Editors' Suggestion

Compatibility of Generalized Noisy Qubit Measurements

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It is a crucial feature of quantum mechanics that not all measurements are compatible with each other. However, if measurements suffer from noise they may lose their incompatibility. Here, we consider the effect of white noise and determine the critical visibility such that all qubit measurements, i.e., all positive operator-valued measures (POVMs), become compatible, i.e., jointly measurable. In addition, we apply our methods to quantum steering and Bell nonlocality. We obtain a tight local hidden state model for two-qubit Werner states of visibility $1/2$. This determines the exact steering bound for two-qubit Werner states and also provides a local hidden variable model that improves on previously known models. Interestingly, this proves that POVMs are not more powerful than projective measurements to demonstrate quantum steering for these states.

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Introduction.—Quantum mechanics provides a remarkably accurate framework for predicting the outcomes of experiments and has led to the development of numerous technological advancements. Despite its successes, it presents us with puzzling and counterintuitive phenomena that challenge our classical notions of reality. One of the key aspects that set quantum mechanics apart from classical physics is the concept of measurement incompatibility. In classical physics, measuring one property of a system need not affect the measurement of another property. In quantum mechanics, however, the situation is radically different. The uncertainty principle, formulated by Heisenberg, establishes a fundamental limit to the precision with which certain pairs of properties can be simultaneously known [1].

A simple and well-known example is the fact that we cannot simultaneously measure the spin of a particle in two orthogonal directions. It is known that incompatible measurements are at the core of many quantum information tasks. For example, they are necessary to violate Bell inequalities [2–4] and necessary to provide an advantage in quantum communication [5–7] or state discrimination tasks [8–10] (see also the reviews [11,12]).

However, measurement devices always suffer from imprecision. Therefore, an apparatus measures in practice only a noisy version of the measurements. If the noise gets too large, these noisy measurements can become compatible even though they are incompatible in the noiseless limit [13]. In that case, the statistics of these noisy measurements can be obtained from the statistics of just a single measurement, and we say that these noisy measurements are jointly measurable. However, a detector that can perform only compatible measurements has limited power. Most importantly, it cannot be used for many quantum information processing tasks like demonstrating Bell nonlocality since these require incompatible measurements. It is therefore important to ask, how much noise can be tolerated before all measurements become jointly measurable?

In this Letter, we study the effect of white noise and show that all qubit measurements become jointly measurable at a critical visibility of $1/2$. This result has direct implications for related fields of quantum information, in particular, Bell nonlocality [14,15] and quantum steering [16–22]. More precisely, we use the close connection between joint measurability and quantum steering [23–25] to show that the two-qubit Werner state [26],

$$\rho_W^\eta = \eta|\Psi^-\rangle\langle\Psi^-| + (1-\eta)\mathbb{1}/4, \tag{1}$$

cannot demonstrate quantum steering if $\eta \leq 1/2$. Here, $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ denotes the two-qubit singlet state. This also implies that the same state does not violate any Bell inequality for arbitrary positive operator-valued measurements (POVM) applied on both sides whenever $\eta \leq 1/2$.

Notation and joint measurability.—Before we introduce the problem, we introduce the necessary notation. Qubit states are described by positive semidefinite 2×2 complex operators $\rho \in \mathcal{L}(\mathbb{C}_2)$, $\rho \geq 0$ with unit trace $\mathrm{tr}[\rho] = 1$. They can be represented as $\rho = (\mathbb{1} + \vec{x}\cdot\vec{\sigma})/2$, where $\vec{x} \in \mathbb{R}^3$ is a three-dimensional real vector such that $|\vec{x}| \leq 1$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the standard Pauli matrices. In this notation, $\vec{x}$ is the corresponding Bloch vector of the qubit state. General qubit measurements are described by a POVM, which is a set of positive semidefinite operators $A_{i|a} \geq 0$ that sum to the identity $\sum_i A_{i|a} = \mathbb{1}$. Here, we use the label "a" to distinguish between different measurements, while

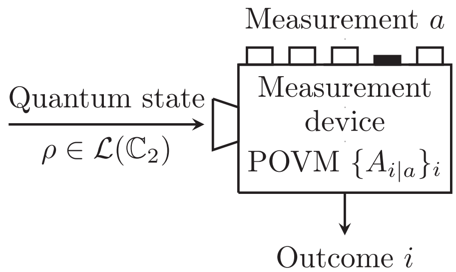

FIG. 1. A measurement device can perform different measurements (labeled with a) that produce an outcome i. If the measurements are too noisy they can be simulated by a device that just performs a single measurement. In this Letter, we address the question of how much white noise can be tolerated before all qubit measurements become jointly measurable.

