Editors' Suggestion

Exact Steering Bound for Two-Qubit Werner States

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Whether positive operator-valued measures (POVMs) provide advantages in demonstrating Bell nonlocality has remained unknown, even in the simple scenario of Einstein-Podolsky-Rosen steering with noisy singlet state, known as Werner states. Here we resolve this long-standing open problem by constructing a local hidden state model for Werner states with any visibility $r \leq 1/2$ under general POVMs, thereby closing the so-called Werner gap. This construction is based on an exact measurement compatibility model for the set of all noisy POVMs and also provides a local hidden variable model for a larger range of Werner states than previously known.

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Quantum entanglement enables behavior that is inconsistent with classical mechanics or any other theory satisfying the principle of locality [1,2]. One such behavior, known as Bell nonlocality, has been extensively studied [3], demonstrated experimentally [4,5], and utilized in different quantum information applications [6–9]. While entanglement is a necessary and sufficient ingredient for demonstrating nonlocality in pure states [10–12], the relationship between entanglement and nonlocality in general mixed states is much less clear [13–16]. Most notable is the existence of Bell local entangled states, which are those that cannot generate Bell nonlocality by themselves. The nature of nonlocality becomes even more interesting when considering a more general nonlocal effect known as quantum steering, which can be realized even for some Bell local states [17,18]. As originally envisioned by Schrödinger in 1935 [17,19], quantum steering involves a type of remote state preparation, and it is arguably the closest realization of Einstein's "spooky action at a distance".

Understanding the differences between entanglement, steering, and Bell nonlocality has been a fundamental challenge in quantum information science. The seminal work of Werner showed that not all entangled states are capable of generating Bell nonlocality by the explicit construction of local hidden variable (LHV) models [14]. The latter refers to theoretical models that reproduce the local measurement statistics of certain quantum states while still satisfying the principle of locality. Werner's original model only considered local projection-valued measures (PVMs), but Barrett later extended it to account for the most general type of local measurements, which are those described by positive operator-valued measures (POVMs) [15]. Moreover, Werner's and Barrett's models are even stronger in that they constitute what is now called a local hidden state (LHS) model. Such models simulate not only the local measurement statistics but also the postmeasurement quantum states for one party conditioned on the local measurement outcome of the other. States satisfying an LHS model are called unsteerable.

Since the work of Werner and Barrett, significant advances have been made in the construction of both LHV and LHS models [20–27]. Some of these models hold only for PVMs, while others encompass POVMs as well. The distinction between PVMs versus POVMs is crucial from a fundamental perspective. While a quantum measurement is traditionally understood as projecting or "collapsing" the given system onto one of many orthogonal subspaces [28], a full accounting of what quantum mechanics allows under local processing should include the use of local ancillas, the enabling ingredient for more general POVMs. The existence of nonprojective POVMs allows for statistical predictions in quantum theory that cannot be simulated using just PVMs and classical randomness [29], and it has led to unique quantum advantages in various state distrimination [30] and communication tasks [31]. Thus, the study of quantum nonlocality and quantum steering is incomplete if it is just limited to PVMs.

The question of whether PVMs are strong enough on their own to separate the class of steerable (nonlocal) states from unsteerable (local) states remained unsolved even for the simplest scenario of with two-qubit Werner states. The Werner states is the canonical family of states for investigating nonlocality due to its analytical simplicity and deep connections to other aspects of quantum information theory such as the potential existence of NPT bound entanglement [32] and channel nonadditivities [33]. So important is the steerability question for two-qubit Werner states that it currently sits on the Open quantum problems list, maintained by the Institute for Quantum Optics and Quantum Information (IQOQI) in Vienna (Problem 39 [34], rephrased as Problem 1 below). Strong numerical evidence [35] and explicit constructions for some special cases [36] suggest that POVMs provide no advantage over PVMs in qubit steering of Werner states. However, a full solution to the problem has remained elusive, as well as a systematic way of constructing an LHS model for POVMs. This Letter resolves these open questions and proves that POVMs and PVMs are indeed equivalent for the steerability of two-qubit Werner states.

