Exceptional-Point Sensors Offer No Fundamental Signal-to-Noise Ratio Enhancement

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Exceptional-point (EP) sensors exhibit a square-root resonant frequency bifurcation in response to external perturbations, making them appear attractive for sensing applications. However, there is an open debate as to whether or not this sensitivity advantage is negated by additional noise in the system. We settle this debate by showing that increased fundamental noises of quantum and thermal origin in EP sensors, and in particular self-excited (or PT-symmetric) EP sensors, negate the sensitivity benefit. Accordingly, EP sensing schemes are only beneficial either with further quantum enhancement or if compared to sensors limited by technical noise. As many modern sensors are limited by technical noise, EP sensors may still find practical uses despite their lack of a fundamental advantage. Alternatively, we propose a quantum-enhanced EP sensor that achieves a sensing advantage even when limited by quantum or thermal fluctuations.

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Introduction.—A classical exceptional-point (EP) system with modes a and b is governed by [1]

$$\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = -i\boldsymbol{H}_{\mathrm{EP}} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{1}$$

where, by definition, the matrix $\boldsymbol{H}_{\mathrm{EP}}$ has a degenerate spectrum with a single complex eigenvalue $\Omega_{\mathrm{EP}} - i\gamma_{\mathrm{EP}}$. Any physical realization of such a system has to be open, i.e., featuring gain and loss.

A classical EP system is said to be an EP sensor if the degeneracy of $\boldsymbol{H}_{\mathrm{EP}}$ is lifted by a perturbation of the form $\boldsymbol{H}_{\mathrm{EP}} + \epsilon\boldsymbol{H}_{\mathrm{pert}}$, such that for $\epsilon \ll 1$, the eigenfrequencies bifurcate as $\sqrt{\epsilon}$; $\epsilon = 0$ is termed the exceptional point. Their claimed utility as a sensor is that if ϵ is proportional to some external parameter of interest, then, in contrast to conventional sensors whose eigenfrequencies bifurcate as ϵ^1, EP sensors exhibit enhanced sensitivity because of the $\epsilon^{1/2}$ scaling near the EP [2–10].

However, the efficacy of a sensor is not decided by large sensitivity to the quantity being sensed, but rather by its imprecision, which depends on both its sensitivity and added noise. Since the sensitivity of EP sensors is well understood, controversy has swirled around the fundamental noise inherent in EP sensors. Some say these sensors offer a fundamental advantage even when limited by fundamental quantum or thermal noise [11,12] or excess fundamental noises can be mitigated by operating somewhat away from the point of degeneracy [13]. Others disagree that EP-based sensors offer a fundamental sensing advantage [14–20].

Part of the controversy is due to the restricted validity of EP sensing models. If an EP sensor operates near an EP with less gain than loss, it has no macroscopic mode amplitudes unless it is excited externally, and can be analyzed as a parameter estimation problem as in Refs. [15,16,19]. However, when an EP sensor is operated near an EP with equal gain and loss, it acts as a two-mode laser above threshold and must be treated differently. This regime has eluded previous theoretical analyses.

In fact, in an experiment demonstrating an EP sensor above its lasing threshold [17]—a Brillouin ring laser gyroscope—excess noise was found to exactly cancel any enhancement from the gyroscope's frequency splitting near its EP. Our analysis shows that this behavior is characteristic of all EP sensors operated above threshold.

Methods of EP sensing.—We break down the class of all EP sensors into subcategories of sensors, which are more amenable to individual treatment. Until we discuss quantum-enhanced EP sensors, we assume these sensors have phase-insensitive gain and reciprocal coupling, i.e., the coupling rate from a to b is the same as from b to a.

EP sensors are primarily divided into passive EP sensors—those with no gain—and active EP sensors—which have gain. References [14,16,19] all conclude that passive EP sensors have no observable $\sqrt{\epsilon}$ bifurcation and thus no fundamental sensing improvement over traditional schemes. We verify these results using the quantum noise formalism of this Letter in the Supplemental Material, Sec. II. A [21].

