Navier-Stokes Equations for Low-Temperature One-Dimensional Quantum Fluids

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We consider one-dimensional interacting quantum fluids, such as the Lieb-Liniger gas. By computing the low-temperature limit of its (generalized) hydrodynamics we show how in this limit the gas is well described by a conventional viscous (Navier–Stokes) hydrodynamics for density, fluid velocity, and the local temperature, and the other generalized temperatures in the case of integrable gases. The dynamic viscosity is proportional to temperature and can be expressed in a universal form only in terms of the emergent Luttinger liquid parameter K and its density. We show that the heating factor is finite even in the zero temperature limit, which implies that viscous contribution remains relevant also at zero temperatures. Moreover, we find that in the semiclassical limit of small couplings, kinematic viscosity diverges, reconciling with previous observations of Kardar-Parisi-Zhang fluctuations in mean-field quantum fluids.

Introduction.—Quantum many-body interacting systems pose an immense technical challenge to modern-day physics due to their exponential complexity. A successful approach in the past years has been in borrowing concepts and ideas from classical hydrodynamic theory and applying them to quantum systems [1–6]. The main idea behind these methods is the same in classical and quantum physics: the exponentially large information determining the state of the system is reduced to few thermodynamic functions, the hydrodynamic fields, that well characterize the local equilibrium state. For quantum gases in one spatial dimension, the theory of generalized hydrodynamics (GHD) [7,8] (see also, e.g., [9–11] for reviews) has shown to be able to perfectly capture the dynamics of integrable, e.g., [12–19] and near-integrable quantum gases [20–25], as well as spin chains, e.g., [26–33], Fermionic systems [34–39], and classical field theories [40–43]. At small values of temperatures and for gapless or gapped systems, GHD recovers historically well-established results on the dynamics of low-temperature systems, in particular the celebrated Luttinger liquid theory [44–50] and the semiclassical approaches [51,52]. In particular, by requantizing the fluctuations on top of a classical background, evolving with GHD at zero temperature, one recovers a Luttinger liquid theory on top of an evolving hydrodynamic fluid, connecting this way GHD with the most relevant field theory description of one dimensional quantum systems [33,53–57].

At zero temperature (or entropy) GHD is effectively a set of equations describing the dynamics of the Fermi points of the fermionized degrees of freedom [12]. In particular, it was shown that, at the level of Euler hydrodynamics, therefore neglecting any viscosity, the GHD evolution of a single Fermi sea (with two Fermi points) is exactly equivalent to the conventional hydrodynamics (CHD) evolution of density ρ and momentum (or fluid velocity η) as the two relevant hydrodynamic modes [12,58]. However, while the latter creates hydrodynamic shocks and fails to be meaningful after the time t^* of the creation of the first shock [59–62], Euler GHD is free of shocks as at time t^* two new modes, i.e., two new Fermi points are created, resolving this way the shock into a simple contact singularity [12]. However, since Euler GHD is valid for strictly integrable systems, it is unclear what is instead the correct hydrodynamics for quasi or even nonintegrable gases at very low temperatures, that can describe different experimental settings.

In this Letter, we show that as Euler CHD is plagued by hydrodynamic shocks, one, on the other hand, cannot neglect viscosity terms that are finite as soon as interparticle interactions are nonzero. Viscous, or diffusive terms, have been indeed incorporated into GHD for a few years [63–67]. They have been shown to be essential for the thermalization of quasi-integrable systems [68] and for well describing spin dynamics in integrable spin chains [69–74]. However, since integrable systems are typically ballistic, they usually account for small perturbative effects on top of the ballistic current. Here we shall show that the picture changes drastically at very low temperature: when $T \to 0$ these terms enter the CHD as a *dynamic viscosity* $\mu(\rho, T)$ and fully regularize its shocks, making the resulting viscous CHD a perfectly valid hydrodynamics for low-temperature gapless systems. By taking a low-temperature limit of GHD we determine a simple and universal expression for the dynamic viscosity which only depends on the density ρ and on the Luttinger liquid parameter $K(\rho)$

for a given interaction strength. Therefore, we claim that our result is universal for any one-dimensional interacting system at low temperatures.