Preliminaries.—In quantum steering [37] a bipartite state ρ_{AB} is shared between two observers, Alice and Bob. Alice implements a local measurement labeled by index x chosen from some family $\{M_{a|x}\}_{a,x}$, with $M_{a|x} \geq 0$, and $\sum_a M_{a|x} = \mathbb{I}$. The possible postmeasurement states for Bob's system are given by the state assemblage $\mathcal{E} = \{\sigma_{a|x}\}_{a,x}$, where $\sigma_{a|x} = \mathrm{Tr}_A[(M_{a|x} \otimes \mathbb{I})\rho_{AB}]$. The state assemblage is unsteerable if it admits an LHS model for all unnormalized states $\sigma_{a|x}$:

$$\sigma_{a|x} = \int d\lambda p(\lambda) p(a|x,\lambda) \rho_\lambda \quad \forall \ a, x, \tag{1}$$

where $p(\lambda)$ is a probability density function over variable λ shared between Alice and Bob, $p(a|x,\lambda)$ is a (stochastic) response function for Alice, and $\{\rho_\lambda\}_\lambda$ is a set of states for Bob satisfying $\int d\lambda p(\lambda)\rho_\lambda = \rho_B$. Essentially, if an assemblage satisfies Eq. (1), then there is arguably no quantum nonlocal effect on Bob's system when Alice measures her part of ρ_{AB} since the entire process can be equivalently simulated using just shared classical randomness λ. A given state is unsteerable if its state assemblage $\mathcal{E}$ prepared by any family of measurements $\mathcal{M}$ admits an LHS model above.

Similarly, a family of POVMs $\{M_{a|x}\}_{a,x}$ is defined to be jointly measurable (compatible) if there exists a compatible model [38]:

$$M_{a|x} = \int d\lambda p(a|x,\lambda) \Pi_\lambda \quad \forall \ a, x, \tag{2}$$

with response functions $p(a|x,\lambda)$ and "parent" POVM $\{\Pi_\lambda\}_\lambda$. The similar forms of Eqs. (1) and (2) is no coincidence, and a one-to-one correspondence between quantum steering and measurement incompatibility has been previously established [39–42]. Here we restate it in terms of the two-qubit Werner state $\rho_W(r)$ and the POVM $\{M^r_{a|x}\}_{a,x}$, which is the noisy version of $\{M_{a|x}\}_{a,x}$ having with the same visibility r, where

$$\rho_W(r) = r|\Psi_-\rangle\langle\Psi_-| + (1-r)\frac{\mathbb{I}\otimes\mathbb{I}}{4}, \tag{3}$$

$$M^r_{a|x} = rM_{a|x} + (1-r)\frac{\mathrm{Tr}(M_{a|x})\mathbb{I}}{2}, \tag{4}$$

with $|\Psi_-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, and index x running over the set of all qubit POVMs.

Lemma 1 [39–42].—The state assemblage obtained by measuring a Werner state $\rho_W(r)$ with $\{M_{a|x}\}_{a,x}$ admits an LHS mode if and only if there exists a parent POVM that can simulate their noisy form $\{M^r_{a|x}\}_{a,x}$.

Now let $\mathcal{M}^r_{\mathrm{POVM}}$ denote the entire collection of noisy qubit POVMs having visibility r [i.e., having the form of Eq. (4)] and let $\mathcal{M}_{\mathrm{POVM}} \coloneqq \mathcal{M}^1_{\mathrm{POVM}}$. We give similar definitions to the sets $\mathcal{M}_{\mathrm{PVM}}$ and $\mathcal{M}^r_{\mathrm{PVM}}$. The central questions answered in this Letter can then be stated as follows.

Problem 1: Werner's problem [34].—Determine the largest visibility r such that $\rho_W(r)$ satisfies an LHS model under all general measurement (POVMs) $\mathcal{M}_{\mathrm{POVM}}$.