Since passive EP sensors have no fundamental sensing benefit, we consider active EP sensing schemes with classical equations of motion,

$$\begin{aligned} \dot{a} &= -i\Omega_0 a - \gamma a + \frac{1}{2}(g+\gamma)b \\ \dot{b} &= -i\Omega_0 b + gb - \frac{1}{2}(g+\gamma)a. \end{aligned} \tag{2}$$

This system has one eigenfrequency, with real part $\Omega_{EP} = \Omega_0$ and imaginary part $\gamma_{EP} = (\gamma - g)/2$. These systems undergo a lasing transition around $\gamma_{EP} = 0$ ("loss" = "gain"). Marginally above threshold ($\gamma_{EP} = 0^-$), or beyond it ($\gamma_{EP} < 0$), the mode amplitudes have a finite value. Below threshold ($\gamma_{EP} > 0$), there is no finite mode amplitude and the sensor is operated by an external drive. As shown in Ref. [16] (see also Sec. II of the Supplemental Material [21]), active EP sensors below threshold have no observable $\sqrt{\epsilon}$ bifurcation.

Thus, the only remaining cases to be analyzed are active EP sensors above threshold with balanced gain and loss, or beyond threshold with more gain than loss [22]. EP sensors operating beyond threshold are unstable as one of their modes is continually amplified. In any physical system, saturation or other nonlinear mechanisms conspire to eventually stabilize this runaway; i.e., the gain g is decreased, or the loss γ increased, until the system reverts to a stable state above threshold with balanced gain and loss rates, i.e., $g = \gamma$ and $\gamma_{EP} = 0$.

Such above threshold EP systems, i.e., EP systems with $\gamma_{EP} = 0$, are "PT-symmetric." Their utility as a sensor is qualitatively different depending on which elements of the coupling matrix are perturbed: the diagonal elements, corresponding to the modes' frequencies and gain or loss rates, or the off-diagonal elements corresponding to the coupling between the modes. Section III in the Supplemental Material [21] analyzes perturbations to the diagonal elements. These perturbations either fail to lift the degeneracy, lead to a linear-in-ϵ response, push the system below threshold where the analysis of [16] and Supplementary Material, Sec. II. B [21] applies, or are equivalent to perturbing the off-diagonal matrix elements.

In sum, the only possible type of EP sensor that has not been ruled out by previous analyses and has an enhanced signal near the EP is a PT-symmetric EP sensor, where the quantity being sensed alters the coupling between the sensor's modes.

The classical mode amplitudes of such a PT-symmetric sensor satisfy

$$\dot{a} = -i\Omega_0 a - \gamma a + \gamma(1+\epsilon)b$$
$$\dot{b} = -i\Omega_0 b + \gamma b - \gamma(1+\epsilon)a, \tag{3}$$

where ϵ is the perturbation being detected by the sensor relative to its EP with $\Omega_{EP} = \Omega_0$ and $\gamma_{EP} = 0$. We must have $\epsilon > 0$ for the sensor to be stable; otherwise one of the system's normal modes is continually amplified, which is unphysical. So the sensor emits an output mode with a finite amplitude whose frequency depends on ϵ, which is, by assumption, coupled to the sensed quantity.

Fundamental frequency noise in PT-symmetric EP sensors.—Since EP sensors are ultimately limited by quantum and thermal fluctuations, the classical model of Eq. (3) is incomplete as far as noise performance is concerned. A minimal quantum model that reproduces Eq. (3) in expectation value can be obtained by promoting the mode amplitudes to operators and adding appropriate noise terms that ensure the preservation of appropriate commutation relations between the operators [23–27]. We also decompose ϵ into its mean value and fluctuations as $\epsilon = \bar{\epsilon} + \delta\epsilon$, where $\bar{\epsilon} \equiv \langle\epsilon\rangle$. This decomposition will allow us to consider the EP sensor as a probe for weak forces (modeled by $\delta\epsilon$) or as a probe to estimate the unknown parameter $\bar{\epsilon}$. The resulting equations of motion for the quantized mode amplitudes are

$$\dot{\hat{a}} = (-i\Omega_0 - \gamma)\hat{a} + \gamma(1 + \bar{\epsilon} + \delta\epsilon)\hat{b} + \sqrt{2\gamma}\delta\hat{a}_{\text{in}}$$
$$\dot{\hat{b}} = (-i\Omega_0 + \gamma)\hat{b} - \gamma(1 + \bar{\epsilon} + \delta\epsilon)\hat{a} + \sqrt{2\gamma}\delta\hat{b}^\dagger_{\text{amp}}. \tag{4}$$