"i" denotes the outcome of a given POVM (see also Fig. 1). In quantum theory, the probability of outcome i when performing the POVM with elements $A_{i|a}$ on the state ρ is given by Born's rule:

$$p(i|a,\rho) = \mathrm{tr}[A_{i|a}\rho]. \tag{2}$$

Because every qubit POVM can be written as a coarse graining of rank-1 projectors [27], we may restrict ourselves to POVMs proportional to rank-1 projectors. (We could also restrict ourselves to POVMs with at most four outcomes [28], but this is not necessary in what follows.) Thus, we write Alice's measurements as $A_{i|a} = p_i|\vec{a}_i\rangle\langle\vec{a}_i|$, where $p_i \geq 0$ and $|\vec{a}_i\rangle\langle\vec{a}_i| = (\mathbb{1} + \vec{a}_i\cdot\vec{\sigma})/2$ for some normalized vector $\vec{a}_i \in \mathbb{R}^3$ ($|\vec{a}_i| = 1$). As a consequence of $\sum_i A_{i|a} = \mathbb{1}$ we obtain $\sum_i p_i = 2$ and $\sum_i p_i\vec{a}_i = \vec{0}$.

These expressions are valid if all measurements are perfectly implemented. However, noise is usually unavoidable in experiments. In this Letter, we study the effect of white noise, where η denotes the visibility. More formally, we define the noisy measurements as

$$A^\eta_{i|a} = \eta A_{i|a} + (1-\eta)\,\mathrm{tr}[A_{i|a}]\mathbb{1}/2. \tag{3}$$

With the notation introduced above the POVM elements become $A^\eta_{i|a} = p_i(\mathbb{1} + \eta\vec{a}_i\cdot\vec{\sigma})/2$. The goal of this Letter is to determine the critical value of η such that all qubit POVMs become jointly measurable.

A set of measurements $\{A_{i|a}\}_{i,a}$ is jointly measurable if there exists a single measurement (so-called parent POVM) $\{G_\lambda\}_\lambda$ such that the statistics of all measurements in the set can be obtained by classical postprocessing of the data of that single parent measurement. More precisely, if for every POVM in the set there exist conditional probabilities $p(i|a,\lambda)$ such that

$$A_{i|a} = \sum_\lambda p(i|a,\lambda)G_\lambda. \tag{4}$$

If this is satisfied, all measurements in the set can be simulated by the single parent POVM with operators G_λ. First, the parent POVM is measured on the quantum state ρ in which outcome λ occurs with probability $p(\lambda|\rho) = \mathrm{tr}[G_\lambda\rho]$. Second, given the POVM labeled by a that we want to simulate, the outcome i is produced with probability $p(i|a,\lambda)$. In total, the probability of outcome i becomes

$$\sum_\lambda p(i|a,\lambda)p(\lambda|\rho) = \sum_\lambda p(i|a,\lambda)\mathrm{tr}[G_\lambda\ \rho] = \mathrm{tr}[A_{i|a}\ \rho]. \tag{5}$$

Here, we used the linearity of the trace. This perfectly simulates a given POVM with elements $\{A_{i|a}\}_i$, since this is the same expression as if the measurement was directly performed on the quantum state ρ given in Eq. (2).

The most prominent example are the two noisy spin measurements $A^\eta_{\pm|x} = (\mathbb{1} \pm \sigma_x/\sqrt{2})/2$ and $A^\eta_{\pm|z} = (\mathbb{1}\pm \sigma_z/\sqrt{2})/2$, where $\eta = 1/\sqrt{2}$. We can consider the following measurement with four outcomes $\lambda = (i,j)$, where $i,j \in \{+1,-1\}$:

$$G_{(i,j)} = \frac{1}{4}\left(\mathbb{1} + \frac{i}{\sqrt{2}}\sigma_x + \frac{j}{\sqrt{2}}\sigma_z\right). \tag{6}$$

One can check that this is a valid POVM and that $A^\eta_{i|x} = \sum_j G_{(i,j)}$ as well as $A^\eta_{j|z} = \sum_i G_{(i,j)}$. Therefore, the statistics of both measurements $\{A^\eta_{i|x}\}_i$ and $\{A^\eta_{j|z}\}_j$ can be obtained from the statistics of just a single parent measurement. Now we consider not only two but the set of all noisy qubit POVMs $\{A^\eta_{i|a}\}_{i,a}$ and show that for $\eta \leq 1/2$ this set becomes jointly measurable.

Protocol.—First, we define two functions. The first one is the sign function, which is defined as $\mathrm{sgn}(x) \coloneqq +1$ if $x \geq 0$ and $\mathrm{sgn}(x) \coloneqq -1$ if $x < 0$. Similarly, the function $\Theta(x)$ is defined as $\Theta(x) \coloneqq x$ if $x \geq 0$ and $\Theta(x) \coloneqq 0$ if $x < 0$ [or $\Theta(x) \coloneqq (|x| + x)/2$].

The parent POVM $\{G_{\vec{\lambda}}\}_{\vec{\lambda}}$ is the measurement with elements

$$G_{\vec{\lambda}} = \frac{1}{4\pi}(\mathbb{1} + \vec{\lambda}\cdot\vec{\sigma}). \tag{7}$$

Here, $\vec{\lambda} \in \mathbb{R}^3$ is a normalized vector uniformly distributed on the unit radius sphere S_2. Physically, this corresponds to a (sharp) projective measurement with outcome $\vec{\lambda}$, where the measurement direction is chosen Haar random on the Bloch sphere [29].

