Universal Measurement-Based Quantum Computation in a One-Dimensional Architecture Enabled by Dual-Unitary Circuits

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(Received 26 October 2022; accepted 14 May 2024; published 17 June 2024)

A powerful tool emerging from the study of many-body quantum dynamics is that of dual-unitary circuits, which are unitary even when read "sideways," i.e., along the spatial direction. Here, we show that this provides the ideal framework to understand and expand on the notion of measurement-based quantum computation (MBQC). In particular, applying a dual-unitary circuit to a many-body state followed by appropriate measurements effectively implements quantum computation in the spatial direction. We show how the dual-unitary dynamics generated by the dynamics of the paradigmatic one-dimensional kicked Ising chain with certain parameter choices generate resource states for universal deterministic MBQC. Specifically, after k time steps, equivalent to a depth-k quantum circuit, we obtain a resource state for universal MBQC on $\sim 3k/4$ encoded qubits. Our protocol allows generic quantum circuits to be "rotated" in space-time and gives new ways to exchange between resources like qubit number and coherence time in quantum computers. Beyond the practical advantages, we also interpret the dual-unitary evolution as generating an infinite sequence of new symmetry-protected topological phases with spatially modulated symmetries, which gives a vast generalization of the well-studied one-dimensional cluster state and shows that our protocol is robust to symmetry-respecting deformations.

DOI: 10.1103/PhysRevLett.132.250601

Introduction.—Recent years have seen significant advances at the frontier of many-body quantum dynamics. A particularly fruitful approach has been to study time evolution induced by quantum circuits, minimal models of dynamics in which degrees of freedom are updated by local unitary gates. Imposing structure on these gates lead to different classes of dynamics—including Clifford [1], matchgate [2–4], and Haar random circuits [5–12]—that allow for the efficient computation of physical quantities while still capturing different interesting regimes of behavior. A recent promising class is that of *dual-unitary circuits* [13–22], which are composed of gates that are not only unitary in the time direction, as required by dynamics in closed quantum systems, but also unitary in the space direction, upon exchanging the role of space and time. Despite this strong property, the class of dual unitaries is broad and rich, capturing both integrable and chaotic systems [13,20]. The versatility of this approach has seen many applications, allowing one to exactly compute spatiotemporal correlation functions [13,20], spectral statistics [23,24], and entanglement dynamics [17,25], thereby providing deep insights into phenomena like quantum chaos, information scrambling, and thermalization.

Intriguingly, the idea of regarding one spatial direction as an effective time direction along which a circuit runs appears already in an older topic, namely that of *measurement-based quantum computation* (MBQC) [26,27]. Here, the idea is that by creating an entangled many-body "resource" state using a finite-depth circuit and subsequently measuring the qubits, it is possible to effectively propagate quantum information through the spatial direction. The desired class of resource states is such that this spatial propagation is, indeed, unitary. Remarkably, there exist resource states in two spatial dimensions (2D) which are *universal*, meaning that this unitary evolution in the spatial direction can efficiently realize *any* quantum operation acting on *any* given number of encoded qubits [28]. Decades of research has uncovered a plethora of such universal resources [29–33], including a fault-tolerant protocol in 3D [34], but the full classification of universal resource states is still ongoing [35–41]. Both computation and fault tolerance in MBQC have been realized in proof-of-principle experiments [42–44].

In this Letter, we show how insights from dual unitarity can shed new light on MBQC, both at a conceptual and

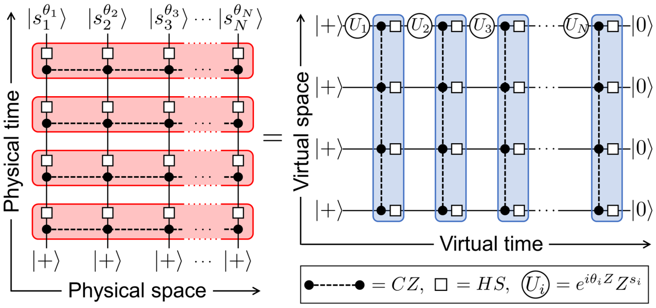


FIG. 1. Graphical depiction of Eq. (2) for $k = 4$. The red (blue) rounded box indicates one application of T_N (T_k). In accordance with the convention of quantum circuits, the order of matrix multiplication in the right figure is from left to right. The left side describes the preparation and measurement of the resource state $|\psi_k\rangle$, while the right side is the simulated universal quantum circuit.