Here, modes $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{amp}}$ have zero expectation value. Their noise properties are quantified by the symmetrized, double-sided spectra $\bar{S}^{\text{in}}_{qq} = \bar{S}^{\text{in}}_{pp} = \frac{1}{2} + n_{\text{in}}$ and $\bar{S}^{\text{amp}}_{qq} = \bar{S}^{\text{amp}}_{pp} = \frac{1}{2} + n_{\text{amp}}$, for their amplitude and phase quadratures, $\hat{q}_{\text{in,amp}} \equiv (\hat{a}^\dagger_{\text{in,amp}} + \hat{a}_{\text{in,amp}})/\sqrt{2}$ and $\hat{p}_{\text{in,amp}} \equiv i(\hat{a}^\dagger_{\text{in,amp}} - \hat{a}_{\text{in,amp}})/\sqrt{2}$, respectively [28]. In these spectra, $n_{\text{in,amp}}$ are the average thermal occupation numbers, which can be zero; however quantum (vacuum) noise gives rise to the $\frac{1}{2}$ terms, which are unavoidable.

In order to observe the system's frequency shifts, it is necessary to consider the sensor's output mode. The natural candidate is the output mode [27] $\hat{a}_{\text{out}} = \sqrt{2\gamma}\hat{a} - \hat{a}_{\text{in}}$, leaking out through the lossy element, as depicted in Fig. 1(a). (A similar input-output relation exists for the mode $\hat{b}_{\text{out}}$, but this is the amplifier's out-coupled mode and is usually unobservable.) A physical EP sensor's loss comes from a beam-splitter interaction, and the output mode is transmitted out of the sensor's feedback loop through this beam splitter, as in Fig. 1(d).

Taking quantum expectation values of Eq. (4) gives the classical amplitudes $a \equiv \langle\hat{a}\rangle$ and $b \equiv \langle\hat{b}\rangle$

$$a(t) = a_+ e^{-i\Omega_+ t} + a_- e^{-i\Omega_- t}$$
$$b(t) = b_+ e^{-i\Omega_+ t} + b_- e^{-i\Omega_- t}, \tag{5}$$

oscillating at the normal-mode frequencies $\Omega_\pm = \Omega_0 \pm \gamma\sqrt{\bar{\epsilon}(2+\bar{\epsilon})}$ featuring the $\bar{\epsilon}^{1/2}$ scaling around the EP. This is depicted in Fig. 1(c). Here, $a_\pm$ and $b_\pm$ are complex constants. As discussed in detail later, the EP sensor has quadrature spectra with poles at $\Omega_\pm$, which will build up coherent oscillations from noise, as in a laser. The coefficients $a_\pm$ and $b_\pm$ will then be determined by saturation effects, not by initial conditions, and will be independent of ϵ as long as the saturation mechanism is.

The operators $\delta\hat{a} \equiv \hat{a} - \langle\hat{a}\rangle$, $\delta\hat{b} \equiv \hat{b} - \langle\hat{b}\rangle$ characterize the system's noise. Their equations of motion follow from linearizing Eq. (4):

