Two-Loop QED Corrections to the Scattering of Four Massive Leptons

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(Received 29 November 2023; revised 16 February 2024; accepted 9 April 2024; published 5 June 2024)

We study two-loop corrections to the scattering amplitude of four massive leptons in quantum electrodynamics. These amplitudes involve previously unknown elliptic Feynman integrals, which we compute analytically using the differential equation method. In doing so, we uncover the details of the elliptic geometry underlying this scattering amplitude and show how to exploit its properties to obtain compact, easy-to-evaluate series expansions that describe the scattering of four massive leptons in QED in the kinematical regions relevant for Bhabha and Møller scattering processes.

DOI: 10.1103/PhysRevLett.132.231904

In recent years, particle physics has seen several interesting developments in experiments at the low-energy precision frontier. Among these are the discrepancy between theory predictions and the experimental value for the muon anomalous magnetic moment, most recently measured to 0.20 ppm at Fermilab [1], as well as the so called "proton radius puzzle." The latter consists in a discrepancy between the proton charge radius as determined in [2,3] compared to previous results [4]. The upcoming PRad-II experiment [5] will perform an independent measurement to attempt to resolve this inconsistency.

This experimental program requires matching efforts on the theoretical side to provide equally precise and reliable predictions. An important part of these efforts is the recent development of the Monte Carlo event generator McMule [6]. Leveraging next-to-soft stabilization [7,8] for real-virtual corrections, McMule can describe Bhabha [9] ($e^+e^- \to e^+e^-$) and Møller scattering ($e^-e^- \to e^-e^-$) at the fully differential level up to next-to-next-to-leading order (NNLO) in quantum electrodynamics (QED). Bhabha scattering is important for luminosity measurements at lepton colliders. Møller scattering is instead the main source of systematic uncertainty for the PRad II experiment [5] and relevant for parity-violation searches and for precise measurements of the weak mixing angle [10]. Precision measurements of Møller scattering at very low energies (2.5 MeV) [11] have also been recently undertaken.

The remaining outstanding ingredient to make theoretical studies in NNLO QED at arbitrary energy scales possible, are the two-loop virtual amplitudes for the scattering of four massive leptons. Their calculation has received much attention in the literature. They were first computed in massless QED around two decades ago [12]. Event simulations at leading and next-to-leading order (NLO) [13–15], as well as power-suppressed mass effects up to NNLO [16–21] have been considered. Fermionic loop corrections were computed with full mass dependence in [22]. Logarithmically enhanced electroweak contributions have also been included to NNLO [23–26]. However, despite the long-lasting efforts [27–38], complete results for the massive two-loop virtual amplitudes are still not available, mostly due to the complexity of the integrals involved.

Here we move an important step towards the inclusion of mass effects to Bhabha and Møller scattering up to NNLO in QED, by performing the first calculation of the two-loop QED corrections to the scattering amplitude of four massive leptons. While the electron mass renders the two-loop integrals considerably more complicated, it will allow us to study the phenomenological impact of so-far neglected mass effects in extreme regions of phase space. In addition, these amplitudes also provide us with an invaluable playground to test recent developments in the theory of elliptic generalizations of multiple polylogarithms [39–45] and about canonical differential equations [46] for arbitrary geometries [47–55].

Kinematics and tensor decomposition.—We work in QED with one single type of massive lepton, which we refer to as the electron. We study higher-order corrections to the scattering of four electrons,

$$0 \to e^+(p_1) + e^-(p_2) + e^-(p_3) + e^+(p_4), \qquad (1)$$

where all momenta are outgoing and satisfy $p_i^2 = m^2$, $i = 1, \ldots, 4$, as well as $p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$. The amplitude $\mathcal{A}(1_{e^+}, 2_{e^-}, 3_{e^+}, 4_{e^-})$ can be parameterized as a function of the fermion mass m and three Mandelstam invariants

$$s=(p_1+p_2)^2,\quad t=(p_1+p_3)^2,\quad u=(p_2+p_3)^2,\quad (2)$$

where $s+t+u=4m^2$.