practical level. Namely, reading a dual-unitary circuit in the time direction describes the preparation of the resource state, while reading it in the spatial direction directly reveals the logical circuit induced by appropriate measurement of the resource state, as pictured in Fig. 1. This provides an accessible alternative approach to MBQC beyond the traditional stabilizer [26], teleportation [45], or matrix product state-based [29] formalisms, and highlights how resource states can emerge naturally under certain classes of quantum many-body dynamics. These results also elevate dual unitarity from an abstract computational tool to a concept with direct practical application.

While Refs. [46–48] explored how measurements can stochastically induce a universal gate set in the spatial direction of dual-unitary circuits, performing scalable quantum computation requires deterministic control without postselection of measurement outcomes. Here, we show that, by using dual-unitary circuits that are additionally chosen to be Clifford, it is possible to implement deterministic MBQC without sacrificing universality. Specifically, we show that a depth-k dual-unitary circuit composed of repeated applications of uniform single-site and nearest-neighbor Ising entangling gates on a chain prepares a resource state for MBQC on k qubits. We then identify an unbounded sequence of k for which the resulting unitary evolution that can be efficiently implemented via measurement is universal on at least $\sim 3k/4$ qubits. Practically, our protocol allows one to trade between space and time resources in quantum computers and to access deeper circuits than are possible with existing MBQC schemes using a given number of measurements, all while requiring only spatially uniform nearest-neighbor and single-qubit operations in a 1D architecture.

From another perspective, the dual-unitary circuits we consider can be viewed as "infinite-order entanglers" for symmetry-protected topological (SPT) orders [49–52]. That is, after k time steps, the resulting state possesses 1D SPT order with a symmetry group and edge degeneracy that grow unboundedly with k. On one hand, this shows that certain dual-unitary circuits provide a new way to generate infinite families of SPT order, which have proven very useful to the study of SPT order in the past [53–61]. On the other hand, this insight connects our results to the large literature on using SPT phases as resources for MBQC [35–40,62–71]. In particular, it allows us to directly apply previous results [37,39,65] to show that our MBQC schemes work not only using the fixed-point states generated by the dual-unitary evolution, but also any generic deformations thereof preserving certain symmetries.

Resource states from a dual-unitary circuit.—We consider N qubits arranged on a line, denote Pauli operators including the identity as I, X, Y, Z, and write the eigenbasis of Z as $|0\rangle, |1\rangle$. Dynamics are generated by the kicked Ising model, which is defined by the Floquet evolution $U_F = e^{-ih\sum_{i=1}^N Y_i} e^{-iH_{\mathrm{Ising}}\tau}$, where $H_{\mathrm{Ising}} = J\sum_{i=1}^{N-1} Z_i Z_{i+1} + g\sum_{i=1}^N Z_i$ [72]. We fix $\tau = 1$ and $J, h = \pm\pi/4$ since the evolution generated by U_F is dual-unitary only for these parameters. The parameter g is chosen such that the evolution is also Clifford, meaning that any product of Pauli operators is mapped to another product of Pauli operators under conjugation by U_F. This important property will allow us compensate for random measurement outcomes and achieve deterministic computation. This property holds for $g = 0$ and $g = \pm\pi/4$, but we focus on the latter and leave the former to the discussions at the end. U_F is then equivalent to the quantum circuit (up to irrelevant phases and Pauli operators [75]),

$$T_N = \prod_{i=1}^{N} H_i S_i \prod_{i=1}^{N-1} CZ_{i,i+1}, \tag{1}$$

where we define the gates $CZ = I - 2|11\rangle\langle 11|$, $H = [(X+Z)/\sqrt{2}]$ and $S = \sqrt{Z} = \mathrm{diag}(1, i)$. We note that the single-qubit gate HS cyclically permutes the three Pauli operators under conjugation.