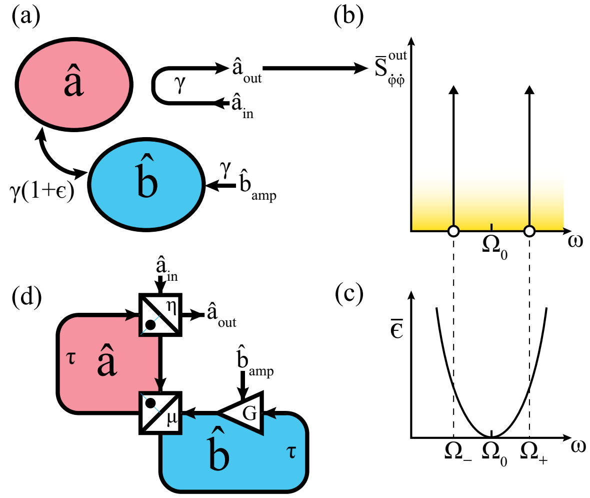

FIG. 1. A schematic of PT-symmetric EP sensors used to detect a weak signal. Mode $\hat{a}$ is coupled to the output by a decay rate γ, while mode $\hat{b}$ is amplified at a rate γ, which introduces a noise mode $\hat{b}_{\mathrm{amp}}$. The modes are coupled to each other at the rate $\gamma(1+\epsilon)$, where ϵ is proportional to the weak signal being measured. (a) shows a Markovian EP sensor with dynamics described by Eq. (4). (b) is the output frequency spectrum, a pair of delta-function spikes surrounded by white noise. The locations of the delta functions in (b) are determined by the small quantity being sensed and split quadratically, as in (c). (d) shows a physical implementation of the sensor using a pair of beam splitters, two delay lines, and an amplifier. The weak signal alters the transmissivity of the beam splitter coupling modes $\hat{a}$ and $\hat{b}$.

$$\delta\dot{\hat{a}} = -i\Omega_0\delta\hat{a} - \gamma\delta\hat{a} + \gamma(1+\bar{\epsilon})\delta\hat{b} + \gamma b\delta\epsilon + \sqrt{2\gamma}\delta\hat{a}_{\mathrm{in}}$$
$$\delta\dot{\hat{b}} = -i\Omega_0\delta\hat{b} + \gamma\delta\hat{b} - \gamma(1+\bar{\epsilon})\delta\hat{a} - \gamma a\delta\epsilon + \sqrt{2\gamma}\delta\hat{b}^{\dagger}_{\mathrm{amp}}. \quad (6)$$

These fluctuations in the internal modes leak out according to $\delta\hat{a}_{\mathrm{out}} = \sqrt{2\gamma}\delta\hat{a} - \delta\hat{a}_{\mathrm{in}}$. To determine the system's output fluctuations, we solve Eq. (6) in the frequency domain, assuming that the fluctuation dynamics are much faster than the mean dynamics of Eq. (3). To simplify the equations, we assume that we operate the sensor near resonance and near the EP such that $\omega - \Omega_0 \ll \gamma$ and $\bar{\epsilon} \ll 1$. (We show in Sec. V of [21] that removing these assumptions does not alter the results presented here.) The sensor's output phase quadrature fluctuations are given by

$$\delta\hat{p}_{\mathrm{out}}[\omega] = \frac{2\gamma^2}{(\omega-\Omega_-)(\omega-\Omega_+)}[\delta\hat{p}_{\mathrm{in}}[\omega] + \delta\hat{p}_{\mathrm{amp}}[\omega] + \sqrt{2\gamma}(p_a^-\delta\epsilon[\omega-\Omega_-] + p_a^+\delta\epsilon[\omega-\Omega_+])], \quad (7)$$

where $p_a^{\pm}$ are the phase quadratures of the constants $a_{\pm}$. A similar result holds for the amplitude quadrature. Given the EP sensor's output fluctuations, we can now evaluate its performance.

Parameter estimation.—We first consider using the EP sensor to estimate the unknown parameter $\bar{\epsilon}$. In this case, we assume ϵ varies slowly and drop the $\delta\epsilon$ terms in Eq. (7). Considering the EP sensor's spectrum near resonance and near the EP, we define $\Delta\omega_{\pm} \equiv \omega - \Omega_{\pm}$ and invoke the near-resonant approximation to work to leading order in $\Delta\omega_{\pm}/\gamma$. We continue to assume that we are near enough to the EP that we can work to leading order in $\bar{\epsilon}$. We need to be careful about how these approximations interact in the denominator of Eq. (7): since PT-symmetric EP sensors will operate around some small constant value of $\bar{\epsilon}$, but a frequency measurement will only be affected by frequency noise infinitesimally close to resonance, $\Delta\omega_{\pm}$ vanishes faster than $\bar{\epsilon}$. With these approximations, the output phase quadrature spectrum takes the form

$$\bar{S}^{\mathrm{out}}_{pp}[\Omega_{\pm} + \Delta\omega_{\pm}] = \frac{\gamma^2(1+2n_{\mathrm{th}})}{2\bar{\epsilon}\Delta\omega_{\pm}^2}. \quad (8)$$

In this expression, quantum zero-point fluctuations contribute the constant term in the numerator, while thermal noise in the amplifier and in-coupled mode contribute identically via the average thermal occupation $n_{\mathrm{th}} \equiv (n_{\mathrm{amp}} + n_{\mathrm{in}})/2$.