GHD and CHD.—We start by deriving CHD by taking the low-temperature limit of GHD, and we consider the Lieb-Liniger model [75] as a reference, although our derivation is fully generic. The Lieb-Liniger model for N contact-interacting bosonic particles in an external potential $V(x)$ is given by the Hamiltonian

$$\hat{H} = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \partial_{x_i}^2 + \frac{\hbar^2 c}{m} \sum_{i<j=1}^{N} \delta(x_i - x_j) + \sum_{i=1}^{N} V(x_i) \quad (1)$$

and it represents a paradigmatic model for one-dimensional interacting systems and cold atomic gases, see, e.g., Refs. [76–78]. For convenience, we consider from thereafter $\hbar = m = 1$, and $k_B = 1$ to express temperature in units of $mk_B/\hbar^2$. In the repulsive regime $c > 0$, its eigenstates are labeled by (fermionic) quasiparticles with bare energy $\varepsilon(\theta) = \theta^2/2$ and momentum $k(\theta) = \theta$, where $\theta \in [-\infty, \infty]$ are the rapidities or the quasimomenta of the particles. In the thermodynamic limit, the state is specified by a filling function $n(\theta)$ fixed by the temperature and chemical potential. Within the framework of GHD, one assumes that at each position x, t there exists a fluid cell where the gas is locally thermodynamic, and where we can introduce a local filling function $n(\theta; x, t)$. The time evolution of the latter then reads, given also the external force $\mathfrak{f} = -\partial_x V$, as

$$\partial_t n(\theta; x, t) + v^{\text{eff}}(\theta; x, t)\partial_x n(\theta; x, t) + \mathfrak{f}\partial_\theta n(\theta; x, t) = 0 \quad (2)$$

where $v^{\text{eff}}(\theta; x, t) = (\partial_\theta \varepsilon)^{\text{dr}}/(\partial_\theta k)^{\text{dr}}$ is the dressed velocity of the quasiparticles on top of the background fixed by the filling $n(\theta; x, t)$. The latter can be found by knowing the explicit dressing operation, which reads as $f^{\text{dr}} = (1 - \varphi n)^{-1} \cdot f$ with the scattering matrix $\varphi(\theta) = c/(\pi(c^2 + \theta^2))$ acting as a convolution operator $\varphi \cdot f = \int \varphi(\theta - \alpha)f(\alpha)d\alpha$.

At low temperatures, the filling function becomes very close to a sharp Fermi sea, such that $\partial_x n(\theta; x, t) = -\sum_\sigma \sigma \partial_x \theta^\sigma(x, t)\delta[\theta - \theta(x, t)^\sigma]$ with θ^σ the two Fermi edges, indexed by $\sigma = \pm 1$, and Eq. (2) can be rewritten as an equation for the two Fermi edges [12]

$$\sum_\sigma \delta(\theta - \theta^\sigma)[\partial_t \theta^\sigma + (\eta + \sigma v_F(\rho))\partial_x \theta^\sigma - \mathfrak{f}] = 0. \quad (3)$$

For convenience, we described the fluid in a co-moving frame with fluid velocity $\eta = (\theta^+ + \theta^-)/2$, such that at each position x the gas is found in a Fermi sea with symmetric Fermi edges $\pm q = \pm(\theta^+ - \theta^-)/2$, and Fermi velocity $v_F = \pi\rho/K$ proportional to the local density ρ, with K the Luttinger liquid parameter, obtained from the dressed charge as $K \equiv 1^{\text{dr}}(\theta^\pm)^2$ [46,79]. Eq. (3) can be

shown, see Supplemental Material [80], to be fully equivalent to CHD for density and fluid velocity readings as

$$\partial_t \rho + \partial_x(\eta\rho) = 0,$$
$$\partial_t \eta + \eta\partial_x \eta + \rho^{-1}\partial_x \mathcal{P}_s^{(0)}(\rho) = \mathfrak{f}, \quad (4)$$

