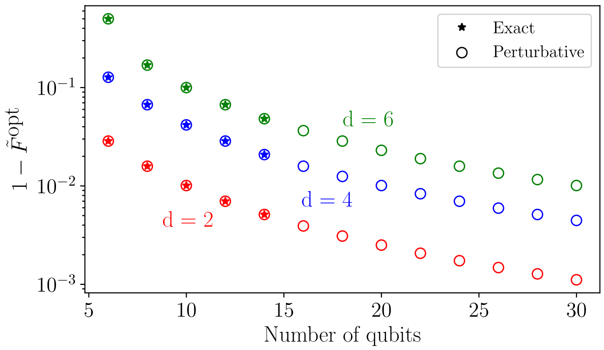

FIG. 2. The near-optimal infidelity of the thermodynamic code with distance d (see Ref. [45] for the definition of the code words) and $l = 1$, i.e., single erasure error. The stars represent the exact approach where the QEC matrix is numerically computed, and the circles represent the perturbative expression given in Eq. (7).

stabilizer codes, their optimal decoders under non-Pauli noise are in general unknown. The shown qubit codes include the classic $[\![9,1,3]\!]$ stabilizer code [75] and approximate $[\![4,1,2]\!]$ [4] code. Another example is the ((5, 6, 2)) code [76], which is a qudit nonstabilizer code.

The contributions of our expression lie not only in numerics but also in analytical aspects. We take the thermodynamic code [39,45] as an example. The code has sparked interest because of its close connections with the eigenstate thermalization hypothesis [77], as well as being an instance of a covariant code [40]. The precise definition of the code words are given in Ref. [45], and they are characterized by the distance d. When considering a constant number of erasure errors and thermodynamic limit, the optimal infidelity was proven to scale as $1 - F^{\rm opt} = \Theta(1/N^2)$ [40], where N is the number of qubits. Beyond a scaling argument, our expression makes it possible to derive the near-optimal infidelity analytically [45],

$$1 - \tilde{F}^{\rm opt} = \frac{l}{16}\frac{d^2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right), \tag{7}$$

for l erasure errors. The derivation of Eq. (7) with our formalism is straightforward and can be easily extended to consider more erasure errors [45]. The comparison of the perturbative and the exact forms of the near-optimal infidelity is shown in Fig. 2, where it is clear they closely follow each other. For the exact form, the QEC matrix is numerically computed, and the simulation stops at 14 qubits because of the increased computational cost. To compare, if we attempt to optimize for the optimal fidelity, it is only possible for less than 5 qubits under the same time constraint.

Bosonic codes, or oscillator codes, are codes that encode a d_L-dimensional logical space in the infinite-dimensional Hilbert space of oscillator(s). For a bosonic code, the average excitation number, $\bar{n} \coloneqq {\rm Tr}(\hat{n}\hat{P}_L)$, affects code properties and is also a parameter of experimental interest. In the inset plot of Fig. 3, we show the optimal performance of a few popular bosonic codes under excitation loss [45], including the binomial code [8], the cat code [7], and the GKP code [5]. An emerging feature is that for the binomial and cat codes, their performance does not improve monotonically with energy with fixed S. Here, S represents the Fock basis spacing, which can be

