Certifying Network Topologies and Nonlocalities of Triangle Quantum Networks

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Quantum networks promise unprecedented advantages in information processing and open up intriguing new opportunities in fundamental research, where network topology and network nonlocality fundamentally underlie these applications. Hence, the detections of network topology and nonlocality are crucial, which, however, remain an open problem. Here, we conceive and experimentally demonstrate to determine the network topology and network nonlocality hosted by a triangle quantum network comprising three parties, within and beyond Bell theorem, with a general witness operator for the first time. We anticipate that this unique approach may stimulate further studies toward the efficient characterization of large complex quantum networks so as to better harness the advantage of quantum networks for quantum information applications.

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Introduction.—The study of quantum networks advances rapidly, which promises advantages in computing, sensing, and secured communications [1–11]. The network topology and network nonlocality fundamentally underlie various applications of quantum networks. Wiring the entanglement sources and parties in a quantum network differently results in different network topology. Existing studies often customize a witness to certify the nonlocality associated with a specific network topology [12–34], which, however, may not uncover the network nonlocality associated with a different network topology. Hence, an open question is to design a measurement to generally determine the network topology and network nonlocality of a quantum network.

The question becomes more complicated by going beyond the conventional Bell theorem [25], since it appears natural for multiple sources to independently distribute entanglement in a quantum network by assuming the network local model with multivariables, which is a subject of recent interest. Hence, quantum networks present a fertile ground for research in fundamental physics such as stronger [12,13] and novel forms [22] of entanglement swapping, nonlocality without inputs [32,35–37], limitations on measurement dependence [38], and distinguishing the role of complex numbers in quantum theory [39] beyond the single source, which has motivated numerous network nonlocality experiments [16–18,20,21,23,27,31,33,38,40–47].

Intriguingly, the triangle quantum networks comprising three parties are an ideal platform to study the problem. As depicted in Fig. 1, every two parties may share a pair of entangled particles or the three parties share a three-particle Greenberger-Horne-Zeilinger (GHZ) state beside the classical connections, creating five different network topologies: (1) in N_0, the three parties are connected classically, i.e., assuming network local models; (2) in N_1, N_2, and N_3, one, two, or three sources of entangled pairs are installed beside the classical connections, respectively; (3) in $\mathsf{N}_{1'}$, the three parties share a three-particle GHZ state beside the classical connections. It is evident that the network nonlocal correlations of the triangle quantum networks with topology N_1 correspond to the conventional two-party Bell

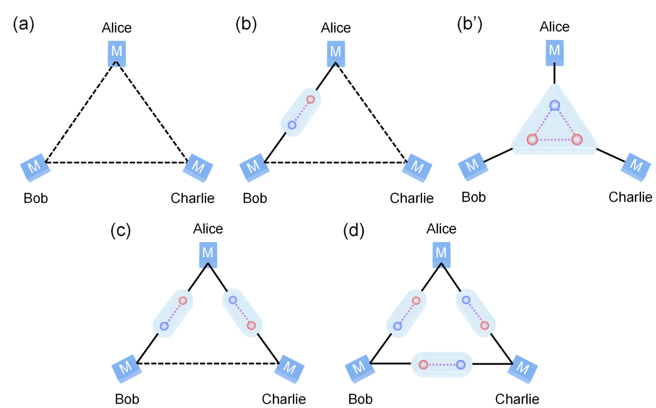

FIG. 1. Triangle quantum networks comprising three parties—Alice, Bob, and Charlie—allow five different network topologies. In (a) N_0, the three parties are classically connected (dashed lines). Beside the classical connections, in (b)–(d) N_1, N_2, N_3, one, two, or three sources of entangled pairs (○ · ·○) are installed, respectively. In (b′) $\mathsf{N}_{1'}$, a three-particle GHZ state is installed. The configurations respect cyclic symmetry.

nonlocality [48]. The triangle quantum networks with topology N_2 contain two sources, with one delivering a pair of entangled particles between Alice and Bob and the other delivering a pair of entangled particles between Alice and Charlie. This scenario has been studied in terms of the chain-Bell nonlocality [48,49], bilocality [12], and full network nonlocality (FNN) [24]. The triangle quantum network with topology N_3, which is known as the ring network, contains three sources, with one source delivering a pair of entangled particles between each pair of parties. Characterizing its nonlocality as a nonconvex problem was a theoretical challenge [32,50,51]. Hence, a triangle quantum network hosts a rich family of network topologies and network nonlocalities for both within and beyond Bell theorem. Here, we showcase in this Letter to detect the network topologies and network nonlocalities hosted by the triangle quantum networks with a general witness. We anticipate that this study may promote our understanding toward characterizing large complex quantum networks efficiently.

