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Noncompletely Positive Quantum Maps Enable Efficient Local Energy Extraction in Batteries

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Energy extraction from quantum batteries by means of completely positive trace-preserving (CPTP) maps leads to the concept of CPTP-local passive states, which identify bipartite states from which no energy can be squeezed out by applying any CPTP map to a particular subsystem. We prove, for arbitrary dimension, that if a state is CPTP-local passive with respect to a Hamiltonian, then an arbitrary number of copies of the same state—including an asymptotically large one—is also CPTP-local passive. We show further that energy can be extracted efficiently from CPTP-local passive states employing noncompletely positive trace-preserving (NCPTP) but still physically realizable maps on the same part of the shared battery on which operation of CPTP maps were useless. Moreover, we provide the maximum extractable energy using local-CPTP operations, and then, we present an explicit class of states and corresponding Hamiltonians, for which the maximum can be outperformed using physical local NCPTP maps. We provide a necessary and sufficient condition and a separate necessary condition for an arbitrary bipartite state to be unable to supply any energy using NCPTP operations on one party with respect to an arbitrary but fixed Hamiltonian. We build an analogy between the relative status of CPTP and NCPTP operations for energy extraction in quantum batteries, and the association of distillable entanglement with entanglement cost for asymptotic local manipulations of entanglement. The surpassing of the maximum energy extractable by NCPTP maps for CPTP-passive as well as for CPTP-nonpassive battery states can act as detectors of non-CPTPness of quantum maps.

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Introduction.—In recent years, research in the miniaturization of devices and in quantum technology (see, e.g., [1–4]) has increased significantly, which has led to the investigation of quantum batteries [5–31]. There have also been attempts to extract maximal energy from these batteries in minimal time, aided by optimal control [6,32,33].

There exist states from which no energy can be extracted using unitaries. They are called "passive states," which depend on the choice of the Hamiltonian [34–40]. In the context of open quantum systems [18,19], shared quantum batteries have been considered, and to extract energy from them, completely positive trace-preserving (CPTP) maps were operated on one fixed part of a bipartite system. In this framework also, some states may exist from which no energy can be extracted by operating CPTP maps on the fixed party. These states are identified in Refs. [18,19] and given the names "strong local passive" and "CP-local passive," respectively.

The study of open systems beyond the realm of CPTP maps is less explored. In this Letter, we explore energy extraction from shared quantum batteries by considering all quantum maps, that include also certain NCPTP maps, in the set of allowed operations on a particular subsystem of the bipartite battery. However, to describe the evolution of physical systems, the NCPTP maps must still be positive, and moreover, not all positive NCPTP maps are quantum mechanically valid ones.

The objective of this work is to look into the hierarchy between physically realizable NCPTP maps and CPTP dynamics in the context of quantum batteries. In this regard, in the first part of our work, we show that energy can be extracted from various two-qubit CPTP-local passive states using NCPTP operations on the same party where application of CPTP operations was unable to extract energy. The successful extraction of energy from such a state can act as a witness for the detection of the non-complete positiveness of these maps. Moreover, we find that whatever be the number of copies, multiple copies of a CPTP-local passive state remain CPTP-local passive with respect to the relevant Hamiltonian. This characteristic of CPTP-local passive states implies that the gap between CPTP and NCPTP local maps, with respect to CPTP-local passive states, persists in the limit of an asymptotic number of copies. Further, we show the advantage of NCP maps over CP maps can also be witnessed for general battery states which may not be CP-local passive.

We draw an analogy between these features and irreversibility in entanglement manipulations as seen in the qualitative and quantitative differences between distillable entanglement and entanglement cost [41,42]. See also

Refs. [43–51] in this regard. Finally we find a necessary and sufficient condition and an independent necessary condition for a bipartite state to be NCPTP-local passive.

Throughout the Letter, we use the following nomenclatures: By local CPTP (NCPTP) operations, we mean CPTP (positive trace-preserving) maps acting on a particular subsystem of a bipartite battery while the other part of the battery is kept unchanged. In shorthand notation, we will refer to such local CPTP and NCPTP maps as LCPTP and LNCPTP maps, respectively. Systems that are allowed to have only local CPTP evolution will be specified as CPTP-open systems, whereas systems that can experience NCPTP dynamics are named NCPTP-open systems. For a detailed discussion on energy extraction from closed and open quantum batteries, one can go through the Supplemental Material [52].