Since it is the frequency shift $\Omega_{\pm}(\bar{\epsilon})$ that is sensed, it is the frequency noise corresponding to the above spectrum that is relevant. The sensor's output frequency spectrum is given by $\bar{S}_{\dot{\varphi}\dot{\varphi}}[\Omega_{\pm}+\Delta\omega_{\pm}] = \Delta\omega_{\pm}^2\bar{S}^{\mathrm{out}}_{pp}[\Omega_{\pm}+\Delta\omega_{\pm}]/(4\gamma|a_{\pm}|^2)$ (see Refs. [29–32] and Supplementary Material, Sec. IV [21]). Converting Eq. (8) into an equivalent frequency spectrum, we find

$$\bar{S}_{\dot{\varphi}\dot{\varphi}}[\Omega_{\pm} + \Delta\omega_{\pm}] = \frac{\gamma(1+2\bar{n}_{\mathrm{th}})}{8|a_{\pm}|^2\bar{\epsilon}}. \quad (9)$$

This can be recognized as the Schawlow-Townes spectrum [33], with the $1/\bar{\epsilon}$ factor interpreted as a Petermann factor, which generically describes excess laser frequency noise in multimode lasers with nonorthogonal modes [17,34].

Importantly, the frequency noise given by Eq. (9) increases as we approach the EP (i.e., $\bar{\epsilon} \to 0$), meaning that PT-symmetric EP sensors pay a penalty in the form of increased quantum and thermal noise near the EP, as shown in Fig. 2(a).

The scaling of the frequency noise $\sqrt{\bar{S}_{\dot{\varphi}\dot{\varphi}}} \sim 1/\sqrt{\bar{\epsilon}}$ in fact *precisely nullifies* the $\sqrt{\bar{\epsilon}}$ scaling in the frequency sensitivity. To wit, consider the scenario where the unknown parameter is to be estimated from, say, the difference in frequency of the modes (measuring a single frequency does not change the conclusion): then the sensitivity to $\bar{\epsilon}$ is $\mathcal{S} = |\partial(\Omega_+ - \Omega_-)/\partial\bar{\epsilon}|$, whereas the noise is $\mathcal{N} = \sqrt{(\bar{S}_{\dot{\varphi}\dot{\varphi}}[\Omega_+ + \Delta\omega] + \bar{S}_{\dot{\varphi}\dot{\varphi}}[\Omega_- + \Delta\omega])\Delta\omega_{\mathrm{meas}}/(2\pi)}$ (here, $\Delta\omega_{\mathrm{meas}}$ is the measurement bandwidth). The imprecision

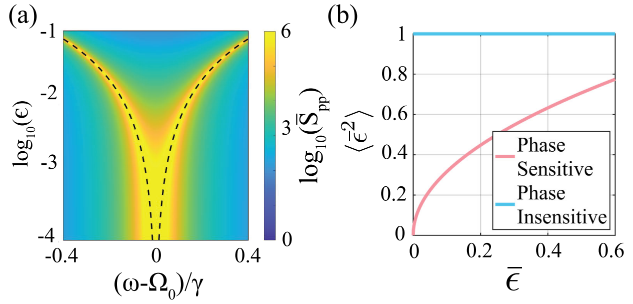

FIG. 2. Markovian PT-symmetric EP sensors' output phase quadrature spectrum, $\bar{S}^{\rm out}_{pp}[\omega]$. (a) The phase quadrature spectrum of a sensor with a phase-insensitive gain as a function of frequency and fractional distance to the EP, ϵ. The dashed black line shows the sensor's resonant frequencies scaling as $\sqrt{\epsilon}$, giving rise to increased sensitivity for $\epsilon \ll 1$. But the fundamental noises (quantum and thermal) also increase with the same scaling so that resonance locations are no easier to distinguish than when the sensor is farther from the EP. (b) While the ability of an EP sensor to estimate the value of small parameter $\bar{\epsilon}$ is independent of $\bar{\epsilon}$ for a sensor with a phase-insensitive gain, sensors with phase-sensitive gain are more sensitive as $\bar{\epsilon}$ decreases. In these plots, we take $\bar{n}_{\rm th} = 0$. In (b), we take $\Delta\omega_{\rm meas}/(16\pi\gamma_a|a_\pm|^2) = 1$ and $\gamma_b \Delta\omega^2_{\rm meas}/[6\gamma_a^2(\gamma_a+\gamma_b)] = 1$.