with $\mathcal{P}_s^{(0)}(\rho)$ is the static pressure of the gas at zero temperature, fixed by the equation of state of the system. In the specific example of the Lieb-Liniger gas, the static pressure at zero temperature is given by the integral of the dressed energy ε^{dr}, $\mathcal{P}_s^{(0)}(\rho) = -\int_{-q}^{q} d\theta \varepsilon^{\text{dr}}(\theta)/2\pi$, and where q is obtained by inverting the relation $\rho = \int_{-q}^{q} 1^{\text{dr}}(\theta)d\theta/2\pi$. Notice also that the fluid velocity η is related to the momentum field p as $\eta = p/\rho$.

The equation for the fluid velocity η in Eq. (4) takes the form of the celebrated Burgers equation, which is known to display hydrodynamic shocks [81]. In Euler GHD such shocks are resolved by adding to Eqs. (3) two extra Fermi edges, i.e., by introducing $2n$ hydrodynamic fields θ^σ with $\sigma = 1, \ldots, 2n$ that describe n split Fermi seas [12]. In the phase space (θ, x) this translates into the existence of a *continuous* one-dimensional contour $\Gamma(s)$ that separates the region where $n(\theta, x) = 1$ from the region where $n(\theta, x) = 0$.

Clearly, this picture is based on the underlying integrability of the model, i.e., in the fact that all the rapidities θ are conserved quantities. For nonintegrable generic systems, it is clear that this phenomenon cannot be the one that regularizes the shocks. As shocks in the Burgers equation are regularized by a finite viscosity, we shall then see in the coming section how to indeed introduce viscosity in CHD (4).

Viscous CHD.—We now wish to include diffusive or viscosity effects. In order to do so, we shall consider local thermodynamic states with finite but small temperature, as there are no viscosity effects at strictly zero temperature. We consider therefore local canonical equilibrium states with given density ρ (or chemical potential μ), fluid velocity η and temperature $T = 1/\beta \ll T_d$, with $T_d = \rho^2/2$ the quantum degenerate temperature of the gas [82]. In the Lieb-Liniger model, this corresponds to filling functions of the form $n(\theta; x, t) = (1 + e^{\beta \varepsilon^{\text{dr}}(\theta; x, t)})^{-1}$ given in terms of its pseudo-energy

$$\varepsilon^{\text{dr}} = \varepsilon[\theta - \eta(x, t)] - [q(x, t)]^2 - \frac{\varphi \cdot \log(1 + e^{-\beta\varepsilon})}{\beta(x, t)}. \quad (5)$$

We then move to introduce diffusion terms into CHD by taking the high β limit of the diffusive GHD, which is known from Refs. [63,64] and it is given by adding to the rhs of Eq. (2) the diffusive part

$$R \cdot \partial_x[R^{-1} \cdot \tilde{D} \cdot \partial_x n(\alpha)], \quad (6)$$

with the kernels $R_{\theta,\theta'}$ and $\tilde{D}_{\theta,\theta'}$ presented in [80]. As diffusive GHD contains ∂_x^2 terms, a description only in terms of the Fermi edges becomes impossible, as we cannot simply use that the derivative of the filling is a delta function at the edges, as this would produce $\delta'(\theta - \theta^\sigma) = \partial\delta(\theta - \theta^\sigma)$, whose meaning outside integration is unclear.