Problem 2: Compatibility problem.—Determine the largest visibility r such that $\mathcal{M}^r_{\mathrm{POVM}}$ is a compatible family of measurements.

By Lemma 1, these problems are two sides of the same coin as their solutions identify the same threshold visibility r. For Problem 1, extensive research has been devoted to understanding the steering and Bell nonlocal bounds for Werner states [14,15,21,43–47]. Compared to the Bell nonlocal bound, the steering bound for two-qubit Werner states is better understood (Fig. 1), where the exact value of $r = 1/2$ has been proven when restricting the steering task to $\mathcal{M}_{\mathrm{PVM}}$ [17]. However, for POVMs, the exact steering bound is still unknown [34], with only the lower bound provided by [15].

Here we will provide a complete solution to Problem 1 by solving the formally equivalent Problem 2. The compatibility problem is fundamentally important in its own right since it helps address the related question of whether POVMs are more noise robust compared to standard PVMs [48–50]. Moreover, as we will exploit below, the compatibility problem is somewhat easier to tackle mathematically due to its relatively simple geometrical representation.

Remark.—In both problems it is sufficient to just consider extremal POVMs (in Problem 1) or their noisy

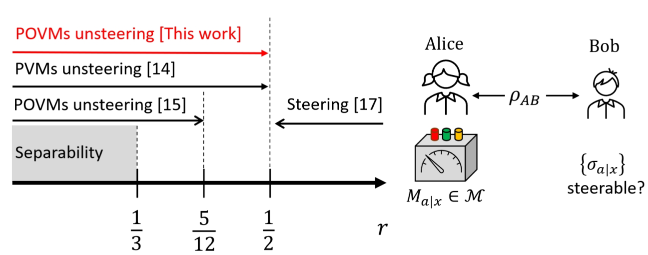

FIG. 1. Left: steering bounds for two-qubit Werner states $\rho_W(r)$ under projective measurements (PVMs) and POVMs. This Letter closes the "Werner gap" by raising the POVM lower bound to $r = 1/2$. Right: schematic of quantum steering, Alice performs measurements chosen from sets $\mathcal{M}$, and Bob checks whether the assemblage of states is steerable or not.

versions (in Problem 2), while nonextremal POVMs can be stochastically simulated from them [29]. For qubits, an extremal POVM and its noisy version consist of at most four effects $\{M_a\}_{a=1}^4$ that are each rank one [29] and represented in the Pauli basis as

$$M_a = \mu_a(\mathbb{I} + \hat{x}_a \cdot \vec{\sigma}) \xrightarrow{\text{noisy}} M_a^r = \mu_a(\mathbb{I} + r\hat{x}_a \cdot \vec{\sigma}),$$

where $\vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$ is the vector of Pauli matrices and $\hat{x}_a$ is the unit Bloch vector of effect M_a. From positivity and normalization, one has $\sum_a \mu_a = 1$, $\sum_a \mu_a \hat{x}_a = \vec{0}$, and $\mu_a \geq 0$, which together also imply that $\mu_a \leq 1/2$. Note that PVMs are a special subset of extremal POVMs consisting of two effects conveniently written in terms of a single Bloch vector $\hat{x}$ as $M_\pm = 1/2(\mathbb{I} \pm \hat{x} \cdot \vec{\sigma})$.

In what follows, we will use the Bloch vectors to parametrize the choice of measurement in a given family of measurements when the association is clear. For example, for a fixed visibility r, the set of all noisy PVMs can be parametrized by unit vectors on the Bloch sphere as $\mathcal{M}_{\text{PVM}}^r \coloneqq \{M_{\pm|\hat{x}}^r = 1/2(\mathbb{I} \pm r\hat{x} \cdot \sigma)\}_{\hat{x}}$. Likewise, any choice of a noisy extremal POVM $\{M_a^r\}_a$ can be identified by a collection of four subnormalized Bloch vectors $\{\vec{x}_a\}_a$, where $\vec{x}_a = \mu_a \hat{x}_a$ satisfy the positivity and normalization conditions specified above.