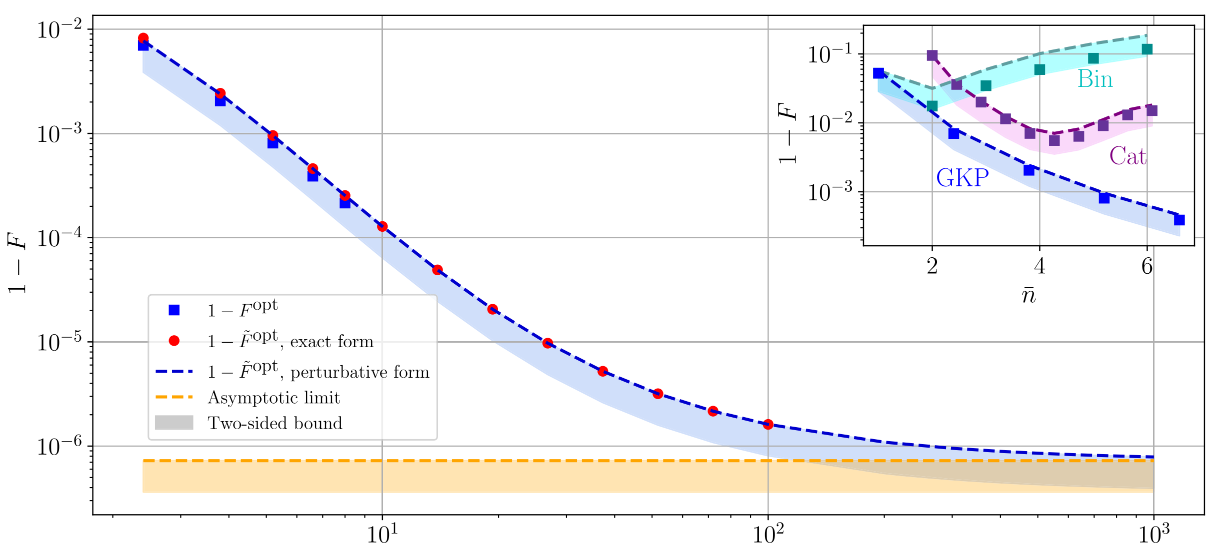

FIG. 3. The channel infidelity of square lattice GKP qubit code with $\gamma = 10\%$ loss. All shaded regions represent the two-sided bound on the optimal fidelity. The red circles represent the exact form of the near-optimal infidelity, where the QEC matrix is computed analytically. The blue dashed line is the perturbative form, evaluated through a closed-form analytical expression [78]. The inset figure shows the performance of GKP codes (square lattice), cat codes ($S = 4$), and binomial codes ($S = 1$) [45]. The squares (dashed lines) represent the optimal (near-optimal) fidelity.

understood as the distance against loss. On the contrary, the GKP code's performance improves monotonically for the range of $\bar{n}$ shown. Similar numerical observations were made in previous works [8,12], where heuristic explanations were given based on the QEC matrix. Our expression Eq. (6) supports their arguments with rigor: like exact qubit codes, the cat code and binomial code exactly correct no more than S excitation loss. However, they are completely unprotected against more than S loss, which is more likely to occur at larger $\bar{n}$. Therefore, the infidelity is not suppressed with energy.

The critical difference of GKP codes is that while they generally cannot even correct a single loss, the uncorrectable elements for all orders of loss are suppressed by a factor of $e^{-\{\gamma/[\gamma+(1/\bar{n})]\}}$ [12]. One open question was whether the suppression of the infidelity holds asymptotically: optimizing for the optimal fidelity was only possible for $\bar{n} \leq 10$ [12] because the Hilbert space size quickly becomes unmanageable. The near-optimal fidelity solves the problem. First, with the QEC matrix being analytically computable, the only complexity cost for the exact form in Eq. (4) is to compute the matrix square root. Thus, it is much cheaper than SDP optimizations, and the numerical results can reach $\bar{n} \sim 10^2$. Second, we can express the near-optimal performance analytically with the perturbative form [78], which reveals the near-optimal fidelity for arbitrarily large $\bar{n}$. In Fig. 3, we demonstrate the results for a GKP square code under loss rate $\gamma = 0.1$. At asymptotically large $\bar{n}$, the perturbative expression converges as

$$\lim_{\bar{n}\to\infty} 1 - \tilde{F}^{\mathrm{opt}} = e^{-\frac{\pi}{2}\frac{1-\gamma}{\gamma}}, \tag{8}$$

which is approximately 7.2×10^{-7} for $\gamma = 0.1$. As a comparison, the best decoder with known performance at infinite energy is the amplification decoder (AD) [12,20]. For the same loss, AD gives a logical error rate of $1 - F^{\mathrm{AD}} \approx e^{-(\pi/8)[(1-\gamma)/\gamma]} = 2.9 \times 10^{-2}$. Therefore, while the exceptionally low near-optimal fidelity highlights the promises of the GKP code construction, its gap with the performance of the known decoders emphasizes the potential gain in improving GKP decoders. Similar analysis can be performed for general multimode GKP encodings [78].

Discussion.—We have derived the near-optimal channel fidelity as a quantitative metric for arbitrary codes and noise channels. Our metric is closely related to optimal code performances through narrow two-sided bounds. Since the near-optimal fidelity only requires the QEC matrix as input, it generalizes the KL conditions beyond distinguishing exact QEC codes with the same resource cost. To conclude, the proposed metric and its perturbative form reduce computational costs for numerical simulations and enable us to obtain analytical descriptions of code performances.

Our new approach opens doors to numerical benchmarking of large-sized codes and oscillators encoding high-energy states. The benchmarking result can, in turn, be used to optimize or guide the discovery of novel encoding and efficient decoding for realistic noises. Moreover, it is critical to understand the near-optimal performance of codes in asymptotic limits for many concepts and settings in quantum information theory, such as the achievable rate of a code family.

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*These authors contributed equally to this work.
†Corresponding author: guozheng@uchicago.edu
‡Corresponding author: liang.jiang@uchicago.edu

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