A general Bell-type witness operator.—There is no universal method to detect network topology and nonlocality for general networks. Triangle networks turn out to be the simplest network structure containing the necessary ingredients to study the problem. Following the convention in the study of Bell theorem, we adopt the assumption of local operation with classical communication and nonsignaling that refers to a party conducting measurements independent of the measurements conducted by other parties. We define a network correlation measurement as

$$\begin{aligned}\mathcal{B} := 2[&A_0B_0C_0 + A_0B_0C_1 + A_1B_1C_0 - A_1B_1C_1 + A_2B_2C_2 + A_2B_3C_2 + A_3B_2C_3 \\ &- A_3B_3C_3 + A_4B_4C_4 + A_5B_4C_4 + A_4B_5C_5 - A_5B_5C_5] \\ &+ A_0C_0 + A_0C_1 + A_1C_0 - A_1C_1 + B_2C_2 + B_3C_2 + B_2C_2 - B_3C_3 + A_4B_4 + A_5B_4 + A_4B_5 - A_5B_5, \end{aligned} \tag{1}$$

in which the correlators $A_xB_yC_z$ are defined according to the joint conditional probability distributions $P(a,b,c|x,y,z)$ as $\sum_{a,b,c=0,1}(-1)^{a+b+c}P(a,b,c|x,y,z)$, and the bipartite correlators such as A_xB_y can be evaluated according to marginal distribution $P(a,b|x,y)$. Given a triangle network without knowledge of the configuration, we brutally perform optimized correlation measurements Eq. (1) considering all possible network topology. By hybridizing the Mermin operator [30] to witness tripartite correlations and the Clauser-Horne-Shimony-Holt (CHSH) operator [52] to witness bipartite correlations, we show below that the correlation measurement (1) provides a method to certify the network topology and witness the Bell nonlocalities hosted by the triangle quantum network at different levels [53].

Let us consider the input quantum states $|\Phi^+\rangle_{AB}$, $|GHZ_3\rangle$, $|\Phi^+\rangle_{AB}\otimes|\Phi^+\rangle_{A'C}$, and $|\Phi^+\rangle_{AB}\otimes|\Phi^+\rangle_{A'C}\otimes|\Phi^+\rangle_{B'C'}$ respectively for network topology $\mathsf{N}_1, \mathsf{N}_{1'}, \mathsf{N}_2$, and N_3, where $|\Phi^+\rangle_{AB} = [(|00\rangle_{AB}+|11\rangle_{AB})/\sqrt{2}]$ and $|GHZ_3\rangle = [(|000\rangle_{ABC}+|111\rangle_{ABC})/\sqrt{2}]$. By optimizing the measurements A_i, B_j, and C_k independently for each network topology, we arrive at the upper bound of the network correlation measurement,

$$\langle\mathcal{B}\rangle \le S_{\mathsf{N}_m}, \quad \text{where } S_{\mathsf{N}_m} \equiv \max_{A_i,B_j,C_k}\langle\mathcal{B}\rangle, \tag{2}$$

with $m = 0, 1, 1', 2, 3$ labeling the network topology, respectively. Intriguingly, we note that each optimal measurement comprises separate and independent local measurements, i.e., Alice, Bob, and Charlie each perform two separate measurements to the two particles at his or her disposal, which greatly eases the experimental realization. For example, we attain $S_{\mathsf{N}_3} = 18\sqrt{2}$ with the set of measurements,