Compatible model.—The compatibility problem for the set of all noisy PVMs $\mathcal{M}_{\text{PVM}}^r$ has been studied in previous works [50,51], including a companion paper to this one [42]. It has been shown that there exists a compatible model for $\mathcal{M}_{\text{PVM}}^r$ using a finite-sized parent POVM whenever $r < 1/2$; while at $r = 1/2$, the infinite-outcome parent POVM $\{\Pi_{\hat{\lambda}} = (1/4\pi)(\mathbb{I} + \hat{\lambda} \cdot \vec{\sigma})\}_{\hat{\lambda}}$ provides a simulation. Explicitly, a compatible model that simulates a noisy PVM at $r = 1/2$ in any spin direction $\hat{x}$ is given by

$$\begin{aligned} M_{\pm|\hat{x}}^{r=1/2} &= \int_{\mathcal{S}} d\hat{\lambda} p(\pm|\hat{x}, \hat{\lambda}) \Pi_{\hat{\lambda}} = \int_{\mathcal{S}} d\hat{\lambda} \Theta(\pm\hat{x} \cdot \hat{\lambda}) \Pi_{\hat{\lambda}} \\ &= \frac{1}{2}\left(\mathbb{I} \pm \frac{1}{2}\hat{x} \cdot \vec{\sigma}\right), \end{aligned} \tag{5}$$

where the response function $p(\pm|\hat{x}, \hat{\lambda}) = \Theta(\pm\hat{x} \cdot \hat{\lambda})$, with Θ being the Heaviside step function. Our goal is to generalize this model for the simulation of all noisy POVMs at the same visibility $r = 1/2$.

As a starting point, we use the same parent POVM $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$ and try to extend the model in Eq. (5) to an arbitrary choice of POVM $\{M_a^r = \mu_a(\mathbb{I} + r\hat{x}_a \cdot \vec{\sigma})\}_a$ (identified by its Bloch vectors $\{\vec{x}_a = \mu_a \hat{x}_a\}_a$) by taking the response function to be $p(a|\{\vec{x}_a\}, \hat{\lambda}) = 2\mu_a \Theta(\hat{x}_a \cdot \hat{\lambda})$ [52]. This does, in fact, provide a decomposition of each effect $M_a^{r=1/2}$ in terms of the parent POVM $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$ since

$$M_a^{r=1/2} = \mu_a\left(\mathbb{I} + \frac{1}{2}\hat{x}_a \cdot \vec{\sigma}\right) = \int_{\mathcal{S}} d\hat{\lambda} 2\mu_a \Theta(\hat{x}_a \cdot \hat{\lambda}) \Pi_{\hat{\lambda}}. \tag{6}$$

Unfortunately, this response function is not normalized; i.e., $\sum_a p(a|\{\vec{x}_a\}_a, \hat{\lambda}) \neq 1$ for all $\hat{\lambda}$. To remedy this, we use the fact that the collection of response functions satisfying Eq. (6) is not unique, since the set of effects in $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$ is underconstrained (linearly dependent), and we can judiciously search it to find a normalized response function. We show below how this can be done, focusing first on three-outcome POVMs and then moving to four outcomes. By the remark above, this covers all extremal qubit POVMs and therefore solves the full problem.

Theorem 1.—The set of all noisy three-outcome POVMs $\mathcal{M}_{\text{3-POVM}}^{r=1/2}$ is compatible at $r = 1/2$.

A detailed proof is presented in the Supplemental Material [53]; two key ideas are backing our method.