For a given POVM with operators $A^{1/2}_{i|a} = p_i(\mathbb{1} + \vec{a}_i\cdot\vec{\sigma}/2)/2$, where $\sum_i p_i = 2$, $|\vec{a}_i| = 1$, and $\sum_i p_i\vec{a}_i = \vec{0}$, we define the following function that associates a real-valued number to each point in $\vec{x} \in \mathbb{R}^3$:

$$f_a\colon\ \mathbb{R}^3 \to \mathbb{R}\colon\ f_a(\vec{x}) \coloneqq \sum_i p_i\Theta(\vec{x}\cdot\vec{a}_i). \tag{8}$$

Now, we choose an orthonormal coordinate frame of the Bloch sphere, defined by the three pairwise orthogonal unit vectors $\vec{x}', \vec{y}', \vec{z}' \in S_2$. In addition, we define the eight vectors $\vec{v}_{s_x s_y s_z} := s_x \vec{x}' + s_y \vec{y}' + s_z \vec{z}'$, where $s_x, s_y, s_z \in \{+1, -1\}$. This frame shall be chosen such that $f_a(\vec{v}_{s_x s_y s_z}) \leq 1$ for all of these eight vectors, and we show below that one can always find such a coordinate frame. Note that the vectors $\vec{v}_{s_x s_y s_z}$ are the vertices of a cube with side length two that is centered at the origin of the Bloch sphere.

After choosing a suitable frame, we can define the conditional probabilities:

$$p(i|a, \vec{\lambda}) = p_i \, \Theta(\vec{a}_i \cdot \vec{v}_{s_x s_y s_z}) + \frac{(1 - f_a(\vec{v}_{s_x s_y s_z}))\alpha_i}{\sum_i \alpha_i}. \quad (9)$$

Here, $\vec{v}_{s_x s_y s_z}$ is the vector with indices $s_k = \mathrm{sgn}(\vec{\lambda} \cdot \vec{k}')$ for $k \in \{x, y, z\}$. Hence, the three signs s_k denote the octant of $\vec{\lambda}$ in the rotated frame defined by $\vec{x}', \vec{y}', \vec{z}'$. (Equivalently, $\vec{v}_{s_x s_y s_z}$ is the vertex of the cube closest to $\vec{\lambda}$.) In addition, α_i is defined as

$$\alpha_i := \frac{p_i}{2}\left(1 - \frac{1}{4}\sum_{s_x, s_y, s_z = \pm 1} \Theta(\vec{a}_i \cdot \vec{v}_{s_x s_y s_z})\right). \quad (10)$$

Idea of the protocol.—Suppose for now that it is possible to find a suitable frame in which $f_a(\vec{v}_{s_x s_y s_z}) \leq 1$ for all eight vectors $\vec{v}_{s_x s_y s_z}$. Since this part is more technical, we discuss it at the end of this section. We can check first that the conditional probabilities are indeed well defined. Namely, they are positive and sum to one. Positivity follows from the fact that $p_i \geq 0$ and $\Theta(x) \geq 0$ (for all $x \in \mathbb{R}$). In addition, $f_a(\vec{v}_{s_x s_y s_z}) \leq 1$, and the proof that $\alpha_i \geq 0$ is given in Supplemental Material Sec. I [see Lemma 1 (2)] [30]. A quick calculation also shows that the probabilities sum to one:

$$\sum_i p(i|a, \vec{\lambda}) = f_a(\vec{v}_{s_x s_y s_z}) + (1 - f_a(\vec{v}_{s_x s_y s_z})) = 1. \quad (11)$$

Now we are in a position to show that

$$A_{i|a}^{1/2} = \int_{S_2} \mathrm{d}\vec{\lambda}\, p(i|a, \vec{\lambda}) G_{\vec{\lambda}}. \quad (12)$$

We give the detailed proof in Supplemental Material Sec. III [30], but sketch the main idea here. It is important to recognize that the function $p(i|a, \vec{\lambda})$ is the same for two different $\vec{\lambda}$ that lie in the same octant of the rotated frame $\vec{x}', \vec{y}', \vec{z}'$. Intuitively speaking, this leads to a coarse graining of the measurement outcomes $\vec{\lambda}$ in each of these octants. These coarse-grained operators $G_{s_x s_y s_z}$ behave like a noisy measurement in the direction of the corresponding vector $\vec{v}_{s_x s_y s_z}$. More precisely, we calculate in Supplemental Material Sec. III A that

$$G_{s_x s_y s_z} := \int_{S_2 | \mathrm{sgn}(\vec{\lambda} \cdot \vec{k}') = s_k} \mathrm{d}\vec{\lambda}\, G_{\vec{\lambda}} = \frac{\mathbb{1}}{8} + \frac{\vec{v}_{s_x s_y s_z} \cdot \vec{\sigma}}{16}. \quad (13)$$