Following [56,57], we work in the 't Hooft-Veltman dimensional regularization scheme [58] (tHV) and decompose the scattering amplitude into eight independent Lorentz-covariant tensors T_i and respective form factors $\mathcal{F}_i$,

$$\mathcal{A}(1_{e^+},2_{e^-},3_{e^-},4_{e^+})=\sum_{i=1}^{8}\mathcal{F}_i T_i. \quad (3)$$

We choose the tensors as

$$\begin{aligned}&T_1=m^2\times t_1,\qquad T_2=m\times[t_2+t_3]\\&T_3=t_4,\qquad T_4=m^2\times t_5,\\&T_5=m\times[t_6+t_7]+t_8,\qquad T_6=m\times[t_6+t_7]-t_8\\&T_7=m\times[t_2-t_3],\qquad T_8=m\times[t_6-t_7],\end{aligned}\quad (4)$$

where the spinor chains t_i are defined as

$$t_i=\bar{U}_e(p_2)\Gamma_i^{(1)}V_e(p_1)\times\bar{U}_e(p_3)\Gamma_i^{(2)}V_e(p_4),\quad (5)$$

and $\mathbf{\Gamma}_i=\{\Gamma_i^{(1)},\Gamma_i^{(2)}\}$ represent the following sets of Dirac matrices:

$$\begin{aligned}&\mathbf{\Gamma}_1=\{1,1\},\qquad \mathbf{\Gamma}_2=\{\not{p}_3,1\},\\&\mathbf{\Gamma}_3=\{1,\not{p}_2\},\qquad \mathbf{\Gamma}_4=\{\not{p}_3,\not{p}_2\},\\&\mathbf{\Gamma}_5=\{\gamma^{\mu_1},\gamma_{\mu_1}\},\qquad \mathbf{\Gamma}_6=\{\not{p}_3\gamma^{\mu_1},\gamma_{\mu_1}\},\\&\mathbf{\Gamma}_7=\{\gamma^{\mu_1},\not{p}_2\gamma_{\mu_1}\},\qquad \mathbf{\Gamma}_8=\{\not{p}_3\gamma^{\mu_1},\not{p}_2\gamma_{\mu_1}\}.\end{aligned}\quad (6)$$

External momenta and polarizations are considered four dimensional, while loop momenta are treated in D dimensions. The number of tensors is then equal to the number of independent chirality configurations to all orders in perturbation theory [56,57], 16, of which only 8 are independent due to parity invariance. Furthermore, the process under consideration is invariant under the simultaneous exchange $p_2\leftrightarrow p_3$ and $p_1\leftrightarrow p_4$. Under this transformation $t_2\leftrightarrow t_3$ and $t_6\leftrightarrow t_7$, cf. Eq. (6), such that $T_{7,8}$ are odd. Accordingly, we conclude that $\mathcal{F}_7=\mathcal{F}_8=0$ to all orders.

Following the standard approach, we compute the form factors in Eq. (3), by defining a set of projectors $\mathcal{P}_i=[(T^\dagger\cdot T)^{-1}]_{ik}T_k^\dagger$ as combinations of dual tensors $T_i^\dagger$. Here "$\cdot$" denotes the scalar product between tensors and their dual, which is realized in practice by summation over spins of the external fermions, such that $\mathcal{F}_i=\mathcal{P}_i\cdot\mathcal{A}$. By applying the projectors on the corresponding relevant QED Feynman diagrams, we can express each form factor as a linear combination of scalar Feynman integrals and organize the one- and two-loop integrals into several integral topologies.