(1.i) For each given $\{M_a^{r=1/2}\}_a$, coarse grain the parent POVM $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$ into a different finite-outcome POVM $\{\Pi_{\mathcal{A}}\}_{\mathcal{A}}$ that is capable of simulating each effect of $\{M_a^{r=1/2}\}_a$ individually (the hidden variable $\hat{\lambda}$ is now replaced with a coarse-grained varible $\mathcal{A}$). A natural choice of coarse-grained POVM $\{\Pi_{\mathcal{A}}\}_{\mathcal{A}}$ is given by which combinations of the Bloch vectors $\hat{x}_a$ are "on" for a given $\hat{\lambda}$ [i.e., $\Theta(\hat{x}_a \cdot \hat{\lambda})$ equaling one]. More precisely, we define

$$\Pi_{\mathcal{A}} = \int_{\mathcal{S}} d\hat{\lambda} \prod_{a \in \mathcal{A}} \Theta(\hat{x}_a \cdot \hat{\lambda}) \prod_{a' \notin \mathcal{A}} [1 - \Theta(\hat{x}_{a'} \cdot \hat{\lambda})] \Pi_{\hat{\lambda}}, \tag{7}$$

with $\mathcal{A} \subseteq \{1, 2, 3\}$ and $\Pi_\emptyset = \Pi_{\{1,2,3\}} = 0$. For the parent measurement $\{\Pi_{\mathcal{A}}\}_{\mathcal{A}}$, the starting (unnormalized) response function $p(a|\mathcal{A})$ for the $M_a^{r=1/2}$ defined in Eq. (6) is given by the values in Table I.

(1.ii) Smooth the response function using the (linear dependent) constraints on $\{\Pi_{\mathcal{A}}\}_{\mathcal{A}}$. Because of the two completion relations, $\sum_a M_a^{r=1/2} = \mathbb{I}$ and $\sum_A \Pi_{\mathcal{A}} = \mathbb{I}$, the six effects $\Pi_{\mathcal{A}}$ are linear dependent and satisfy

$$\sum_{a=1}^{3} q_a (\Pi_{\{a\}} - \Pi_{\{a\}^c}) = 0, \tag{8}$$

where $q_a \coloneqq 1 - 2\mu_a$ and $\mathcal{A}^c = \{1, 2, 3\} \backslash \mathcal{A}$ is the set complement of A. Moreover, the spherical symmetry in the coarse graining implies that for every $a, a' \in \{1, 2, 3\}$,

$$\frac{1}{\alpha_a}(\Pi_{\{a\}} + \Pi_{\{a\}^c}) - \frac{1}{\alpha_{a'}}(\Pi_{\{a'\}} + \Pi_{\{a'\}^c}) = 0, \tag{9}$$

TABLE I. Unnormalized response function $p(a|\mathcal{A})$ for simulating $\{M_a^{r=1/2}\}_{a=1}^3$ with $\{\Pi_{\mathcal{A}}\}_{\mathcal{A}}$.

	$\Pi_{\{2,3\}}$	$\Pi_{\{1,3\}}$	$\Pi_{\{1,2\}}$	$\Pi_{\{3\}}$	$\Pi_{\{2\}}$	$\Pi_{\{1\}}$
$M_1^{r=1/2}$	0	$2\mu_1$	$2\mu_1$	0	0	$2\mu_1$
$M_2^{r=1/2}$	$2\mu_2$	0	$2\mu_2$	0	$2\mu_2$	0
$M_3^{r=1/2}$	$2\mu_3$	$2\mu_3$	0	$2\mu_3$	0	0

TABLE II. Normalized response function $p'(a|\mathcal{A})$ for simulating $\{M_a^{r=1/2}\}_{a=1}^3$ with $\{\Pi_\mathcal{A}\}_\mathcal{A}$.