With this definition, Eq. (12) becomes

$$A_{i|a}^{1/2} = \sum_{s_x, s_y, s_z = \pm 1} p(i|a, \vec{\lambda}) G_{s_x s_y s_z}. \quad (14)$$

Using the definition of $p(i|a, \vec{\lambda})$ in Eq. (9) and some algebra (details in Supplemental Material Sec. III), this reduces to

$$A_{i|a}^{1/2} = \sum_{s_x, s_y, s_z = \pm 1} p_i \, \Theta(\vec{a}_i \cdot \vec{v}_{s_x s_y s_z}) G_{s_x s_y s_z} + \alpha_i \mathbb{1}. \quad (15)$$

In the end, we prove this identity by using a closely related geometric formula that decomposes $\vec{a}_i$ into the vectors $\vec{v}_{s_x s_y s_z}$ (see Supplemental Material Sec. I):

$$\sum_{s_x, s_y, s_z = \pm 1} \Theta(\vec{a}_i \cdot \vec{v}_{s_x s_y s_z}) \vec{v}_{s_x s_y s_z} = 4\vec{a}_i. \quad (16)$$

The identity in Eq. (15) can be seen as the main idea of the protocol. We want to find a set of coarse-grained operators $G_{s_x s_y s_z}$ that can be used to decompose all the POVM elements $A_{i|a}^{1/2}$. The conditional probabilities $p(i|a, \vec{\lambda})$ given in Eq. (9) are then constructed according to this decomposition. The first term in $p(i|a, \vec{\lambda})$, namely, $p_i \, \Theta(\vec{v}_{s_x s_y s_z} \cdot \vec{a}_i)$, is the coefficient that comes from the decomposition of $A_{i|a}^{1/2}$ in terms of $G_{s_x s_y s_z}$. The second term in $p(i|a, \vec{\lambda})$ is constructed to add the noise term $\alpha_i \mathbb{1}$.

To give an example, consider the blue vector in Fig. 2 for which $\vec{a}_1 = (0, 0, 1)^T$ and $p_1 = 1/2$; hence, $A_{1|a}^{1/2} = p_1(\mathbb{1} + \vec{a}_1 \cdot \vec{\sigma}/2)/2 = \mathbb{1}/4 + \sigma_z/8$. It turns out that we can use the standard coordinate frame in which the cube vertices are simply $\vec{v}_{\pm\pm\pm} := (\pm 1, \pm 1, \pm 1)^T$. Direct calculation shows that $p_1 \, \Theta(\vec{a}_1 \cdot \vec{v}_{s_x s_y s_z}) = 1/2$ if $s_z = +1$ (and zero if $s_z = -1$) as well as $\alpha_1 = 0$. In addition, the coarse-grained operators become $G_{s_x s_y s_z} = \mathbb{1}/8 + (s_x \sigma_X + s_y \sigma_Y + s_z \sigma_Z)/16$. It is then easy to check that $1/2(G_{+++} + G_{+-+} + G_{-++} + G_{--+}) = A_{1|a}^{1/2}$.

However, while the identity in Eq. (15) holds for any orthonormal frame, it can be translated into a protocol with well-defined probabilities only if $\sum_i p_i \, \Theta(\vec{v}_{s_x s_y s_z} \cdot \vec{a}_i) = f_a(\vec{v}_{s_x s_y s_z}) \leq 1$ for all eight vertices of the cube. We show now that such a frame always exists. The proof has two steps. First, we show that for any such cube, it holds that

$$\sum_{s_x, s_y, s_z = \pm 1} f_a(\vec{v}_{s_x s_y s_z}) \leq 8. \quad (17)$$

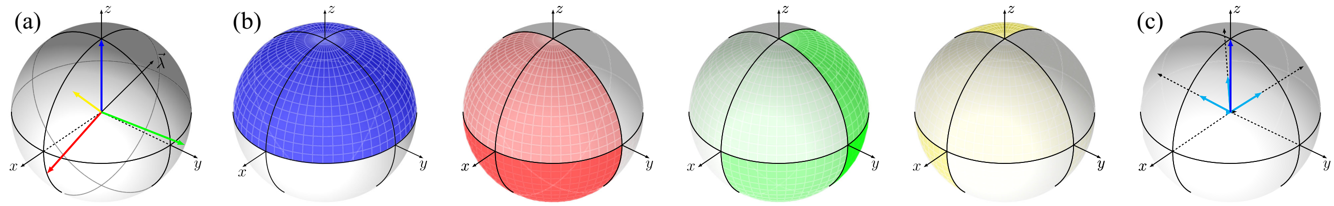

FIG. 2. An illustration for a SIC-POVM [31]. (a) The different outcomes i are represented with different colors and the colored vectors represent $\vec{a}_i$ (note also $p_i = 1/2$ for $i = 1, 2, 3, 4$). (b) The opacity of the colors represents the probability to output i given that $\vec{\lambda}$ lies in that region of the sphere, hence $p(i|a, \vec{\lambda})$. This function is constant in each octant of the chosen frame, which is simply the standard coordinate frame in this case. For the $\vec{\lambda}$ shown in the left-hand sphere ($s_x = -1$, $s_y = s_z = +1$), the outcome is most likely blue (50%) or green (49%). (c) Collecting all results $\vec{\lambda}$ from one octant behaves like the operator $G_{s_x s_y s_z}$ represented by the cyan arrows for the blue outcome. The sum of these operators simulates the desired (blue) operator $A^{1/2}_{1|a}$. (More details in Supplemental Material Sec. IV [30]).