$$\begin{aligned}&A_0 = \mathbb{1}\otimes\sigma_Z, \quad A_1 = \mathbb{1}\otimes\sigma_X, \quad A_2 = A_3 = \mathbb{1}\otimes\mathbb{1}, \\ &A_4 = \frac{1}{\sqrt{2}}(\sigma_Z+\sigma_X)\otimes\mathbb{1}, \quad A_5 = \frac{1}{\sqrt{2}}(\sigma_Z-\sigma_X)\otimes\mathbb{1}, \\ &B_0 = B_1 = \mathbb{1}\otimes\mathbb{1}, \quad B_2 = \mathbb{1}\otimes\frac{1}{\sqrt{2}}(\sigma_Z+\sigma_X), \\ &B_3 = \mathbb{1}\otimes\frac{1}{\sqrt{2}}(\sigma_Z-\sigma_X), \quad B_4 = \sigma_Z\otimes\mathbb{1}, \quad B_5 = \sigma_X\otimes\mathbb{1}, \\ &C_0 = \mathbb{1}\otimes\frac{1}{\sqrt{2}}(\sigma_Z+\sigma_X), \quad C_1 = \mathbb{1}\otimes(\sigma_Z-\sigma_X), \\ &C_2 = \sigma_Z\otimes\mathbb{1}, \quad C_3 = \sigma_X\otimes\mathbb{1}, \quad C_4 = C_5 = \mathbb{1}\otimes\mathbb{1}, \end{aligned} \tag{3}$$

where $\sigma_{x,y,z}$ are Pauli matrices and $\mathbb{1}$ is the identity operation.

We list the upper bounds in Table I, which are distinctive from each other and are noise-tolerant, hence evidencing

TABLE I. Upper bounds of network correlation measurement (1) for triangle quantum networks with noise tolerance shown by the lowest visibilities of quantum states to beat the respective bounds (see Supplemental Material [53] for the calculation of the respective bounds).

Network topology	Upper bound S	Visibility
N_0	$S_{\mathsf{N}_0} = 18$	...
N_1	$S_{\mathsf{N}_1} = 12 + 6\sqrt{2}$	0.8787
$\mathsf{N}_{1'}$	$S_{\mathsf{N}_{1'}} = 6\sqrt{13}$	0.9469
N_2	$S_{\mathsf{N}_2} = 12\sqrt{2} + 6$	0.9419
N_3	$S_{\mathsf{N}_3} = 18\sqrt{2}$	0.9023

TABLE II. Witnessing network topology of triangle quantum networks. '…' for no known results and ✗ for inapplicable.

Bell-type correlation measurement	N_1	$N_{1'}$	N_2	N_3
Ref. [30]	✓	✗	✗	✗
Ref. [29]	✗	✓	…	…
Refs. [12,14,15,19]	✓	✗	✓	✗
Ref. [32]	✗	✗	✗	✓
Ref. [22]	✓	✗	✓	✗
Ref. [33]	✓	✗	✓	✓
Ref. [34]	✓	✗	✓	✗
Refs. [24,31]	✓	✗	✓	✗
Ref. [26]	✓	✗	✓	✗
This work	✓	✓	✓	✓

TABLE III. Witnessing quantum nonlocalities of triangle quantum networks. '…' for no known results and ✗ for inapplicable.

Bell-type correlation measurement	MN	FCNN	FTNN
Ref. [30]	✗	✗	✗
Ref. [29]	✓	…	…
Refs. [12,14,15,19]	✗	✓	✗
Ref. [32]	✗	✗	✗
Ref. [22]	✗	✓	✗
Ref. [33]	✗	✓	✓
Ref. [34]	✓	✗	✗
Refs. [24,31]	✗	✓	✗
Ref. [26]	✗	✓	✓
This work	✓	✓	✓

that the network correlation measurement (1) witnesses the entire set of topology of a triangle quantum network. We list in Table II many existing network correlation measurements as a comparison, which shows that the Mermin operator [30] identifies N_1, the Svetlichny operator [29] identifies $\mathsf{N}_{1'}$, the network operator proposed by Branciard *et al.* [12,14,15,19,22] and others [24,26,31] identify N_1 and N_2, the one by Renou *et al.* [32] identifies N_3, the one by Suprano *et al.* [33] identifies N_1,N_2, and N_3, and the one proposed in [34] identifies N_1 and N_2.