	$\Pi_{\{2,3\}}$	$\Pi_{\{1,3\}}$	$\Pi_{\{1,2\}}$	$\Pi_{\{3\}}$	$\Pi_{\{2\}}$	$\Pi_{\{1\}}$
$M_1^{r=1/2}$	0	$2\mu_1$	$2\mu_1$	0	0	$2\mu_1$
$M_2^{r=1/2}$	$2\mu_2 - X$	0	$1-2\mu_1$	0	1	Y
$M_3^{r=1/2}$	$2\mu_3 - Y$	$1-2\mu_1$	0	1	0	X

where $\alpha_a \coloneqq \mathrm{Tr}(\Pi_{\{a\}})/2 = \mathrm{Tr}(\Pi_{\{a\}^c})/2 > 0$. We can then smooth the response function $p(a|\mathcal{A})$ in Table I by adding or subtracting Eqs. (8) and (9), thereby changing the weights of the different effects $\Pi_\mathcal{A}$ while maintaining a simulation of the $M_a^{r=1/2}$. In particular, a new normalized function $p'(a|\mathcal{A})$ is specified in Table II, where

$$X = \frac{\alpha_1 q_1 - \alpha_2 q_2 + \alpha_3 q_3}{2\alpha_1}, \qquad Y = \frac{\alpha_1 q_1 + \alpha_2 q_2 - \alpha_3 q_3}{2\alpha_1}.$$

For $p'(a|\mathcal{A})$ to be a well-defined response function, all the values in Table II must be non-negative and each column must be normalized. Normalization can be verified by inspection since $X + Y = q_1 = (1-2\mu_1)$ and $\sum_a \mu_a = 1$; non-negativity is proven in the Supplemental Material [53]. Therefore, we conclude that

$$M_a^{r=1/2} = \sum_A p'(a|\mathcal{A})\Pi_\mathcal{A}, \tag{10}$$

and so any arbitrary three-outcome noisy measurement $\{M_a^{r=1/2}\}_a \in \mathcal{M}_{\text{3-POVM}}^{r=1/2}$ can be simulated by a common POVM $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$ with a response function $p'(a|\mathcal{A})$ defined above. Note that Eq. (7) can be analytically computed as shown in the Supplemental Material [53]. Along with the response function in Table II, the compatible model is explicitly constructed. We also emphasize a key property in the proof of Theorem 1.

Observation 1.—To renormalize the response function for the $\{\Pi_\mathcal{A}\}_\mathcal{A}$ of a three-outcome POVM $\{M_a^{r=1/2}\}_{a=1}^3$, it is sufficient to change the response function for two of the effects and leave the third one untouched.

Observation 1 is simply based on the construction given in Table II, where the first row is left unchanged, and it will be critical in extending Theorem 1 to the four-outcome case.

Theorem 2.—The set of all noisy four-outcome POVMs $\mathcal{M}_{\text{4-POVM}}^{r=1/2}$ is compatible at $r = 1/2$.

Since this covers all extreme qubits POVMs, $\mathcal{M}_{\text{POVM}}^{r=1/2}$ is a compatible family of measurements. The full proof is given in the Supplemental Material [53], including codes for the explicit construction of compatible models for any extreme POVMs [54,55], but conceptually it can also be broken into the following two steps, as depicted in Fig. 2.

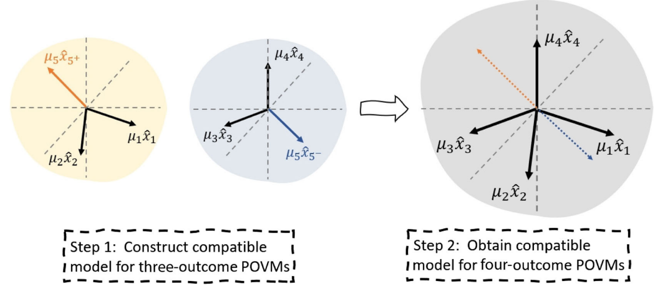

FIG. 2. A schematic for constructing a compatible model for a four-outcome measurement $\{M_a^{r=1/2}\}_{a=1}^4$. Step 1: construct compatible models for POVM$_+$ and POVM$_-$ without changing the response function of "pseudo effects" $M_{5^\pm}^{r=1/2}$. Step 2: build a compatible model for the four-outcome measurement by combining the response function from step 1 and the coarse-grained POVM defined by all six vectors $\{\hat{x}_{a=1}^4\} \cap \{\hat{x}_{5^\pm}\}$.