The second part of the proof uses a theorem by Hausel *et al.* [32] (see Theorem 1 in that reference) that applies to continuous real-valued functions on S_2 that have the additional property that $f(\vec{x}) = f(-\vec{x})$. By using similar techniques as in Ref. [33], we show in Supplemental Material Sec. II that the function $f_a(\vec{x})$ fulfills these conditions [30]. In their theorem, they show that there always exists a rotation of the cube such that the functional values coincide at all eight vertices of that cube. Hence, choosing the orthonormal frame according to that rotation, we obtain $f_a(\vec{v}_{s_x s_y s_z}) = C$ for all $s_x, s_y, s_z \in \{+1, -1\}$. Combining this with the above bound in Eq. (17), we get $8C \leq 8$ and, therefore, $f_a(\vec{v}_{s_x s_y s_z}) \leq 1$ for that specific cube (see Supplemental Material Sec. II for more details).

The theorem in Ref. [32] is a special case of a family of so-called Knaster-type theorems. They state that for a given continuous real-valued function on the sphere, a certain configuration of points can always be rotated such that the functional values coincide at each of these points. Other interesting related results concerning S_2 are due to Dyson [34], Livesay [35], and Floyd [36]. Also, the well-known Borsuk-Ulam theorem is of this type [37].

We want to remark that we do not necessarily have to choose a cube in which all of these eight values coincide. It is only required that all of these eight values are smaller than one. Note that we do not give an explicit way to construct such a coordinate frame. However, in many cases, for instance, for POVMs with two or three outcomes, it turns out that an explicit construction can be found. We discuss this further in the Appendix as well as in Supplemental Material Sec. IV (see also there for further examples and more illustrations) [30].

Local models for entangled quantum states.—Now we apply the developed techniques to Bell nonlocality and quantum steering. Suppose Alice and Bob share a two-qubit Werner state [26],

$$\rho_W^\eta = \eta|\Psi^-\rangle\langle\Psi^-| + (1-\eta)\mathbb{1}/4, \tag{18}$$

where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ denotes the two-qubit singlet. They can apply arbitrary local POVMs on their qubit. As before, we denote Alice's measurement operators with $A_{i|a} = p_i|\vec{a}_i\rangle\langle\vec{a}_i| = p_i(\mathbb{1} + \vec{a}_i \cdot \vec{\sigma})/2$ (where $p_i \geq 0$, $|\vec{a}_i| = 1$, $\sum_i p_i = 2$, and $\sum_i p_i \vec{a}_i = \vec{0}$). Similarly, Bob can perform an arbitrary POVM with elements $B_{j|b}$ that are defined analogously. Note that Alice's and Bob's measurements are now completely arbitrary; i.e., they are not noisy. Instead, the entangled state is not pure but has a certain amount of white noise. The correlations when Alice and Bob apply local POVMs to this state become

$$p(i, j|a, b) = \mathrm{tr}[(A_{i|a} \otimes B_{j|b})\rho_W^\eta]. \tag{19}$$

It is a fundamental question in Bell nonlocality, for which η these correlations can violate a Bell inequality. It is known that two-qubit Werner states violate the CHSH inequality [38] for $\eta > 1/\sqrt{2} \approx 0.7071$. Vertesi showed that they violate another Bell inequality whenever $\eta > 0.7056$ [39].

On the other hand, Werner constructed in his seminal paper from 1989 a local model for all bipartite projective measurements if $\eta \leq 1/2$ [26] albeit these states are entangled if $\eta > 1/3$. Later, this bound was improved by Acin *et al.*, who showed that the state is local whenever $\eta \leq 1/K_G(3)$ [40]. Here, $K_G(3)$ is the so-called Grothendieck constant of order three and the best current bound is by Designolle *et al.*, $1.4367 \leq K_G(3) \leq 1.4546$ [41]. This implies that ρ_W^η is local if $\eta \leq 0.6875$ and violates a Bell inequality if $\eta \geq 0.6961$. However, these local models apply only to projective measurements (where $p_1 = p_2 = 1$ and $\vec{a}_2 = -\vec{a}_1$).

Considering general POVMs, Barrett found a local model for all POVMs whenever $\eta \leq 5/12$ [27]. Using a technique developed in Refs. [42,43], the best bound is again by Ref. [41] which shows that ρ_W^η is local for all POVMs if $\eta \leq 0.4583$. Based on the connections made in Refs. [23–25], we can now show that whenever $\eta \leq 1/2$ we cannot violate any Bell inequality since all correlations can be described by the following local model.