The resource states $|\psi_k\rangle$ are defined by acting on an initial product state with the unitary circuit k times, $|\psi_k\rangle = T_N^k(\otimes_{i=1}^N |+\rangle)$, where $|+\rangle = [(|0\rangle + |1\rangle)/\sqrt{2}]$. These states are pictured in Fig. 1. Since T_N is a Clifford circuit, the states $|\psi_k\rangle$ are stabilizer states that are uniquely defined by the equations $S_i^{(k)}|\psi_k\rangle = |\psi_k\rangle$, where $S_i^{(k)} = T_N^k X_i T_N^{k\dagger}$. Equivalently, $|\psi_k\rangle$ is the unique ground state of the gapped Hamiltonian $H_k = -\sum_i S_i^{(k)}$. For example, we have $S_i^{(1)} = X_{i-1}Y_i X_{i+1}$ (with modifications near the ends of the chain), so $|\psi_1\rangle$ is the 1D cluster state [26]. The 1D cluster state is a prototypical resource for MBQC on a single encoded qubit, and also a simple example of 1D SPT order [76].

Now we will describe a protocol for MBQC using the states $|\psi_k\rangle$ as resource states. The equivalence of our measurement-based scheme to the traditional unitary gate-based model will follow directly from the dual unitarity of T_N^k. In our protocol, each qubit in the chain is measured sequentially from left to right in a rotated basis $\{|0^\theta\rangle, |1^\theta\rangle\}$ defined by an angle θ where $|s^\theta\rangle = e^{-i\theta X}|s\rangle$. The output of the quantum computation is determined by the probabilities of obtaining different measurement outcomes, which, according to the Born rule, are given by the inner products $|\langle s_1^{\theta_1}, \ldots, s_N^{\theta_N}|\psi_k\rangle|^2$, where each $s_i = 0$, 1. To determine these overlaps, we utilize the dual-unitary property of the circuit T_N^k to read it "sideways," which gives [77]

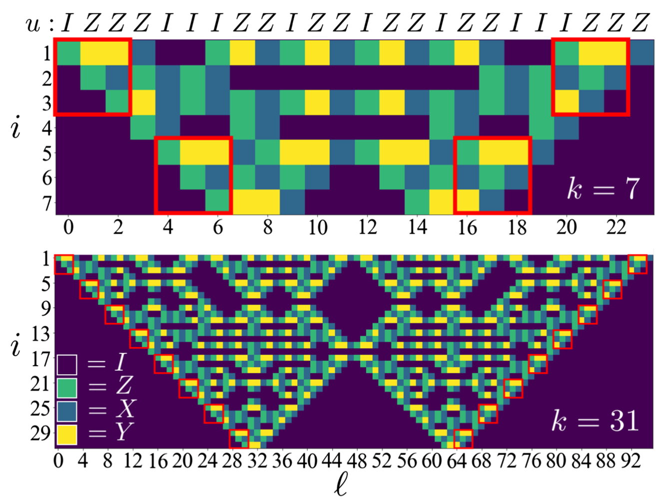
$$\langle s_1^{\theta_1}, \ldots, s_N^{\theta_N}|\psi_k\rangle = \langle R|U(\theta_N, s_N)\ldots U(\theta_1, s_1)|L\rangle, \quad (2)$$

where we have defined the vectors $|L\rangle = \otimes_{i=1}^k |+\rangle_i$ and $|R\rangle = \otimes_{i=1}^k |0\rangle_i$ and the unitary operator $U(\theta, s) = T_k e^{i\theta Z_1} Z_1^s$, where T_k is as defined in Eq. (1). This equation, which is depicted in Fig. 1, is the first step in proving universality of our protocol. It shows that the statistics arising from measuring the resource states $|\psi_k\rangle$ can be reinterpreted as describing a process in which k "virtual" qubits are initialized in a state $|L\rangle$, evolved by unitaries $U(\theta, s)$, and then projected onto a final state $|R\rangle$. The evolution during this process depends on the choice of measurement bases defined by the angles θ_i. Thus, the dual unitarity provides a natural perspective on how measurement of physical qubits translates into controllable unitary evolution of the virtual qubits [84].