in the estimate of $\bar{\epsilon}$ is [35], $\langle \bar{\epsilon}^2_{\rm imp}\rangle \equiv \mathcal{N}/\mathcal{S}$, which in this case

$$\langle \bar{\epsilon}^2_{\rm imp}\rangle = \sqrt{\frac{(1+2\bar{n}_{\rm th})\Delta\omega_{\rm meas}}{16\pi\gamma|a_\pm|^2}} \tag{10}$$

is *independent* of the sensor's proximity to the EP. Thus, operating a sensor near a PT-symmetric EP offers no fundamental imprecision reduction for parameter estimation since the sensitivity and noise are enhanced equally.

Weak force sensing.—Instead of using the EP sensor to estimate the value of $\bar{\epsilon}$, we could instead consider using it to measure a weak force coupled to $\delta\epsilon$. Assuming the fluctuations in $\delta\epsilon$ are weak stationary near the EP, the sensor's output quadrature spectra can be computed from Eq. (7) using the known spectra of the fundamental input noises. Near the EP and near resonance, the output quadrature spectra are given by

$$\bar{S}^{\rm out}_{xx}[\omega] = 4\gamma^4 \frac{(1+2n_{\rm th}) + 2\gamma(|x_-|^2+|x_+|^2)\bar{S}_{\epsilon\epsilon}[\omega-\Omega_0]}{(\omega-\Omega_-)^2(\omega-\Omega_+)^2}. \tag{11}$$

Here, $x\in\{q,p\}$ is the amplitude or phase quadrature. From Eq. (11), we see that as in the parameter estimation case, there is no sensing advantage to using an EP sensor to measure a weak force: the fluctuations in the weak force $\bar{S}_{\epsilon\epsilon}$ and the fundamental noises $\sim(1+2n_{\rm th})$, are both transduced to the output via the same response $1/(\omega-\Omega_\pm)^2$, so that any scaling of the response with ϵ is immaterial in distinguishing the weak force from the fundamental noises. Relaxing the assumptions of weak stationarity and/or proximity to resonance and/or EP do not alter this conclusion (see Sec. V of [21]).

In sum, neither parameter estimation nor weak-force sensing benefits from using an EP sensor because unavoidable quantum frequency noises in an EP system precisely nullify the enhanced sensitivity.

Practical advantage of PT-symmetric sensors.—While PT-symmetric sensors do not offer any advantage in overcoming the limitation set by fundamental noises of quantum and thermal origin, they can potentially be advantageous when the sensor is limited by technical noises. By technical noise, we mean any noise that is uncorrelated with the fundamental noises and independent of ϵ (an example is apparent frequency noise in the readout of the sensor). Suppose technical noise is characterized by its spectrum $\bar{S}^{\rm tech}_{\dot\varphi\dot\varphi}$, the imprecision with which $\bar{\epsilon}$ can be estimated is

$$\langle \bar{\epsilon}^2_{\rm imp}\rangle = \sqrt{\frac{\bar{S}_{\dot\varphi\dot\varphi}+\bar{S}^{\rm tech}_{\dot\varphi\dot\varphi}}{2\gamma^2/\bar{\epsilon}}}. \tag{12}$$

If technical noises dominate (i.e., $\bar{S}^{\rm tech}_{\dot\varphi\dot\varphi} \gg \bar{S}_{\dot\varphi\dot\varphi}$), it is advantageous to operate the sensor by approaching the EP; however, since the spectrum of fundamental noises is characterized by $\bar{S}_{\dot\varphi\dot\varphi} \sim 1/\bar{\epsilon}$, approaching the EP will eventually make the fundamental noises dominate, saturating the imprecision in estimating $\bar{\epsilon}$ to the value given in Eq. (10).