CPTP-local passivity in the asymptotic limit.—Here we present an interesting property of CPTP-local passive states.

Theorem 1.—If a state ρ is CPTP-local passive with respect to a fixed Hamiltonian H, then $\rho^{\otimes n}$ will also be CPTP-local passive for any positive integer n, with respect to $\sum_n H$.

Remark.—Here, $\sum_n H$ denotes a sum of Hamiltonians of independent systems having H as the Hamiltonian for each of them, so that $\sum_n H = H \otimes I^{\otimes(n-1)} + I \otimes H \otimes I^{\otimes(n-2)} + \cdots + I^{\otimes(n-1)} \otimes H$, where I denotes the identity operator on the Hilbert spaces of the individual copies.

Proof.—Suppose a state, ρ_{AB}, which acts on a composite Hilbert space, $\mathcal{H}_A \otimes \mathcal{H}_B$, is CPTP-local passive with respect to a Hamiltonian, H_{AB}. Let us consider two copies of ρ_{AB}, i.e., $\rho_2 = \rho_{A_1B_1} \otimes \rho_{A_2B_2}$, and the corresponding Hamiltonian is $H_{A_1B_1A_2B_2} = H_{A_1B_1} \otimes I_{A_2B_2} + I_{A_1B_1} \otimes H_{A_2B_2}$. Let us introduce two more Hilbert spaces, $\mathcal{H}_{A'_1}$ and $\mathcal{H}_{A'_2}$, which are copies of $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$, respectively, and define an operator $C_{A_1A_2A'_1A'_2} = \mathrm{Tr}_{B_1B_2}[(\rho_2^{\mathrm{T}_{A_1}\mathrm{T}_{A_2}} \otimes I_{A'_1A'_2})(I_{A_1A_2} \otimes H_{B_1B_2A'_1A'_2})]$. Though $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$ have the same dimension, let us denote the dimension of each of these Hilbert spaces as d_{A_1} and d_{A_2}, respectively. It is easy to check that $C_{A_1A_2A'_1A'_2} = C_{A_1A'_1} \otimes M_{A_2} \otimes I_{A'_2} + M_{A_1} \otimes I_{A'_1} \otimes C_{A_2A'_2}$, where $M_{A_i} = \mathrm{Tr}_{B_i}(\rho_{A_iB_i}^{T_{A_i}}) = [\mathrm{Tr}_{B_i}(\rho_{A_iB_i})]^T \geq 0$ and $C_{A_iA'_i} = \mathrm{Tr}_{B_i}[(\rho_{A_iB_i}^{\mathrm{T}_{A_i}} \otimes I_{A_i})(I_{A'_i} \otimes H_{B_iA'_i})]$. Since both $\rho_{A_1B_1}$ and $\rho_{A_2B_2}$ are CPTP-local passive, from the condition of CPTP-local passivity we have

$$C_{A_iA'_i} - \mathrm{Tr}_{A'_i}(d_{A_i}^2|\phi^+\rangle\langle\phi^+|C_{A_iA'_i}) \otimes \mathcal{I}_{A'_i} \geq 0 \quad \text{for } i = 1, 2. \tag{1}$$

We want to check if the condition of CPTP-local passivity also holds for ρ_2, i.e., if the inequality still remains satisfied if we replace A and A' by A_1A_2 and $A'_1A'_2$, respectively. By doing so on the left-hand side of the CPTP-local passivity condition, we get

$$\begin{aligned} &C_{A_1A_2A'_1A'_2} - \mathrm{Tr}_{A'_1A'_2}(d_{A_1A_2}^2|\phi^+\rangle\langle\phi^+|C_{A_1A_2A'_1A'_2}) \otimes \mathcal{I}_{A'_1A'_2} \\ &\quad = (M_{A_1} \otimes I_{A'_1}) \otimes Q_{A_2A'_2} + Q_{A_1A'_1} \otimes (M_{A_2} \otimes I_{A'_2}), \end{aligned} \tag{2}$$