The virtual computation described by the right-hand side of Eq. (2) currently has two issues. First, the unitary evolution depends on the measurement outcomes s_i, which are random. Second, the computation ends with a projection onto a fixed state $|R\rangle$ rather than a full projective measurement. It turns out that both issues are solved by adjusting future measurement bases depending on past measurement outcomes (as is common to all schemes of MBQC). We describe this in detail in Supplemental Material (SM) [77], and for the rest of the main text we always assume the outcome $|0^\theta\rangle$ is obtained. In short, the effect of obtaining the "wrong" measurement outcome $|1^\theta\rangle$ is to insert the byproduct operator Z_1 at that step in the computation. To deal with this unwanted operator, we imagine pushing it through to the end of the circuit. Importantly, because T_k is Clifford, Z_1 will remain a product of Pauli operators as it is pushed through each layer of the circuit, which therefore only has two controlled effects. First, depending on where the wrong outcome occurred, a subset of the rotation angles at later times will be flipped, $\theta_i \to -\theta_i$. This can be counteracted by flipping θ_i in the corresponding bases of future measurements. Second, once the byproduct operator is pushed to the end, it acts on $|R\rangle$ in such a way that $|R\rangle$ gets mapped onto a random product state in the $|0/1\rangle$ basis depending on the complete history of all measurement outcomes. Therefore, when accounting for the random measurement outcomes, repeating the protocol many times while recording the measurement statistics $|\langle s_1^{\theta_1}, \ldots, s_N^{\theta_N}|\psi_k\rangle|^2$ allows us to garner the measurement statistics $|\langle i_1, \ldots, i_k|\phi_{\text{out}}\rangle|^2$ for all $|i_j\rangle = |0/1\rangle$, where $|\phi_{\text{out}}\rangle = U(\theta_N)\ldots U(\theta_1)|L\rangle$ with $U(\theta_i) = T_k e^{i\theta_i Z_1}$ is the output state of the virtual computation. This constitutes a complete scheme of quantum computation, where we initialize a quantum register in a known state $|L\rangle$, perform deterministic unitary evolution on it, and read out the final output state in a fixed basis.

Determining the set of gates.—What remains is to determine which unitary circuits can be implemented using products of the unitaries $U(\theta)$. To understand these circuits, we make the important observation that, since T_k is unitary and Clifford, it has a finite period, meaning there is a smallest integer p_k such that $T_k^{p_k} \propto I$. The periods p_k for $k \leq 7$ are given in Fig. 2. Now, consider breaking the computation into blocks of length p_k. The net effect of measuring all spins in one block is

$$\prod_{\ell=0}^{p_k-1} U(\theta_\ell) = \prod_{\ell=0}^{p_k-1} T_k e^{i\theta_\ell Z_1} \propto \prod_{\ell=0}^{p_k-1} e^{i\theta_\ell O_k(\ell)}, \quad (3)$$



k	1	2	3	4	5	6	7
p_k	3	4	12	10	24	18	24
dim($\mathcal{A}_k$)	3	10	63	120	496	4095	8256
$\mathcal{A}_k$	$su(2^k)$	$sp(2^k)$	$su(2^k)$	$so(2^k)$	$so(2^k)$	$su(2^k)$	$sp(2^k)$

FIG. 2. Top: depiction of the operators $O_k(\ell)$ up to a phase for $k = 7$, 31 with respective periods $p_k = 24$, 96. Each column indexed by ℓ represents one product of Pauli operators acting on the k virtual qubits, which are indexed by $i = 1, \ldots, k$. The operator in one column is obtained from the previous via the local rules in Eq. (4). The string of I and Z at the top indicates the repeating pattern found in the symmetry of $|\psi_k\rangle$ for $k = 7$, where Z coincides with X and Y operators in the top row of the space-time operator evolution. Bottom: numerically determined computational power for small k. The lower row lists Lie algebras with dimensions consistent with the numerical calculations, where $su(n)$, $so(n)$, and $sp(n)$ denote the algebras of special unitary, orthogonal, and symplectic matrices.

where $O_k(\ell) \coloneqq T_k^{\ell\dagger} Z_1 T_k^{\ell}$. Therefore, the elementary gates in our scheme are k-qubit rotations generated by the operators $O_k(\ell)$, which are determined by the space-time evolution of Z_1 under conjugation by $T_k^\dagger$ a number ℓ times. Again, as T_k is a Clifford circuit, these operators will all be k-qubit Pauli operators. The evolution is determined (up to a possible factor of -1) from the following local rules:

$$T_k^\dagger X_i T_k = Z_i, \quad T_k^\dagger Z_i T_k = \begin{cases} Y_1 Z_2 & i = 1 \\ Z_{i-1} Y_i Z_{i+1} & 1 < i < k. \\ Z_{k-1} Y_k & i = k \end{cases} \tag{4}$$