Suppose Alice performs her measurement $\{A_{i|a}\}_i$ on the Werner state with $\eta = 1/2$ and obtains outcome i. After doing so, Bob's qubit is precisely in the (unnormalized) postmeasurement state:

$$\rho_B(i) = \mathrm{tr}_A[(A_{i|a} \otimes \mathbb{1})\rho_W^{1/2}] = p_i(\mathbb{1} - \vec{a}_i \cdot \vec{\sigma}/2)/4. \quad (20)$$

It is now important to recognize that this state can be simulated with the same techniques as before, due to the duality of states and measurements. More precisely, consider the following protocol.

(1) Bob's system is in a well-defined pure qubit state $\rho_{\vec{\lambda}} = (\mathbb{1} + \vec{\lambda} \cdot \vec{\sigma})/2$, where $\vec{\lambda} \in \mathbb{R}^3$ is a normalized vector distributed Haar random on the unit radius sphere S_2.

(2) Alice chooses her POVM with operators $A_{i|a} = p_i(\mathbb{1} + \vec{a}_i \cdot \vec{\sigma})/2$. Now, she applies precisely the same steps as in the previous protocol for the given values of p_i, vectors $-\vec{a}_i$ ("$-$" to account for the anticorrelations in the singlet), and $\vec{\lambda}$. Namely, she chooses a suitable frame and produces her outcome i according to the conditional probabilities in Eq. (9).

(3) Bob chooses his POVM with elements $B_{j|b}$ and performs a quantum measurement on his state $\rho_{\vec{\lambda}}$.

The distribution of the state $\rho_{\vec{\lambda}}$, namely $(1/8\pi)(\mathbb{1} + \vec{\lambda} \cdot \vec{\sigma})$, is the same expression as the one for the parent POVM in Eq. (7) (up to a factor of 2 since states and measurements are normalized differently). Hence, if we sum over all the states where Alice outputs i, she samples precisely the state $p_i(\mathbb{1} - \vec{a}_i \cdot \vec{\sigma}/2)/4$ [analog to $A_{i|a}^{1/2} = p_i(\mathbb{1} + \vec{a}_i \cdot \vec{\sigma}/2)/2$ before]. This matches exactly the expression in Eq. (20). Intuitively speaking, there is no difference for Bob's qubit if Alice performs the protocol above or performs the measurement on the actual Werner state for $\eta = 1/2$. Therefore, when Bob applies his POVM, the resulting statistics are the same in both cases. Hence, the protocol above simulates the statistics of arbitrary POVMs applied to the state $\rho_W^{1/2}$ in a local way:

$$\mathrm{tr}[(A_{i|a} \otimes B_{j|b})\rho_W^{1/2}] = \frac{1}{4\pi}\int_{S_2} \mathrm{d}\vec{\lambda}\, p(i|-a, \vec{\lambda})\mathrm{tr}[B_{j|b}\rho_{\vec{\lambda}}]. \quad (21)$$

This model is even a so-called local hidden state model which implies that the state $\rho_W^{1/2}$ is not steerable [16,17,21]. In the most fundamental steering scenario, we consider two parties, Alice and Bob, that share an entangled quantum state. The question is whether Alice can steer Bob's state by applying a measurement on her side. However, Bob wants to exclude the possibility that his system is prepared in a well-defined state that is known to Alice. Then, Alice could just use her knowledge of the "hidden state" $\rho_{\vec{\lambda}}$ to pretend to Bob that she can steer his state. However, in reality, they do not share any entanglement at all. This is precisely the case in the above protocol, proving that the state ρ_W^η cannot demonstrate quantum steering whenever $\eta \leq 1/2$. This was known before for the restricted case of projective measurements $A_{\pm|a} = (\mathbb{1} \pm \vec{a} \cdot \vec{\sigma})/2$ [17]. When general POVMs are considered, the best model so far is the one from Barrett [27], which was shown to be a local hidden state model by Quintino *et al.* [44]. That model shows that ρ_W^η cannot demonstrate steering if $\eta \leq 5/12$. Numerical evidence suggested that the same holds for all $\eta \leq 1/2$ [45–47]. Our model shows that this is indeed the case.

On the other hand, if such a local hidden state model cannot exist, we say that the state is steerable. It is known that the two-qubit Werner state can demonstrate steering whenever $\eta > 1/2$ [17]. Therefore, the bound of $\eta = 1/2$ is tight. Because of the connection between steering and joint measurability [23–25], $\eta = 1/2$ is also tight for the joint measurability problem, ensuring the optimality of our construction.

Conclusion.—In this Letter, we provided tight bounds on how much white noise a measurement device can tolerate before all qubit measurements become jointly measurable. We considered the most general set of measurements (POVMs) and applied our techniques to quantum steering and Bell nonlocality. Exploiting the connection between joint measurability and steering [23–25], we found a tight local hidden state model for two-qubit Werner states of visibility $\eta = 1/2$. This solves Problem 39 on the page of Open quantum problems [48] (see also Ref. [49]) and Conjecture 1 of Ref. [46]. An important direction for further research is the generalization to higher dimensional systems [50,51].