where $Q_{A_iA'_i} = C_{A_iA'_i} - \mathrm{Tr}_{A'_i}(d_{A_i}^2|\phi^+\rangle\langle\phi^+|C_{A_iA'_i}) \otimes \mathcal{I}_{A'_i} \geq 0$ [see Eq. (1)]. Here we have used the relation that $\mathrm{Tr}_{A'_1}[(|\phi^+\rangle\langle\phi^+|)_{A_1A_2A'_1A'_2}] = (1/d_{A_1A_2})(|i\rangle\langle i|)_{A_1} \otimes (|\phi^+\rangle\langle\phi^+|)_{A_2A'_2}$. The above relation also holds if we swap A_1 (A'_1) with A_2 (A'_2). Since $Q_{A_iA'_i}, M_{A_i} \geq 0$, we can say the entire operator presented on the right-hand side or left-hand side in Eq. (2) is positive. This implies C_{A_1,A_2,A'_1,A'_2} also satisfies the condition of CPTP-local passivity. Therefore, if we take two copies of CPTP-local passive states, ρ_{CP}, then $\rho_{CP}^{\otimes 2} = \rho_{CP} \otimes \rho_{CP}$ is also a CPTP-local passive state. Since this proof is valid for arbitrary dimension of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_{A'}$, following similar argument the CPTP-local passivity of $\rho_{CP}^{\otimes 2^n}$, for arbitrary integer, n, can be shown by induction. Further, if 2^n copies of a state is CPTP-local passive, it is obvious that any $x < 2^n$ copies of the same state will also be locally passive for any positive integer, x. Hence also in the asymptotic limit, i.e., for $n \to \infty$, the state would turn out to be CPTP-local passive. ■

Comparison between CPTP-open and NCPTP-open batteries.—Henceforth, in this Letter, by NCPTP maps, we will mean only those which are physically realizable, i.e., valid quantum maps. The maximum amount of energy that can be extracted from a battery state ρ_{AB} using local NCPTP maps can be written as

$$\begin{aligned} \Delta W_{\max}^{NCP}(\rho_{AB}, H_{AB}) = \mathrm{Tr}[\rho_{AB}H_{AB}] \\ - \min_{\Lambda_A}\mathrm{Tr}[\Lambda_A \otimes I_B(\rho_{AB})H_{AB}]. \end{aligned} \tag{3}$$

Here the minimization is performed over all NCPTP maps Λ_A. Before considering general NCPTP maps Λ_A let us first focus on energy extraction using a particular type of LNCPTP evolution. To form the LNCPTP dynamics, consider a tripartite system consisting of a bipartite battery, AB, and an environment E. Let the tripartite initial state, ρ_{EAB}, be pure and entangled within the bipartition $E{:}AB$. The battery state, ρ_{AB}, can be obtained by tracing out E, i.e., $\rho_{AB} = \mathrm{Tr}_E(\rho_{EAB})$. We want to evaluate the maximum extractable energy from ρ_{AB} for a given Hamiltonian H_{AB}, using NCPTP operations on the system A. The generation of such an LNCPTP evolution can be ascertained by a global unitary evolution of the environment E and the system A followed by tracing out the environment. Therefore, the maximum extractable energy from ρ_{AB}, with respect to Hamiltonian H_{AB}, using this kind of LNCPTP map is

$$\begin{aligned}\Delta W_p^{NCP}(\rho_{AB},H_{AB}) = {} & \mathrm{Tr}[\rho_{AB}H_{AB}] \\ & - \min_{U_{EA}}\mathrm{Tr}[\mathrm{Tr}_E(U_{EA}\otimes I_B\rho_{EAB} \\ & \times U_{EA}^{\dagger}\otimes I_B)H_{AB}], \end{aligned} \tag{4}$$