The only way to clarify their role is to plug them within the hydrodynamic equation for the only two modes that are relevant here, i.e., density and momentum (fluid velocity). This already clarifies the essential difference of viscous CHD from Euler GHD: while the latter can be simply obtained as an equation for the evolution of the Fermi edges (3) it can be trivially extended to any number of them during time evolution. Viscous CHD instead requires fixing the number of Fermi edges from the start, and, as we shall see, does not require producing new Fermi edges, as hydrodynamic shocks are regularized by viscosity. By means of a long but straightforward calculation deferred to the Supplemental Material [80], we then arrive to

$$\partial_t\rho + \partial_x(\eta\rho) = 0,$$
$$\partial_t\eta + \eta\partial_x\eta + \frac{\partial_x\mathcal{P}_s(\rho,T)}{\rho} = \frac{\partial_x[\mu(\rho,T)\partial_x\eta]}{\rho} + \mathfrak{f}, \quad (7)$$

with the dynamic viscosity at the leading order is linear in temperature, as already remarked previously [83]. In terms of dimensionless interaction $\gamma = c/\rho$ and temperature $\tau = T/c^2$, the dynamic viscosity reads as

$$\frac{\mu(\gamma,\tau)}{c} = \frac{K\tau\gamma^3}{4\pi}(\partial_\gamma\log K)^2 + \mathcal{O}(\tau^3), \quad (8)$$

which can be also rewritten for fixed interaction c as $\mu(\rho,T) = (\rho T K/4\pi)(\partial_\rho\log K)^2 = T(1 - K\partial_\rho v_F/\pi)^2/4v_F$. This result can be understood similarly as in the kinetic picture of diffusion in integrable models proposed in [84]: the excitations close to the Fermi edges move with Fermi velocity $v_F(\rho) = \pi\rho/K(\rho)$ but as there are local density fluctuations $\rho \to \rho + \delta\rho$ within the fluid, the Luttinger parameter (i.e., the Fermi velocity) also fluctuates $K \to K + \partial_\rho K\delta\rho$, giving a diffusive spreading to the trajectories. The dependence of the Luttinger parameter on density (and therefore on the position in an inhomogeneous fluid) is what makes inhomogeneous Luttinger liquids for interacting systems nonconformal invariant [85] and it is a direct effect of nontrivial interactions. In the so-called Tonks regime of strong repulsion $\gamma \gg 1$, the dependence of the Luttinger parameter on the density trivializes, $K \to 1$, and viscosity vanishes, as expected for noninteracting particles. The first correction at $\gamma \gg 1$ is obtained from (8) using $K \simeq (1 + 4/\gamma)$ [47], and reads as

$$\frac{\mu(\gamma,\tau)}{c} \simeq \frac{4\tau}{\pi\gamma} + \mathcal{O}(\gamma^{-3}). \quad (9)$$

The pressure in (7) of the interacting gas is given by taking the first correction to static pressure in T^2, $\mathcal{P}_s^{(0)}(\rho) \to \mathcal{P}_s(\rho,T)$ with

$$\mathcal{P}_s(\rho,T) = \mathcal{P}_s^{(0)}(\rho) + T^2\rho\tilde{\chi}_e/2, \quad (10)$$

with $\tilde{\chi}_e = K/(3\rho^2)$. Equation (7) gives the evolution of density and momentum of the fluid, which is one-to-one with the chemical potential q and the boost η. Given that the system is at a finite temperature, and this also represents a hydrodynamic variable, an extra equation is needed: the one for the evolution of the energy density e. We define energy density at rest as $e = E/\rho - \eta^2/2$, where E is the total energy of the fluid, which in the Lieb-Liniger model is computed via $E = \int d\theta n(\theta)(\theta^2/2 + V(x))^{dr}$. By proceeding in an analogous manner as to derive the Eqs. (7), we obtain

$$\partial_t e + \eta\partial_x e + \frac{\mathcal{P}_s(\rho,T)}{\rho}\partial_x\eta = \frac{\mu(\rho,T)}{\rho}(\partial_x\eta)^2. \quad (11)$$

As expected, the *kinematic viscosity* $\nu = \mu/\rho$ now enters the equation. We can convert this equation into an equation for the evolution of the temperature field $T(x,t) = 1/\beta(x,t)$ using the definition of the energy susceptibility at fixed density $\delta e/\delta T|_\rho = \tilde{\chi}_e T$. Given the expression at zero temperature for generic interacting systems, we then obtain

$$\partial_t T + \eta\partial_x T = -\frac{\mathcal{P}_s^T}{\rho\tilde{\chi}_e}T\partial_x\eta + \frac{\mu(\rho,T)}{T\rho\tilde{\chi}_e}(\partial_x\eta)^2. \quad (12)$$