The space-time evolution of Pauli operators starting with Z_1 subject to these rules generates a fractal pattern that is pictured in Fig. 2. Let $\mathcal{O}_k$ denote the set of all $O_k(\ell)$ for $\ell = 0, \ldots, p_k - 1$. For a small angles $d\theta$, we have $e^{id\theta P} e^{id\theta Q} \approx e^{id\theta(P+Q)}$ and $e^{id\theta P} e^{id\theta Q} e^{-id\theta P} e^{-id\theta Q} \approx e^{-(d\theta)^2 [P,Q]}$, where $[P, Q] = PQ - QP$ for $P, Q \in \mathcal{O}_k$. Therefore, by concatenating our elementary gates, we can perform any rotation of the form $R = e^{iA}$ where A is an element of the Lie algebra $\mathcal{A}_k$ generated by $\mathcal{O}_k$ using commutators. The set of such rotations is our set of implementable unitaries.

Ideally, we desire $\mathcal{A}_k = su(2^k)$, i.e., $\mathcal{A}_k$ contains all $2^{2k} - 1$ Pauli operators acting on k qubits, which gives universal computation on all k virtual qubits. At first glance, this seems impossible, since the only operations we use are Z rotations of the first qubit and the application of a fixed T_k to all qubits, so it is not clear how to selectively control a generic target qubit. However, we find that the persistent application of T_k allows us to convert temporal control into spatial control. A similar concept was used in Ref. [86]. Indeed, looking at Fig. 2, we see that each operator $O_k(\ell)$ acts differently on different qubits. By judiciously combining these operators, we can selectively control all virtual qubits.

As a first investigation into the form of $\mathcal{A}_k$, we numerically generate the operators in $\mathcal{O}_k$ for small k and repeatedly take commutators until no new operators are found. This is shown in Fig. 2 for $k \leq 7$. In each case, we get an algebra with a dimension that scales exponentially with the number of qubits, but we only get the full $su(2^k)$ in certain cases. While the full algebra $\mathcal{A}_k$ appears to depend on k in a complicated manner, we are able to prove the following lower bound on computational power:

Theorem 1.—For every $k \geq 3$, let $m = \lfloor (k+1)/4 \rfloor$. Then, the set of gates implementable in MBQC using the state $|\psi_k\rangle$ as a resource is universal on at least $3m$ qubits. That is, $su(2^{3m}) \subset \mathcal{A}_k$. If we further have $k = 2^r - 1$ for some $r \geq 0$, then the universal circuit model is guaranteed to be implemented with at most a linear overhead in k.

Therein, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. This is the main result of this Letter, as it means that the resource states $|\psi_k\rangle$ can be used for universal MBQC on $\sim 3k/4$ qubits. The proof of this universality, given in SM [77], uses the self-similar fractal nature of the space-time evolution of the operators $O_k(\ell)$. Namely, we make use of repeating structures in this evolution indicated in Fig. 2 to extend results for small values of k to arbitrarily large values of k. To prove efficiency, we show that the period p_k—which essentially sets a clock speed for our computation since each elementary gate $e^{i\theta O_k(\ell)}$ can only be applied once per period—is linear in k when $k = 2^r - 1$. Furthermore, the elementary gates in our scheme, namely the rotations $e^{i\theta O_k(\ell)}$, differ significantly from the standard gate set consisting of single-qubit rotations and nearest-neighbor two-qubit gates. Nevertheless, the proof of Theorem 1 shows how to efficiently construct any rotation of the form $e^{i\theta P}$ for an arbitrary Pauli string P—which includes the standard gate set—using a number of elementary gates $e^{i\theta O_k(\ell)}$ that is at most linear in k, such that our protocol can simulate the universal circuit model with polynomial overhead. While reducing to the standard gate set is convenient to implement existing quantum algorithms, we note that it does not take full advantage of our gate set, which, for example, also contains rotations that generate long-range entanglement in a single step (i.e., those for which $O_k(\ell)$ is supported on a large fraction of the virtual qubits).