Note added.—Recently, we became aware of the work by Zhang and Chitambar [52] that proves the same results with a different approach.

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Appendix.—Here, we provide some additional information about the nonconstructive nature of our protocol. We stress again that the theorem of Hausel *et al.* [32] implies only that a suitable coordinate frame exists but does not imply how to find one. However, in some cases, we can explicitly find a frame.

Consider, for instance, the important special case of a POVM with only two outcomes which corresponds to a projective measurement. In that case, we have $p_1 = p_2 = 1$ and $\vec{a}_2 = -\vec{a}_1$; hence, $A_{1|a}^{1/2} = (\mathbb{1} + \vec{a}_1 \cdot \vec{\sigma}/2)/2$ and $A_{2|a}^{1/2} = (\mathbb{1} - \vec{a}_1 \cdot \vec{\sigma}/2)/2$. We can express the function $f_a(\vec{x})$ as $f_a(\vec{x}) = \Theta(\vec{x} \cdot \vec{a}_1) + \Theta(-\vec{x} \cdot \vec{a}_1) = |\vec{x} \cdot \vec{a}_1|$. To find a suitable frame, we can choose the x' axis to be aligned

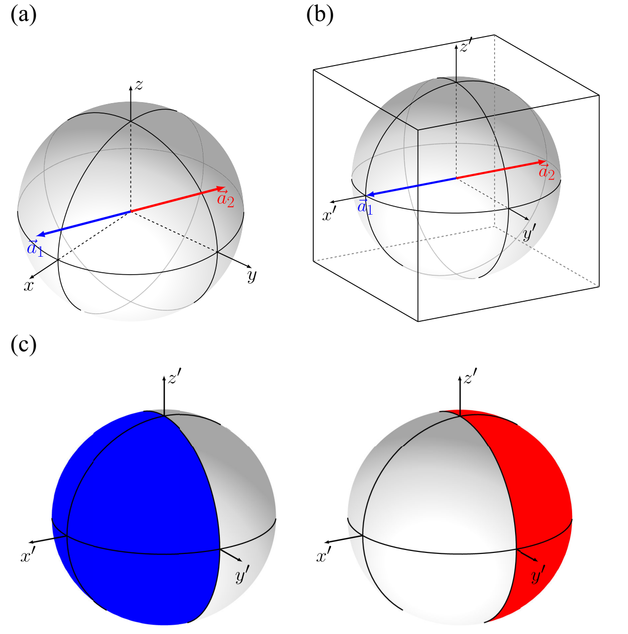

FIG. 3. Construction for the two-outcome POVM with operators $A^{1/2}_{i|a} = (\mathbb{1} + \vec{a}_i \cdot \vec{\sigma}/2)/2$ ($\vec{a}_2 = -\vec{a}_1$). (a) Here, $\vec{a}_1$ can be an arbitrary direction in the Bloch sphere. (b) We can choose the rotated frame such that the x' axis is aligned with $\vec{a}_1$. We also show the corresponding cube here. (c) The conditional probabilities $p(i|a,\vec{\lambda})$ reduce precisely to $p(1|a,\vec{\lambda}) = 1$ if $\vec{\lambda} \cdot \vec{a}_1 \geq 0$ and $p(1|a,\vec{\lambda}) = 0$ if $\vec{\lambda} \cdot \vec{a}_1 < 0$, as indicated with the two colors. Hence, if the outcome $\vec{\lambda}$ of the parent POVM lies in the hemisphere centered around $\vec{a}_1$ (blue region), the outcome is always $i = 1$, and if it lies in the hemisphere centered around $\vec{a}_2$ (red region), the outcome will be $i = 2$.

with $\vec{a}_{1/2}$, while the y' and z' axes are orthogonal to $\vec{a}_1$. In this way, $\vec{x}' = \vec{a}_1$, and direct calculation shows that $f_a(\vec{v}_{s_x s_y s_z}) = 1$ for all eight vertices $\vec{v}_{s_x s_y s_z} = s_x \vec{x}' + s_y \vec{y}' + s_z \vec{z}'$ as required. In addition, note that $\Theta(\vec{v}_{s_x s_y s_z} \cdot \vec{a}_1) = 1$ if $s_x = +1$ and $\Theta(\vec{v}_{s_x s_y s_z} \cdot \vec{a}_1) = 0$ if $s_x = -1$. Therefore, $\alpha_1 = \alpha_2 = 0$, and the conditional probabilities translate precisely to $p(1|a,\vec{\lambda}) = 1$ if $\vec{\lambda} \cdot \vec{a}_1 \geq 0$ and $p(1|a,\vec{\lambda}) = 0$ if $\vec{\lambda} \cdot \vec{a}_1 < 0$ (and the analog expression for $i = 2$). See Fig. 3 for an illustration.