Technically our computation proceeds as follows. We generate the Feynman diagrams with QGRAF [59]. Using the computer algebra system FORM [60–63], we insert Feynman rules and apply projectors. We employ the public tool REDUZE2 [64] to find mappings onto topologies and expose their symmetries. Finally, we use Kira [65–67] to solve the required integration-by-parts (IBP) relations [68,69] and reduce all integrals to 267 master integrals. This is achieved following Laporta's algorithm [70], improved by finite field techniques [71,72].

Canonical bases for the nonplanar Feynman integrals.—While all planar two-loop topologies have been known in fully analytic form for some time [37,38], their nonplanar counterparts have remained elusive due to the appearance of new mathematical functions of elliptic type. In particular, we are interested in the nonplanar family of integrals displayed in the left graph of Fig (1). We define the integrals as

$$\mathrm{I}_{a_1a_2a_3a_4a_5a_6a_7a_8a_9}\left(D,\frac{s}{m^2},\frac{t}{m^2}\right)=e^{2\gamma_E\epsilon}(\mu^2)^{\sum_{j=1}^{9}a_j-D}\int\frac{\mathrm{d}^Dk_1}{i\pi^{\frac{D}{2}}}\frac{\mathrm{d}^Dk_2}{i\pi^{\frac{D}{2}}}\prod_{j=1}^{9}\frac{1}{P_j^{a_j}},\quad (7)$$

where γ_E denotes the Euler-Mascheroni constant, $D=4-2\epsilon$ is the dimension of space-time, and μ is an auxiliary scale that renders Feynman integrals dimensionless. The propagators are

$$\begin{aligned}&P_1=k_1^2-m^2,\quad P_2=(k_1-k_2-p_2)^2-m^2,\\&P_3=k_2^2-m^2,\quad P_4=(k_2+p_1+p_2)^2-m^2,\\&P_5=(k_1+p_1)^2,\quad P_6=(k_1-k_2)^2,\quad P_7=(k_2-p_3)^2,\\&P_8=(k_2+p_1)^2,\quad P_9=(k_1-p_3)^2.\end{aligned}\quad (8)$$

The integrals in Eq. (7) are functions of homogeneous coordinates $[s:t:m^2]$ on $\mathbb{CP}^2$ and, without loss of generality, we set $\mu=m$, or equivalently work on the patch $[y:z:1]$ with $y=s/m^2$ and $z=t/m^2$. For definiteness, we display our formulas in the region $s>4m^2$, $t<0$, though all results can also be easily continued to any other kinematic region. All integrals can be expressed through IBP reduction in terms of 52 independent master integrals which fulfil a system of first-order partial-differential equations [73–77] in the invariants

$$\mathrm{d}\vec{\mathrm{I}}=A(\epsilon,y,z)\vec{\mathrm{I}}. \quad (9)$$

Notice that 52 is the number of master integrals associated only to the nonplanar family in Eq. (7). To solve this system, we search for a basis transformation to a so-called ϵ-factorized form,

$$\mathrm{d}\vec{\mathrm{J}}=\epsilon A(y,z)\vec{\mathrm{J}},\qquad \vec{\mathrm{J}}=\mathrm{U}(y,z,\epsilon)\vec{\mathrm{I}},\quad (10)$$

which can be formally solved by a path-ordered exponential

$$\vec{\mathrm{J}}(y,z,\epsilon)=\mathbb{P}\exp\left[\epsilon\int_{\gamma}A\right]\vec{\mathrm{J}}_0(\epsilon,y_0,z_0), \qquad (11)$$

where the path γ connects the initial boundary point (y_0, z_0) to a generic point (y, z). If the matrix A can be expressed only through logarithmic differential forms, Eq. (10) is said to be in canonical form, and the new integral candidates $\vec{\mathrm{J}}$ are called a canonical basis [46]. While canonical bases beyond polylogarithms are not yet fully understood, advances have been made in extending ϵ-factorized bases to arbitrary geometries [48,49,52–54,78].