On the rhs of (12), $\mathcal{P}_s^T = \pi\rho(1/\rho + \partial_\rho v_F/v_F)/(3v_F)$ is the low temperature correction to the stationary pressure at constant density, and we omitted the thermal conduction $\partial_x[\kappa(\rho,T)\partial_x T]$ with $\kappa(\rho,T) \sim T^2K\mu(\rho,T)[\partial_\rho(\tilde{\chi}_e K)]^2/32$, which is subleading in temperature. Both terms are derived in the Supplemental Material [80].

Equations (7) and (12) take exactly the same form as the standard Navier-Stokes equations for a fluid, i.e., continuity equation, conservation of momentum and conservation of energy, and it is quite remarkable that we can derive them in an exact, nonperturbative way. We first notice that the heating factor in Eq. (12) is $\sim\mu/T$, namely, it is order zero in temperature. Therefore, even if the system is initially at zero temperature, the rapid growth of the velocity gradient $\partial_x\eta$ heats the system, giving therefore finite viscosity μ to the dynamics of η and regularizing its shocks. Moreover, given that the dynamic viscosity in Eq. (8) is expressed only in terms of the universal features of the low-temperature effective field theory, namely, the Luttinger liquid constant K and its density dependence (which is nontrivial only for interacting quantum gases), we conjecture that its form is universal for generic one-dimensional quantum

fluids. The argument is simple: at low temperatures, all quantum fluids become effectively integrable, as their description is in terms of Luttinger liquid freely propagating bosonic modes. Their diffusion is therefore expected to be the same as for integrable quasiparticles, where the kinetic picture explained above applies. We also stress that there is no fundamental difference in the nature of transport at small temperatures between integrable and nonintegrable systems, contrary to what is claimed in previous literature [86].

Front dynamics in integrable gases.—We now focus on the fate of an initial density bump, for example, in a system at a given temperature $T_0 \ll T_d$ and density ρ_0. Such a setting is paradigmatic to understanding the response of a system to external perturbations, and we here use it to establish the main differences between viscous CHD and integrable diffusive GHD at low temperature, see Fig. 1. When the system is integrable, one could expect that not only temperature and chemical potential can characterize a local stationary state, but a large number of Lagrange multipliers γ_n associated to higher conserved quantities $\langle \hat{Q}_n \rangle = \int d\theta n(\theta)(\theta^n)^{\mathrm{dr}}/n!$, i.e., a generalized Gibbs
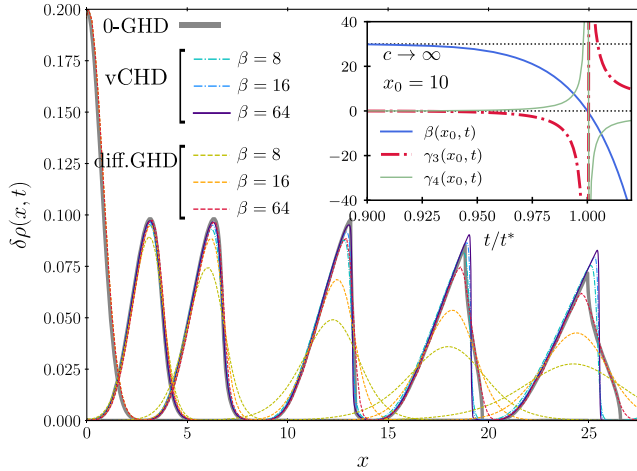