The choice of the frame is unique up to an arbitrary rotation around the x' axis (and a relabeling of the axes). To see this, note that the angle α between $\vec{a}_{1/2}$ and each cube vertex $\vec{v}_{s_x s_y s_z}$ must be at least $\alpha \geq \cos^{-1}(1/\sqrt{3})$ since $|\vec{a}_{1/2}| = 1$, $|\vec{v}_{s_x s_y s_z}| = \sqrt{3}$, and $f_a(\vec{v}_{s_x s_y s_z}) = |\vec{v}_{s_x s_y s_z} \cdot \vec{a}_1| = |\vec{v}_{s_x s_y s_z}| \cdot |\vec{a}_1| \cdot \cos(\alpha)$. Geometrically, this defines the cube uniquely up to a rotation around $\vec{a}_1$. Note, however, that such a rotation would not change the conditional probabilities $p(i|a,\vec{\lambda})$ in the end.

It is worth pointing out that this construction becomes equivalent to the one of Werner [26], which is known to be a tight local hidden state model for projective measurements [17] (and therefore also tight for the problem of joint measurability due to the close connection of these two fields [23–25]).

It turns out that for the case of three outcome POVMs, we can also construct a suitable coordinate frame without relying on the theorem of Hausel *et al.* [32] but only on the intermediate value theorem for continuous functions (see Supplemental Material Sec. IV [30]). For general POVMs, we want to point out that a suitable frame is computationally easy to find for many cases. For instance, we can parametrize a rotation by its three Euler angles. When we discretize the three angles into equally spaced values, we can search through many possible rotations and calculate the functional values for the corresponding cube. If we find a cube, for which all of these eight values are smaller than one, we have found a suitable frame. We provide a MATLAB code for this simple algorithm via GITHUB [53]. It turns out that even this brute-force method finds a suitable frame for most POVMs almost immediately. (However, more sophisticated algorithms, are likely to perform better).

We did some numerical simulations with random POVMs. For that, we generate random points on the sphere $\vec{a}_i \in S_2$ and find p_i by solving $\sum_i p_i \vec{a}_i = \vec{0}$ and $\sum_i p_i = 2$. Then we use our algorithm to find a suitable frame. These numerical simulations strongly suggest that it is the hardest to find a frame if all directions are almost collinear ($|\vec{a}_i \cdot \vec{a}_j| \approx |\vec{a}_i| \cdot |\vec{a}_j|$ for all pairs i, j). Note that the two outcome POVMs from above are precisely of that form. However, even in these cases, a frame was always found in which the largest of the eight values $f(\vec{v}_{s_x s_y s_z})$ is only slightly larger than 1 and this value can be further decreased by further discretizing the Euler angles. There is an intuitive explanation for this effect. For the simulation of a given POVM, it is always advantageous if a given $\vec{\lambda}$ is mapped to an outcome i that is close, meaning that the angle between $\vec{\lambda}$ and $\vec{a}_i$ is small. Consider, for instance, the case of a projective measurement discussed before (or a POVM with almost collinear vectors $\vec{a}_i$). If $\vec{\lambda}$ is (almost) orthogonal to $\vec{a}_{1/2}$, the outcome of the parent measurement $\vec{\lambda}$ is (almost) uncorrelated to the $\vec{a}_{1/2}$ measurement, but it still has to be mapped to either $\vec{a}_1$ or $\vec{a}_2$.

Contrary to that, for a POVM with more outcomes $\vec{a}_i$ spread over the Bloch sphere (like the symmetric, informationally complete (SIC) POVM in Fig. 2), there are more options a given $\vec{\lambda}$ can be mapped to. Roughly speaking it is then more likely to find a measurement outcome $\vec{a}_i$ that is highly correlated with the actual measurement outcome of the parent POVM $\vec{\lambda}$. Based on this intuition, it is reasonable to expect that these POVMs are easier to simulate. In our construction, this expresses itself in the fact that for these POVMs many different coordinate frames are suitable, and therefore several different simulations for such a POVM and $\eta = 1/2$ exist. We can even prove that for the case of the four-outcome SIC POVM [31], any rotation can be chosen (see Supplemental Material Sec. IV [30]). On the contrary, $\eta = 1/2$ is known to be tight for the special case of two-outcome POVMs [17], and therefore only very particular coordinate frames are possible (similar for collinear POVMs).

Note also that we do not exclude the possibility that for certain POVMs better constructions with $\eta > 1/2$ exist. For instance, SIC POVMs are by definition very symmetric and one would expect that a symmetric model gives an even better bound $\eta > 1/2$ (e.g., one can map $\vec{\lambda}$ to the closest outcome $\vec{a}_i$ of the SIC POVM). However, in this Letter, we are merely concerned with finding one construction that works for all POVMs and $\eta = 1/2$. Hence, it is more important for our approach to recover the hemisphere construction of Fig. 3 (which is known to be tight for projective measurements) than to maintain the symmetry of a given POVM.

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