(2.i) Introduce two additional "pseudo effects" $M_{5^\pm}^{r=1/2}$ for the purpose of construction only:

$$M_{5^\pm}^{r=1/2} \coloneqq \mu_5\left(\mathbb{I} + \frac{1}{2}\hat{x}_{5^\pm}\cdot\vec{\sigma}\right), \tag{11}$$

with $-\hat{x}_{5^+} = \hat{x}_{5^-} = [(\mu_1\hat{x}_1 + \mu_2\hat{x}_2)/|\mu_1\hat{x}_1 + \mu_2\hat{x}_2|]$, $\mu_5 = |\mu_1\hat{x}_1 + \mu_2\hat{x}_2|$. The given POVM $\{M_a^{r=1/2}\}_{a=1}^4$ can then be split into two POVMs, each of three effects,

$$\mathrm{POVM}_+ = \frac{1}{\kappa_+}\{M_{5^+}^{r=1/2}, M_1^{r=1/2}, M_2^{r=1/2}\},$$
$$\mathrm{POVM}_- = \frac{1}{\kappa_-}\{M_{5^-}^{r=1/2}, M_3^{r=1/2}, M_4^{r=1/2}\}, \tag{12}$$

where $\kappa_+ = \mu_1 + \mu_2 + \mu_5$ and $\kappa_- = \mu_3 + \mu_4 + \mu_5$ to ensure normalization of POVM$_\pm$.

Define outcome sets $\mathcal{O}_+ = \{3, 4, 5^+\}$ and $\mathcal{O}_- = \{1, 2, 5^-\}$. By Theorem 1, POVM$_\pm$ can be simulated by a coarse-grained POVM $\{\Pi_{\mathcal{A}_\pm}^\pm\}_{\mathcal{A}_\pm}$ whose effects are

$$\Pi_{\mathcal{A}_\pm}^\pm = \int_\mathcal{S} d\hat{\lambda} \prod_{a\in\mathcal{A}_\pm} \Theta(\hat{x}_a\cdot\hat{\lambda}) \prod_{a'\notin\mathcal{A}_\pm}[1 - \Theta(\hat{x}_{a'}\cdot\hat{\lambda})]\Pi_{\hat{\lambda}}, \tag{13}$$

with $\mathcal{A}_\pm \subset \mathcal{O}_\pm$ and compatible models as

$$\frac{1}{\kappa_\pm}M_a^{r=1/2} = \sum_{\mathcal{A}_\pm\subset\mathcal{O}_\pm} p_\pm(a|\mathcal{A}_\pm)\Pi_{\mathcal{A}_\pm}^\pm \quad \text{for } a \in \mathcal{O}^\pm. \tag{14}$$

Crucially, by Observation 1 and Table II, the response function for $M_{5^\pm}^{r=1/2}/\kappa_\pm$ can be taken as $p_\pm(5^\pm|\mathcal{A}_\pm) = 2\mu_5/\kappa_\pm$ if $5^\pm \in A_\pm$ and $p_\pm(5^\pm|\mathcal{A}_\pm) = 0$ otherwise.

(2.ii) Use the combined set of Bloch vectors $\{\hat{x}_a\}_{a=1}^4 \cup \{\hat{x}_{5^\pm}\}$ to define an 18-effect POVM $\{\Pi_\mathcal{A}\}_\mathcal{A}$:

$$\Pi_\mathcal{A} = \int_\mathcal{S} d\hat{\lambda} \prod_{a\in\mathcal{A}} \Theta(\hat{x}_a \cdot \hat{\lambda}) \prod_{a'\notin\mathcal{A}} [1 - \Theta(\hat{x}_{a'} \cdot \hat{\lambda})]\Pi_{\hat{\lambda}}, \quad (15)$$

where $A \subset \{1, 2, 3, 4, 5^+, 5^-\}$. The construction of nonnegative and normalized response functions $p(a|\mathcal{A})$ based on Eq. (14) is carried out in the Supplemental Material [53].