FIG. 1. Evolution of the excess density $\delta\rho = \rho - \rho_\infty$ for an initial bump obtained as the ground state density of (1) with interaction $c = 1$ and Gaussian potential $V(x) = -a_1 - a_2 \exp(-x^2/\sigma^2)$, released at $t > 0$; here $\sigma = 1$, $a_{1,2}$ are set such that the background density $\rho_\infty = 1$ and maximum $\rho_0 = 1.2$. We compare the result by zero-entropy GHD (0-GHD, solid thick lines), diffusive GHD (dashed lines) and viscous CHD (vCHD, dash-dotted lines) at different values of inverse temperatures β (see legend). $K(\gamma \equiv 1) \simeq 3.4$ is obtained from Bethe ansatz [79] and used in (8). Times are $t = 0, 3, 6, 12.5, 18.5, 25$ and increase from the leftmost to rightmost peak in the figure. Inset: Evolution of the inverse temperature and generalized temperatures starting from a thermal case at low temperature in the integrable gas, here for simplicity the Lieb-Liniger gas at large coupling (Tonks limit). The temperatures are obtained by fitting the quasiparticle occupations $n(\theta; x, t)$ at a given position x_0 during the front dynamic expansion. We clearly see the moment when the Fermi sea split as the moment of temperature inversion.

ensembles (GGE), deviating this way from the behavior of a generic nonintegrable system. This amounts to replacing the bare energy $\varepsilon = \theta^2/2$ with a higher-order polynomial

$$\varepsilon = \frac{\theta^2}{2} + \sum_{n \geq 3} \gamma_n(x, t) \frac{\theta}{n!} \tag{13}$$

in Eq. (5), still yielding a valid stationary state of the Lieb-Liniger gas, due to its integrability. By extending the result for the temperature field in (12) to the higher potentials $\gamma_n(x, t)$, we find quite lengthy partial differential equations, see [80] for their expression. The main relevant fact is that while $\partial_t T \sim O(T)$, we instead find $\partial_t \gamma_n \sim O(T^0)$. Namely, even if we prepare a thermal gas at low temperature [where $\gamma_n(x, 0) = 0$], the integrable gas will generate finite generalized temperatures in the post-shock dynamics. For instance, at the shock time t^* for a certain position x_0, we find that $\beta(x_0, t) < 0$ and $\gamma_{3,4}(x_0, t) \neq 0$, leading to the splitting of Fermi seas defined via (13) (cf. inset of Fig. 1). Similarly, higher potentials are activated when the Fermi seas further split. Such temperature dynamics leads to a significantly different shock regularization in GHD compared to the CHD one, even if both are regular hydrodynamics. Indeed, while GHD displays a growing shock region, viscous CHD converges to a given profile, with a finite front width, and it never develops a shock as temperature is lowered. Namely, even if temperature $T \to 0$, the heating factor μ/T entering (12) remains finite, and leads to a self-regulation of the shock driven by the kinematic viscosity.

In Fig. 1, zero-temperature GHD agrees with diffusive GHD at finite temperature, signaling how the zero-temperature approximation is often able to capture out-of-equilibrium fluids. On the other hand, observed deviations from viscous CHD are attributed to the absence of higher conservation laws, as discussed above.

The small coupling limit $c \to 0^+$ and KPZ physics.—As already discussed in the first Lieb-Liniger paper [75], the limit of small coupling of the Lieb-Liniger gas does not simply recover free bosons. Indeed, when $c \to 0^+$, its ground state becomes the one describing the so-called (semi-classical) condensate solution of the nonlinear Schrödinger equation (NLS) [87]. This is characterized by vanishing Fermi momentum $q \sim \sqrt{c}$ but with diverging dressed functions $1^{\mathrm{dr}} \sim 1/\sqrt{c}$ in order to keep the density ρ finite in the limit. The relevant excitations become then the Bogoliubov excitations with spectrum given by $\varepsilon_k \sim |k|$ and therefore with a degenerate group velocity $v_k \sim \mathrm{sgn}(k)$. As $\partial_\rho v_F(\rho) \sim \lim_{q \to 0} \partial_q v_q$ the latter diverge, giving a divergent dynamic viscosity at low coupling as