where U_{EA} are global unitaries acting on E and A. Here we have introduced a *particular type* of LNCPTP channel, assuming that ρ_{AB} is part of a tripartite state, ρ_{EAB}, which as a whole is pure. The subscript p, used on the left-hand side, helps to keep this fact in mind. The dynamics will be different if ρ_{EAB} is mixed. In such cases, depending on the mixedness, and the dimension of the attached environment E, many other LNCPTP channels can be applied to ρ_{AB} but application of the channels discussed above considering ρ_{EAB} pure may not be implementable. Note that in effect we are not assuming here a part of the assumption referred to as the church of the higher Hilbert space, viz. the existence of an entangled pure state in a larger Hilbert space for every mixed quantum state. Since the expression of $\Delta W_{\max}^{NCP}(\rho_{AB},H_{AB})$ involves maximization over all LNCPTP maps, $\Delta W_{\max}^{NCP}(\rho_{AB},H_{AB}) \geq \Delta W_p^{NCP}(\rho_{AB},H_{AB})$ for all bipartite states, ρ_{AB}, and Hamiltonian, H_{AB}.

Such an LNCPTP map can be implemented by first generating genuine three-party entangled states whose two-party reduced densities are entangled, such as in ρ_{EAB}, and then applying a nonlocal unitary operator on the parties, E and A. The W state [53] or cluster states [54,55] are suitable three-party entangled states, of which the experimental generation of W-type states is proposed in optical systems [56–58], trapped ions [59], atomic ensembles [60], and integrated nanophotonic circuits [61]. Cluster states are highly entangled quantum states which can be prepared in spin qubit lattices by interacting them with Ising type Hamiltonian [54], or in photonic systems [62–64]. Whereas, nonlocal unitaries have been experimentally implemented in optical systems [65], solid state systems [66], molecular electron spin qubits [67], etc.

In particular, three-mode W-type entangled coherent states (ECS) [68,69] can be produced experimentally in photonic systems [56]. The prepared three-mode ECS can then be fed as input to an optical multiport, generating a nonlocal unitary [65] acting on two of the modes. The environmental mode can ultimately be discarded from the final state obtained. This can lead to an implementation of an LNCPTP map on a two-mode optical system.

The main objective of this Letter is to compare $\Delta W_{\max}^{NCP}(\rho_{AB},H_{AB})$ with $\Delta W_{\max}^{CP}(\rho_{AB},H_{AB})$, where $\Delta W_{\max}^{CP}(\rho_{AB},H_{AB})$ is defined in the same way as $W_{\max}^{NCP}(\rho_{AB},H_{AB})$ [see Eq. (3)] with the only difference being that the minimization involved in the second term is performed over all CPTP maps acting on subsystem A.

Extraction of energy from CPTP-local passive states.— Let us begin by considering the two-Bell mixture battery state, given by $\tilde{\rho}_{AB} = p_1|\psi^+\rangle\langle\psi^+| + (1-p_1)|\psi^-\rangle\langle\psi^-|$, where $|\psi^\pm\rangle$ are maximally entangled states: $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. Let us also fix the Hamiltonian to be $\tilde{H}_{AB} = p(\sigma^z\otimes\mathcal{I}) + q(\mathcal{I}\otimes\sigma^z) + r(\sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y)$. Here σ^x, σ^y, and σ^z are the Pauli matrices, and r is the strength of the interaction between A and B. For all the numerical calculations, we take $p = q = e/2$, where e has the unit of energy. The state, $\tilde{\rho}_{AB}$, is CPTP-local passive with respect to the Hamiltonian, $\tilde{H}_{AB}$, for $p_1 \leq 1/2$ and $r \geq p/(1-2p_1)$ [19]. We now consider the system, AB, to be NCPTP-open and examine if energy extraction is possible through LNCPTP operations. The details on the construction of the LNCPTP map are given in the Supplemental Material [52].