Theorem 2 implies that all noisy extreme qubit POVMs in $\mathcal{M}_{\mathrm{POVM}}^{r=1/2}$ can be simulated by a common parent POVM $\{\Pi_{\hat{\lambda}}\}_{\hat{\lambda}}$. Together with the fact that all nonextremal POVMs can be stochastically obtained from those extremal POVMs, one can immediately conclude that the $\mathcal{M}_{\mathrm{POVM}}^r$ is compatible at $r = 1/2$, which coincides with the compatibility threshold for $\mathcal{M}_{\mathrm{PVM}}^r$. Not only does this solve both Problems 1 and 2, but it implies that POVMs offer no advantage over PVMs in terms of incompatibility noise robustness.

Implications for Bell nonlocality.—Bell nonlocality is no weaker than steerability in the sense that every LHS model for one-sided measurements on a bipartite state ρ_{AB} can be converted into an LHV model for two-sided measurements [17,18]. Such a connection has been implicitly used by Werner [14] and Barrett [15] in deriving their original LHV models for the Werner state with visibility $r = 1/2$ and $r = 5/12$ under PVMs and POVMs, respectively. Since these initial results, a breakthrough was made in the construction of LHV models under PVMs by relating the problem to finding upper bounds on Grothendieck's constant [44,47,56]. This ensures that $\rho_W(r)$ is Bell local under PVMs whenever $r \leq \approx 0.6875$. However, for general POVMs this method does not directly apply, and the best known locality bound under POVMs is $r \leq \approx 0.4583$ [47], which is derived by simulating noisy POVMs with PVMs [57]. This leaves a gap in the known locality range of $\rho_W(r)$ for PVMs versus POVMs. Our Theorem 2 makes substantial progress toward closing that gap.

Proposition 1.—There exists an LHV model for general measurements on the Werner state $\rho_W(r)$ when $r \leq 1/2$.

Conclusions.—In this Letter we have derived an exact bound for steering two-qubit Werner states under positive operator-valued measures and closed an open question in the literature [34]. Understanding the role of general POVMs in Bell nonlocality has been a puzzle for over three decades and no satisfactory answer has been given. The absence of examples demonstrating the need for nonprojective POVMs in Bell nonlocality implies that using projective measurements may be adequate to uncover this most bizarre prediction of quantum theory. Our result makes an important step in this direction by giving the first concrete example indicating that POVMs and PVMs are equivalent in the simplest and most well-studied Bell scenario of Einstein-Podolsky-Rosen steering with Werner states. However, there is undoubtedly a lot more to understand about general measurements and their significance in the context of nonlocality, and many of these unsolved problems are related to the subject of our Letter. First, it is natural to ask whether our results in deriving LHS (compatibility) models under general measurements can be extended from qubits to systems with arbitrary dimensions. The second interesting problem involves exploring the distinction between PVMs and POVMs in other restricted quantum steering or Bell nonlocality scenarios. For instance, one could consider quantum steering and Bell nonlocality when using finite amounts of shared randomness or when ρ_{AB} is a state with no symmetry. Finally, even among the Werner family of states, our methods are not strong enough to construct LHV models $r > 1/2$. Consequently, when $r > 1/2$ it remains an open question whether PVMs and POVMs are equally powerful for realizing nonlocality.

Note added.—Recently, we learned of the independent work of Renner [58] that solves Werner's problem using a different approach.

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[52] For simplicity, we remove the index label $\{\hat{x}_a\}_a$ for the rest of the Letter, but it is implicitly assumed when $\{M_a^r\}$ is defined. Similarly for $p(a|\lambda)$.

[53] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.250201, which includes Refs. [54,55], for detailed proof on Theorems 1 and 2 and discussion on numerical calculation.

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