$$\mu(\rho, T) \sim \frac{T}{\sqrt{c}}, \tag{14}$$

signaling the breakdown of the viscous CHD and the emergence of Kardar-Parisi-Zhang physics [88]. The latter

is well known to emerge in the stochastic Burgers equation, i.e., given a white δ-correlated noise w, a local perturbation of a hydrodynamic field ϕ satisfying $\partial_t\phi + \partial_x(v\phi + \kappa\phi^2 + w) = 0$, moves with finite velocity v and *spreads superdiffusively*, as opposed to the diffusive case of Eq. (7) whenever μ is finite. Such a phenomenon is known to appear in generic finite-component one-dimensional fluids, as described by the nonlinear fluctuating hydrodynamic (NLFH) theory [89], which can be successfully applied also to lattice (i.e., nonintegrable) NLS [90]. Again, the argument is simple: whenever the Euler currents contain nonlinearities, one should expect that the introduction of a small noise (which, for example, can describe the interaction with other nonhydrodynamic modes) always leads to the KPZ universal fixed point. However, this is not the case at finite coupling since, although it is true that Eq. (7) contains the nonlinearity η^2, there exists a continuous, thermally activated, spectrum of modes around the Fermi points with velocities $v_F \pm \delta v(\theta)$ [with $\delta v(\theta) \ll v_F$ peaked at $\theta = \theta^\sigma$] that are responsible for a finite diffusion constant in the system. It is only in the small coupling limit $c \to 0^+$ that the velocities of all such excitations become degenerate, therefore diminishing the number of effective hydrodynamic modes to only the three macroscopic ones. In this limit, therefore, the theory of NLFH applies and KPZ physics emerges, as signalled by the divergent dynamic viscosity. We should stress that the existence of KPZ physics in the NLS at low temperature was first established in [91] and we here give a first analytical proof of its divergent diffusion constant. Moreover, it is interesting to notice that the degeneracy of the hydrodynamic modes leads to superdiffusive KPZ physics similarly also in the Heisenberg spin chain (and any other integrable model with non-Abelian symmetry) [74,92] at finite temperatures. There the relevant degenerate excitations are not the ones around the Fermi points but the so-called giant magnons [93,94], namely, magnonic excitations with large spin and vanishing velocities.

Conclusion.—We have here shown that the standard Navier-Stokes equations for the evolution of density, momentum, and temperature can be derived from the low-temperature expansion of the GHD for the Lieb-Liniger gas. We have found universal expressions for the linear part in temperature of the dynamic viscosity, which we conjecture to apply to generic one-dimensional fluids. We have shown that the viscosity is zero in the free fermionic limit, as expected, and that it diverges in the semiclassical limit of weakly interacting bosons at small temperatures, which despite many numerical works, it was never established from first principles. The divergent viscosity signals the emergence of KPZ superdiffusive spreading [91], in analogy to the one observed in integrable spin chains [95].

We have shown that the viscous terms regularize hydrodynamic instabilities in one-dimensional gases, although the inclusion of generalized temperatures is necessary in order to predict the full form of the shock front in the integrable gas. Moreover, one should also expect that when the system is strongly out of equilibrium and thus gradients of η become large at the shock points, the system strongly heats locally, invalidating, therefore, the zero-entropy approximation. Our findings therefore suggest that zero-entropy hydrodynamics becomes harder to physically realize whenever interactions are present.

Differently from previous attempts to derive viscosity in quantum fluids by perturbative corrections to Luttinger liquids, see, for example, [6,83,86,96], we here derive nonperturbatively using the generalized hydrodynamic of a specific model, the Lieb-Liniger gas, and we extend our result to generic systems, given the universality of its formulation. Clearly, a different derivation only involving Luttinger liquid modes would also be desirable in the future. Our result is ready to be checked by means of numerical simulations [97–99] and to apply to different quantum fluids as such as chiral edge modes [100–102], to open the way for a full fluctuating hydrodynamic theory [103] of Luttinger liquid field theories, and to unveil Burgers-like turbulent phases [104] in low-dimensional quantum fluids.

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