Quantum networks create interesting new possibilities that go beyond the conventional Bell theorem. General quantum networks with only local measurements already reveal novel network nonlocalities. We show below that surpassing the upper bounds listed in Table I respectively certifies a variety of triangle quantum network nonlocalities.

First, the network correlation measurement (1) is a hybrid of Mermin and CHSH operators. Hence, surpassing the upper bound S_{N_0} certifies that the triangle quantum networks possessing topology N_1, $\mathsf{N}_{1'}$, N_2, or N_3 display Bell nonlocality by rejecting the local realistic model.

Second, surpassing the upper bound S_{N_1} certifies that the triangle quantum networks with topology $\mathsf{N}_{1'}$, N_2, or N_3 display multipartite nonlocality (MN, here tripartite) according to Svetlichny's definition [29] to rule out all biseparable quantum correlations without assuming the source independence.

Third, under the assumption of casually independent sources of entangled pairs, new types of network nonlocalities emerge that are beyond Bell theorem. For example, under the assumption, the triangle quantum network with topology N_2 contains two casually independent sources of entangled photon pairs. This scenario was originally discussed in the context of bilocality [12,14,15,19,22]. Recently it was shown that surpassing the upper bound of the bilocality can be interpreted with a hybrid model employing a nonlocal resource and a source of local variable nature [22,24]. Then FNN was introduced, which requires that all sources in the network must be of nonlocal nature, hence excluding the hybrid model [24,26,33]. Viewing the triangle quantum network with topology N_1 as an implementation of the hybrid model, surpassing the upper bound S_{N_1} certifies that the triangle quantum network with topology N_2 exhibits FNN. Here, we term it as full-chain network nonlocality (FCNN). So does the triangle quantum network with topology N_3.

Fourth, the triangle quantum network possessing topology N_3 is known as the ring network, characterizing its network nonlocality has been a challenge. Renou *et al.* first proposed a nonlocality test without inputs to witness the genuine nonlocality by performing only one local measurement per party, thereby ruling out all classical models [32], which, however, is not experimentally viable without noise tolerance. Employing multiple measurement settings, one may define the *full triangle network nonlocality* (FTNN) to rule out all hybrid models that assume at least one source to be of local nature [24,26], as another exhibition of FNN. Hence, surpassing the upper bound S_{N_2} certifies that the triangle quantum network with topology N_3 is FTNN, i.e., the triangle quantum network with topology N_3 cannot be emulated by another triangle quantum network with topology N_2.

We list in Table III the scenarios of certifying MN, FCNN, or FTNN of triangle quantum networks with many existing network measurements. It is shown that the Mermin inequality [30] is not applicable at all. The proposal by Luo can certify MN [34]. The Svetlichny inequality [29] can also certify MN. Many recent network inequalities under the assumption of source independence [12,14,15,19,22,33] or with postquantum sources [24,26,31] can certify FCNN. To certify FTNN, one may decompose the triangle quantum network with topology N_3 into a chain network and a Bell network [26] or utilize the wired CHSH inequality [33].

Summarizing the results in Tables II and III shows that our network correlation measurement (1) can certify the entire set of network topology and a variety of network nonlocalities hosted by the triangle quantum networks, while the others cannot. We may activate or deactivate a triangle quantum network with a certain type of network topology. For example, we may create a triangle quantum network with topology N_2 (N_1) out of a triangle quantum network with topology N_3 (N_2) by eliminating a source of entangled pair, displaying a kind of hierarchy. We can also convert a triangle quantum network with topology N_2 to a triangle quantum network with topology $\mathsf{N}_{1'}$ with a local operation with classical communication operation, creating a three-particle GHZ state whose nonlocality property falls in a separate category and cannot be emulated by bipartite nonlocal correlations [61], but can be simulated at a higher echelon of the nonlocality hierarchy [62]. We remark that each network topology has its own merit. For example, the triangle quantum networks with topology N_3 or N_2 are robust against particle-loss channel [63] or entanglement-breaking channel [64], i.e., the triangle quantum network with N_3 or N_2 is robust against losing two specific qubits or against losing a single particle, respectively.