Computational phases of matter.—We have described a universal scheme of MBQC using the states $|\psi_k\rangle$ as resource states. It turns out that these states can also be interpreted as fixed-point states of certain 1D symmetry-protected topological (SPT) phases of matter [49–51]. To describe the SPT order, we first need to identify the symmetries. For this, we notice that Eq. (2) defines a matrix product state representation of the wave function [87] from which the symmetries can be straightforwardly determined (see SM [77]). We find that the symmetry group is generated by operators that form a string of Z and I that repeats along the chain with period p_k, similar to so-called spatially modulated symmetries [88,89]. For $k = 1$, the symmetry has the form $ZZIZZI\ldots$ that repeats with a unit cell of size $p_1 = 3$. In general, the repeating pattern mirrors the top row of the space-time evolution of the operators $O_k(\ell)$, such that the symmetry is also deeply linked to the dual-unitary structure; see Fig. 2. These symmetry operators and their translations generate the total symmetry group $\mathbb{Z}_2^{2k}$.

The same matrix product state analysis also reveals the nature of the SPT order of the state $|\psi_k\rangle$ under the $\mathbb{Z}_2^{2k}$ symmetry. We find that the protected zero-energy edge mode that is characteristic of the SPT order has dimension 2^k, which is the maximal possible value for this symmetry group. Because of this, the general results of Refs. [37,39,65] can be directly applied to our context to show that the MBQC protocol we have developed for the fixed-point states $|\psi_k\rangle$ works, with some modification, for any resource state coming from the same SPT phase. Therefore, the ability to perform universal MBQC using single-site measurements is a property not only of the fine-tuned states $|\psi_k\rangle$, but also of the entire SPT phases of matter in which they reside.

Discussion.—We have defined a protocol, enabled by dual unitarity, to generate a new class of universal resource states for MBQC in a one-dimensional architecture using spatially uniform controls. Practically, our protocol represents a new way to embed quantum circuits in space-time. Namely, while the process implemented "in the lab" involves a circuit of depth k on N physical qubits, the simulated circuit has depth N and k qubits (up to constant factors) as in Fig. 1. This allows one to trade between qubit number and coherence time in quantum computers, thereby making optimal use of available resources. Furthermore, simulating a depth-N circuit on k qubits using typical MBQC protocols would require measuring Nk physical qubits [26], so our resource states can access deeper circuits using the same number of measurements.

From a fundamental standpoint, our results show that the dual-unitary circuit T_N^k can also be interpreted as an "infinite-order SPT entangler." Namely, consecutive applications of T_N to an initial product state generates an infinite sequence of 1D SPT orders with exponentially growing edge modes (when $N \to \infty$). While all previously defined circuits that generate SPT phases (i) have finite order such that $U^p = I$ for some p independent of N and (ii) generate phases with a fixed symmetry group G [52], our circuit T_N (i) has infinite order and (ii) generates SPT phases with a growing symmetry group $\mathbb{Z}_2^{2k}$. Our results therefore suggest the existence of a deep relationship between dual-unitary circuits, infinite-order SPT entanglers, and resource states for MBQC. We give a second example of this relationship in SM [77] by replacing $HS \to H$ in T_N, corresponding to $g = 0$ in the kicked Ising model. This circuit is also dual-unitary and generates an infinite family of 1D SPT ordered states that can be used for MBQC on k virtual qubits. However, in this case, the states are not universal resources since the set of gates implementable in MBQC generates an efficiently classically simulable matchgate circuit. This behavior is likely fine-tuned, and we give a more general study in SM [77]—where either H or HS is applied depending on the spatial location and time step—which suggests the conjecture that generic dual-unitary Clifford circuits will generate resources for universal MBQC. We leave a deeper exploration into the relationships between these three concepts, as well as the induced classification of MBQC resource states, for future work.

This work was initiated at the Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611. R. V. is supported by the Harvard Quantum Initiative Postdoctoral Fellowship in Science and Engineering. W. W. H. acknowledges support from the National University of Singapore startup Grant No. A-8000599-00-00 and No. A-8000599-01-00. This work was also partly supported by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (651440, DTS; 651440, RV). T.-C. W. acknowledges support by the Materials Science and Engineering Divisions, Office of Basic Energy Sciences of the U.S. Department of Energy under Contract No. DESC0012704. R. R. acknowledges funding from NSERC, USARO (W911NF2010013) and the Alexander von Humboldt Foundation.

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