Effects of non-Markovian dynamics.—The analysis above considers PT-symmetric EP sensors in the Markovian limit where the coupled-mode equations of Eq. (4) are valid. This case is depicted in Fig. 1(a). While the Markovian assumption leading to these input-output equations [26] is often an excellent assumption, physical optical systems are more accurately described by a set of non-Markovian coupled mode equations. Such a physical system is depicted in Fig. 1(d). We analyze the non-Markovian case in depth in the Supplemental Material, Sec. VI [21], and find that fundamental noise still increases as $1/\bar{\epsilon}$ in this regime, preventing any fundamental sensing enhancement.

Quantum-enhanced EP sensors.—While we have shown that EP sensors have no fundamental advantage without quantum enhancement, we can recover this advantage by engineering the sensor's quantum states. This could be done by replacing the phase-insensitive amplifier in Fig. 1(d) with a phase-sensitive one such as an optical parametric amplifier [23,36,37] or a Josephson parametric amplifier [38–40]. A phase-sensitive PT-symmetric sensor is described by the coupled-mode equations (see Supplementary Material, Sec. VII. A [21])

$$\dot{\hat{a}} = -i\Omega_0\hat{a} - \gamma_a\hat{a} + \gamma_a(1+\bar{\epsilon}+\delta\epsilon)\hat{b} + \sqrt{2\gamma_a}\delta\hat{a}_{\rm in}$$
$$\dot{\hat{b}} = -i\Omega_0\hat{b} + (\gamma_a - r)\hat{b} + e^{-2i\Omega_0 t} r\hat{b}^\dagger - \gamma_a(1+\bar{\epsilon}+\delta\epsilon)\hat{a} + \sqrt{2(\gamma_a+\gamma_b-r)}\delta\hat{b}^\dagger_{\rm amp} + \sqrt{2\gamma_b}\delta\hat{b}_{\rm in}. \quad (13)$$

We assume that the out-coupled mode $\hat{a}_{\rm out}$ is observable, whereas $\hat{b}_{\rm out}$ is not and represents loss. Here, r is the rate of phase-sensitive amplification; for stability, $0 \le r \le \gamma_a + \gamma_b$.

In the most optimistic case the gain is purely phase-sensitive, i.e., $r = \gamma_a + \gamma_b$ (see Supplemental Material, Sec. VII [21] for a more general analysis). The output phase spectrum near resonance is then given to leading order in $\Delta\omega_\pm$ and $\bar{\epsilon}$ by

$$\bar{S}^{\rm out}_{pp}[\Omega_\pm + \Delta\omega_\pm] = \frac{\gamma_b}{2(\gamma_a+\gamma_b)}(1+2\bar{n}_{\rm th}). \quad (14)$$

This spectrum is independent of $\bar{\epsilon}$, so this sensor will have an imprecision that decreases as $\sqrt{\bar{\epsilon}}$. The right panel of Fig. 2 shows the phase quadrature spectrum of a phase-sensitive EP sensor.

We could also consider using the phase-sensitive EP sensor for weak-force sensing. We show in Supplemental Material, Sec. VIII. A. 4 [21], that there is still no sensing enhancement in this regime. Note that the advantage due to this phase-sensitive strategy is different from strategies relying on nonreciprocal coupling of the modes [15].

Conclusion.—Using a self-consistent theory of fundamental noises in PT-symmetric EP sensors, and plugging a range of potential loopholes, we have shown that PT-symmetric EP sensors do not offer any advantage for parameter estimation or weak-force sensing if the sensor is limited by fundamental (i.e., quantum and thermal) noises. That is because these noises scale in exactly the same manner as the scaling of the sensitivity near the EP, thus nullifying any improvement in signal-to-noise ratio. However, an advantage exists if PT-symmetric EP sensors are limited by technical noises, as is most common. Beating the limits set by quantum noise requires quantum resources: we outline a phase-sensitive generalization of an EP sensor that does confer an advantage by harnessing the square-root bifurcation near an EP. A final loophole is a nonstationary EP sensor—for example, an EP analog of Ref. [41]; we leave this for future studies.

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