Let us consider the set of $\tilde{\rho}_{AB}$, which obeys $p_1 \leq 1/2$ and $r \geq p/(1-2p_1)$. Since these states are CPTP-local passive, $\Delta W_{\max}^{CP} = 0$. The behavior of $\Delta W_p^{NCP}(\tilde{\rho}_{AB},\tilde{H}_{AB})$, is depicted with respect to the entanglement, S, of $\tilde{\rho}_{EAB}$, the system-environment joint state, in the bipartition $E{:}AB$, for different values of interaction strength, r, in Fig. 1. We measure S using entanglement entropy. We obtain that ΔW_p^{NCP} is zero at $S = 0$. This is evident as for $S = 0$, $\tilde{\rho}_{EAB}$ is a pure product state in the bipartition $E{:}AB$ and therefore the generated map is still completely positive. Further, we realize that with an increase in S, extractable energy using the LNCPTP evolution also increases. Hence, we can conclude that energy extraction using LNCPTP dynamics is potentially useful and advantageous over energy extraction using LCPTP processes. The optimization is performed over all two-qubit unitaries (see Ref. [70]), which includes nonlocal ones, and the maximal extracted energy corresponds to the optimal unitary. Such an unitary can be, for example, generated in phtononic systems using the mechanism given in [65]. We conclude that a state that seemed to be "passive" under LCPTP maps is no longer passive if more general positive maps, the physical LNCPTP maps, are allowed. Note that by passive, we mean that the state

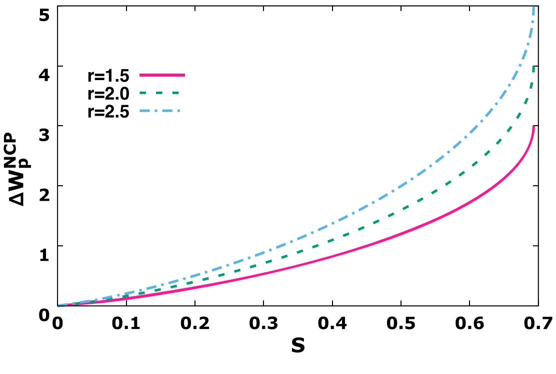

FIG. 1. Maximum extractable energy from CPTP-local passive states using LNCPTP operations. The behavior of ΔW_p^{NCP} is shown for various values of r, i.e., the interaction strength, corresponding to the numerical values $p = q = e/2$. The horizontal and vertical axes are in units of ebits and energy, e, respectively.

is unable to provide any energy through the application of the corresponding map under consideration. Moreover, we observe that the maximum energy extracted using the LNCPTP operation, ΔW_p^{NCP}, increases with increasing strength of the interaction, r. We have proved in Theorem 1 that CPTP-local passive state remains so when an asymptotically large number of copies of the state is considered. Therefore we can conclude that the gap in the energy extraction using LCPTP and LNCPTP maps will remain intact or be more even if we move towards the asymptotic limit, at least in the case of CPTP-local passive states.

Maximum extractable work using LCPTP operations: Surpassing the CPTP-maximum using LNCPTP operations.—The maximum extractable work using LCPTP operations is given by

$$\Delta W_{\max}^{CP} = \mathrm{Tr}(H_{AB}\rho_{AB}) - c_{\min}, \tag{5}$$

where $c_{\min}$ is the minimum eigenvalue of $C_{AA'}$. The maximum is valid for states and Hamiltonians for which $C_{AA'}$ has a nondegenerate eigenspectrum. The proof of Eq. (5) is given in the Supplemental Material [52].

We find that the extractable energy using LNCPTP operations exceeds the CPTP maximum for a class of states $\tilde{\rho}_{AB}$ and Hamiltonian $\tilde{H}_{AB}$ under the condition $\frac{3}{8}|r||x| > |r| + (p+q)/2$, where $r < 0$. Here $x = \cos(\alpha_2 - \beta_2 - \gamma_2) + 2\cos(\beta_2 + \gamma_2) + \cos(\alpha_2 + \beta_2 + \gamma_2) - 4$, where α_2, β_2 and γ_2 are parameters of a unitary, $\mathbb{U}_2$, using which, energy is being extracted. The details of the derivation are given in the Supplemental Material [52].