Experiment.—We now present the experimental verifications. The experimental schematics to realize triangle quantum networks with different entanglement network topologies are depicted in Fig. 2. We use the standard quantum optical techniques to generate the source of entangled photon pairs via the spontaneous parametric down-conversion process (SPDC). We synchronize the creations of three independent sources [20] and generate the three-photon GHZ state with the fusion technique [53,65]. By keeping the production rate of photon pairs as low as 0.0025 per pulse in SPDC, we detect two-photon entangled states $|\Phi^+\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2}$ at a rate of 3000 s^{-1} with quantum state fidelity better than 0.98 and three-photon GHZ state $|\mathrm{GHZ}_3\rangle = (|HHH\rangle + |VVV\rangle)/\sqrt{2}$ at a rate of 1 s^{-1} with fidelity better than 0.97, where H and V stand for horizontal and vertical polarizations, respectively.

We realize photonic triangle quantum networks with four network topologies N_1, $\mathsf{N}_{1'}$, N_2, and N_3, and perform the optimized measurements, respectively. Different from many existing studies of network nonlocality with multiparities, in which each party performs joint measurements to multiphotons at his or her disposal, as noted in the above, a distinct feature of our experiment is that each party performs independent measurements to each of his or her photons. Hence, we separately detect the two-photon or three-photon coincidence between relevant parties sharing a pair of entangled photons or the three-photon GHZ state and synchronize all of the detections in the experiment. A successful measurement event yields a coincidence of six bits comprising both quantum and classical bits, regardless

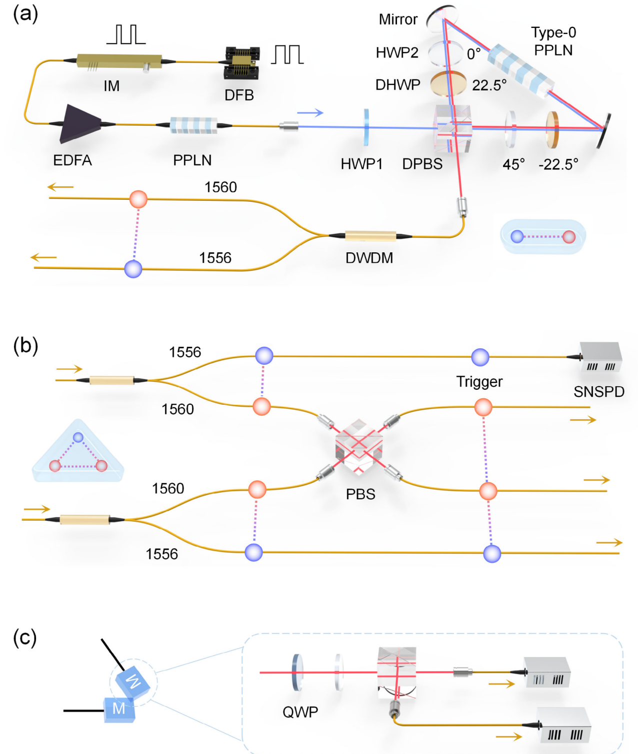

FIG. 2. Schematics to generate entangled photon pairs and three-photon GHZ states and to realize single-photon polarization measurement. (a) Generation of entangled photon pairs. We inject 90 ps laser pulses with a central wavelength of 779 nm to a periodically poled MgO doped lithium niobate (PPLN) crystal enclosed in a Sagnac interferometer to induce the type-0 SPDC process, which emits photon pairs in state $|\Phi^+\rangle$, respectively, with signal and idler photons at the wavelengths of 1556 nm and 1560 nm, which are separated spatially by a dense wavelength division multiplex (DWDM) filter. We arrange three synchronized but independent SPDC processes to generate three sources of polarization-entangled photon pairs. (b) Generation of the three-photon GHZ state via fusion. We interfere two signal photons at a polarizing beam splitter (PBS). The detection of an idler photon heralds the presence of three-photon GHZ state in a postselection fashion [31]. (c) Realization of single-photon polarization measurements ($\sigma_{x,y,z}$). DPBS, dual-wavelength PBS; (DH)HWP/QWP, (dual-wavelength) half/quarter-wave plate; SNSPD, superconducting nanowire single-photon detector.

of the network topology. For example, for the network topology $\mathsf{N}_{1'}$, there are three quantum bits and three classical bits. We introduce the classical bits in the data postprocessing process [53].

We calculate the ensemble average of the network correlation measurement with the accumulated experimental statistics. We obtain $\langle \mathcal{B} \rangle = 20.0470 \pm 0.1650$, 20.9525 ± 0.3553, 22.1903 ± 0.2074, and 25.2077 ± 0.2106, respectively, in experiments with network topologies N_1, $\mathsf{N}_{1'}$, N_2, and N_3, which nicely fall between the theoretical upper bounds as shown in Fig. 3, hence confirming that our network correlation measurement

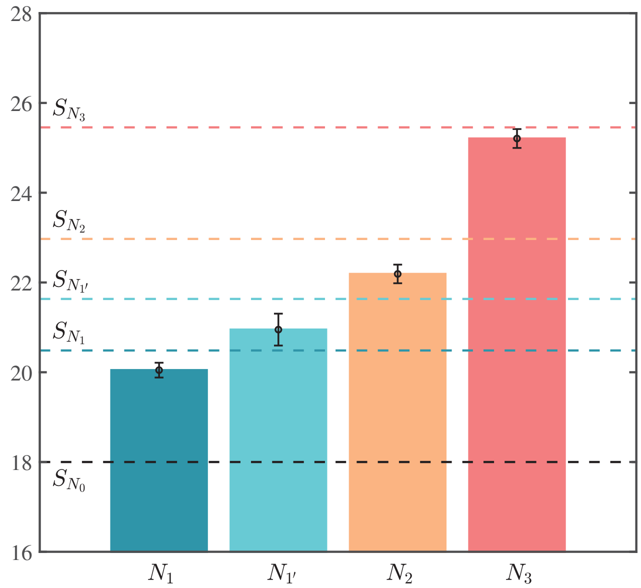

FIG. 3. Experimental results of network correlation measurement $\langle \mathcal{B} \rangle$ for different network topologies N_1, $\mathsf{N}_{1'}$, N_2, and N_3. Error bars represent 1 standard deviation in experiments.

witnesses the network topologies and nonlocalities hosted by the triangle quantum networks.

Conclusions.—A common scenario is that we do not know the configuration when we encounter a quantum network. Detecting the network topology and certifying the network nonlocality remain a challenge. In this Letter, we consider triangle quantum networks comprising three parties connected by the least number of resources such as classical channels, two-particle entangled states, and three-particle GHZ states. Although we do not know the configuration, we know the network must have one of the five topologies. Employing the Mermin or CHSH operator separately cannot completely distinguish the five topologies. Instead, we propose to hybrid the Mermin and CHSH operator as a new Bell-type operator. In the experimental realization of the Bell-type operator, we first conduct the set of measurements optimized for the N_3 network [53]. If the measurement outcomes produce the upper bound S_{N_3} (or surpass S_{N_2}), we determine that the network has the N_3 topology. Otherwise, we perform the set of measurements optimized for the N_2 network, and so on and so forth. In addition, note that our method is device-independent similar to Bell experiments, which only consider the statistics of outcomes by assuming black-box measurement devices for each party. Using an adaptive automated optimization procedure [66,67], we can obtain the maximal violation of witness correlator Eq. (1) without any prior knowledge of the quantum system and measurements by tuning local measurements according to joint statistics. Our study may stimulate further studies toward characterizing large complex quantum networks so as to better harness the advantage of quantum networks for a broad range of applications in quantum computing, quantum sensing, and quantum communications. Nevertheless, our current study only considers sources of quantum entanglement and local variables in the quantum network. The results may be further strengthened and broadened by including postquantum sources and employing entangle measurement in the future [62]. For example, the certification of MN in this study is weaker than that of Svetlichny, which allows partial postquantum correlations among biseparable correlations [29], and the certifications of FCNN and FTNN in this work are weaker than that including postquantum correlations [24,26]. We note that the measurement as a hybrid of the Mermin operator and the CHSH operator with a ratio of $\alpha = 2:1$ may be further optimized to better distinguish different topologies and nonlocalities with α as a free tuning parameter [53].

Note added.—While this work is in press, we became aware of a related work [68] which presents a scheme to certify network topology in the six-qubit photonic networks.

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