Energy extraction from CPTP nonpassive batteries using LNCPTP maps.—Here we consider the same Hamiltonian $\tilde{H}_{AB}$, and find the rank-two state $\bar{\rho}_{AB}^r$, which provides the maximum amount, among all rank-two states, of extractable energy when LNCPTP operations are applied. A discussion on its motivation is provided in the Supplemental Material [52]. We denote the amount of extractable energy from $\bar{\rho}_{AB}^r$ using LNCPTP operations by $\Delta\bar{W}_M^{NCP}(r)$, which is equal to $\Delta\bar{W}_M^{NCP}(r) = \max_{\bar{\rho}_{AB}^r} \Delta W_p^{NCP}(\bar{\rho}_{AB}^r, \tilde{H}_{AB})$, where r is the coupling between A and B, and the maximization is done over all rank two states, $\bar{\rho}_{AB}^r$. $\Delta\bar{W}_M^{NCP}(r)$ is being written as a function of r because it of course depends on the Hamiltonian $\tilde{H}_{AB}$, but for all the numerical calculations, we fix p and q to $\epsilon/2$ and only vary r. The dependency of $\Delta\bar{W}_M^{NCP}(r)$ on r, is illustrated in Fig. 2 using a pink line. The details of the construction of the LNCPTP map are given in the Supplemental Material [52]. We also calculate the maximum amount of energy, $\Delta W_{\max}^{CP}(\bar{\rho}_{AB}^r, \tilde{H}_{AB})$, that can be extracted using LCPTP operations, from the same state, $\bar{\rho}_{AB}^r$ and present the nature of $\Delta W_{\max}^{CP}(\bar{\rho}_{AB}^r, \tilde{H}_{AB})$ in the same figure, Fig. 2, using the green dashed line, with respect to the corresponding value of r for which $\bar{\rho}_{AB}^r$ has been obtained. It can be noticed from the figure that $\Delta W_{\max}^{CP}(\bar{\rho}_{AB}^r, \tilde{H}_{AB})$ is always smaller than $\Delta\bar{W}_M^{NCP}(r)$ which is equal to $\Delta W_p^{NCP}(\bar{\rho}_{AB}^r, \tilde{H}_{AB})$ for all the considered values of r.

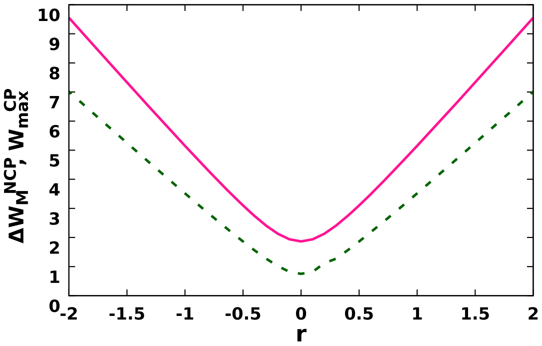

FIG. 2. Comparison between LCPTP and LNCPTP operations as functions of interaction strength, r. The numerical values of p and q are the same as in Fig. 1. All quantities plotted have the units of energy, e.

It should be noted that if instead of rank-two states, rank-one states were considered in place of $\bar{\rho}_{AB}$ or $\tilde{\rho}_{AB}$, the energy extraction operation would reduce to a CPTP map. To get the flavor of NCPTP maps, one must assure that the rank of the battery state is higher than unit. For numerical simplicity, we have restricted ourselves to rank-two states. We expect that considering higher rank states would not affect the results qualitatively.

Detecting NCPTP maps.—One can notice that Eq. (5) can be utilized to formulate a witness for the detection of the noncomplete positivity of the governing dynamics. That is, to check if an unknown map is CP or not, we can take a particular state and try to extract energy from it by operating the unknown map on a particular subsystem of the state. If the amount of energy that can be extracted by operating the map surpasses $\Delta W_{\max}^{CP}$, then the map is certainly non-CPTP. If the energy extraction is less than that, then no inference can be drawn about the non-CPTP nature of the unknown map. A particularly interesting case is of a nonzero NCPTP energy extraction from CPTP-passive batteries.

Analogy with asymptotic manipulations of entanglement.—On a similar footing with the relation between distillable entanglement, E_D, and entanglement cost, E_C [see Fig. 2(a) of Supplemental Material [52]], we find that when $\Delta W_{\max}^{CP} = 0$, i.e., for CPTP-local passive states, ΔW_p^{NCP} can be positive. Moreover, even when $\Delta W_{\max}^{CP} > 0$, we have seen that ΔW_p^{NCP} can also be positive and strictly greater than $\Delta W_{\max}^{CP}$. This fact is depicted in Fig. 2(b) of Supplemental Material [52]. We have considered only those LNCPTP maps that can be generated by applying unitaries on the pure state $|\bar{\Psi}\rangle_{EAB}$. The situation can be generalized by considering mixed states of the entire system, EAB, which may further enhance the extractable energy. It can therefore be potentially inferred that $\Delta W_{\max}^{CP}$

and $\Delta W_{\max}^{NCP}$ exhibit a relation similar to that between E_D and E_C, respectively. This is merely a coincidence as of now. We, however, believe that there might be connections between the two. This is because the target state in analyzing entanglement cost is a mixed state in cases when there is a finite gap between entanglement cost and distillable entanglement. This is similar to what happens in NCPTP energy extraction—the state of AB is necessarily mixed, as otherwise, the operation on A cannot be a (physical) NCPTP map. The target state in entanglement distillation is a pure state (asymptotically), and in parallel, if AB is in a pure state, the operation on A is necessarily CPTP.

Necessary and sufficient condition for NCPTP-local passivity.—Next we provide a necessary and sufficient condition for a state to be NCPTP-local passive for an arbitrary but fixed Hamiltonian, H_{AB}. The pair, $\{H_{AB}, \rho_{AB}\}$, is NCPTP-local passive, with respect to subsystem A, where ρ_{AB} is attached to an environment E and the complete state of the composite system ABE is ρ_{ABE}, if and only if $C - C' \geq 0$, where $\langle \alpha\beta | C' | \alpha'\beta' \rangle = \delta_{\alpha\alpha'} \sum_i \langle ii | C | \beta'\beta \rangle$. Here $|i\rangle \in \mathcal{H}_{E_i} \otimes \mathcal{H}_{A_i}$, $|\beta^{(')}\rangle \in \mathcal{H}_{E_{1(2)}} \otimes \mathcal{H}_{A_{1(2)}}$, $C \equiv C_{E_1A_1E_2A_2} = \langle \phi | (\mathcal{I}_{E_1} \otimes H_{A_1B_1} \otimes \rho_{E_2A_2B_2}^T) | \phi \rangle_{B_1B_2}$, and $|\phi\rangle_{B_1B_2}$ is the unnormalized maximally entangled state, given by $|\phi\rangle_{B_1B_2} = \sum_j |jj\rangle$. $H_{A_1B_1}$ and $\rho_{E_2A_2B_2}$ are copies of H_{AB} and ρ_{ABE}, respectively, which act on the Hilbert spaces $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$ and $\mathcal{H}_{E_2} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$, respectively, where $\mathcal{H}_{E_i}$, $\mathcal{H}_{A_i}$ and $\mathcal{H}_{B_i}$ denote the Hilbert spaces of, respectively, the environment and subsystems A and B for $i = 1$ and 2. The proof of this statement is given in Supplemental Material [52] (see Refs. [71,72] for references). Additionally, we provide an independent necessary condition for NCPTP-local passivity using a different approach in the Supplemental Material [52].

Conclusion.—We examined the contrast between CPTP-open and NCPTP-open quantum batteries. The actions of CPTP and NCPTP maps have been restricted to only a subsystem. CPTP-local passive states of a bipartite quantum battery, described by a fixed Hamiltonian, are those states from which no energy can be extracted by applying CPTP maps to a particular part of the battery. One of our main goals in this Letter is to go beyond CPTP maps and compare the efficacy of physical NCPTP maps with that of CPTP maps in the context of energy extraction from quantum batteries. In this regard, we first proved that CPTP-local passive states remain CPTP-local passive even in the asymptotic limit and then showed that energy can be extracted from these CPTP-local passive states using LNCPTP maps. The extracted energy using NCPTP maps from CPTP-local passive states can be considered as a witness for the detection of the NCPTP nature of the applied map. Moreover, we maximized the energy extractable from a quantum battery using CPTP-local operations, and found even if this maximum is positive, the extractable energy using LNCPTP operations can exceed that positive value. These features are analogous to relative values of distillable entanglement and entanglement cost, where there exist shared quantum states for which the former vanishes while the latter is nonzero, as well as quantum states for which the former is nonzero and yet the latter is even larger. Finally, we presented a necessary and sufficient condition and an independent necessary condition for an arbitrary state to be NCPTP-local passive, which depends on the form of the Hamiltonian.

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