We obtained ϵ factorization by leveraging many of these developments. For the planar topologies, and for all polylogarithmic subsectors of the nonplanar topology, we used unitarity cuts and multivariate residue analysis [79] to select integral candidates with unit leading singularities [37,38]. Starting from the six-propagator nonplanar integrals, generalizations of these methods to genus-one geometries become necessary. In fact, one can see that the maximal cut of the irreducible six-propagator nonplanar four-point graph (see right panel in Fig. 1) in Baikov representation [80,81] can be expressed as

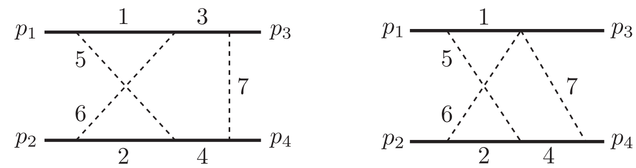

FIG. 1. The nonplanar topology (left) and its next-to-top sector (right) with six master integrals. Solid lines correspond to massive propagators, dashed lines to massless ones.

$$\mathrm{MaxCut}_{\mathcal{C}}[\mathrm{I}_{110111100}]\sim\int_{\mathcal{C}}\frac{\mathrm{d}z_2\wedge\mathrm{d}z_1}{z_2\sqrt{(z_1-s-z_2)(z_1-s+4m^2-z_2)}\sqrt{(tz_1-st+sz_2)^2-4m^2[tz_1^2+s(t-z_2)^2]}}. \qquad (12)$$

By further taking the residue at $z_2 = 0$ in (12), one is left with an integral over a family of elliptic curves

$$\mathcal{E}_4\colon\ Y^2=(X-e_1)(X-e_2)(X-e_3)(X-e_4), \qquad (13)$$

with the four roots given by

$$e_1=y-4,\qquad e_2=-\frac{yz+2\sqrt{yz(y+z-4)}}{4-z},$$
$$e_3=-\frac{yz-2\sqrt{yz(y+z-4)}}{4-z},\qquad e_4=y. \qquad (14)$$

We choose as first *period* for $\mathcal{E}_4$ the integral

$$\Psi_0(y,z)\equiv 2\int_{e_2}^{e_3}\frac{\mathrm{d}X}{Y}=\frac{4K(\lambda)}{\sqrt{(e_1-e_3)(e_2-e_4)}}, \qquad (15)$$

where $K(\lambda)$ is the complete elliptic integral of the first kind of argument

$$\lambda=\frac{4}{2+\sqrt{\frac{-y(y+z-4)}{-z}}}. \qquad (16)$$

To determine an ϵ-factorized form, we first notice that the integral sector $\mathrm{I}_{110111100}$ contains six master integrals. We therefore expect two master integrals which satisfy coupled differential equations and map to the generators of the first de Rham cohomology group $H^1_{\mathrm{dR}}(\mathcal{E}_4)$, plus four additional ones corresponding to independent punctures on the elliptic curve [54]. Candidates for the first two integrals can be found starting from the ansatz [82–84]

$$\mathrm{J}_{47}=\frac{\mathrm{I}_{110111100}}{\Psi_0},\qquad \mathrm{J}_{46}\sim\frac{1}{\epsilon}\frac{\Psi_0^2}{2\pi i W_z}\partial_z\mathrm{J}_{47}+\ldots, \qquad (17)$$

where $W_z=(1/2\pi i)[1/z^2(y+z)(y+z-4)]$ is the Wronskian of the second-order Picard-Fuchs equation associated with the elliptic curve. The explicit expression of J_{46} is immaterial for this discussion, and is given in the Supplemental Material [85]. The remaining four candidates can be identified by analyzing their integrand representation and the resulting differential equations. As a last step, in order to obtain a fully ϵ-factorized form, one needs to integrate out some inhomogenous entries in the differential equation matrix, which leads to the appearance of additional transcendental integrals. In this way, the final ϵ-factorized system (10) is expressed in terms of 87 distinct one-forms ω_i. It is easy to verify that the integrability condition $\mathrm{d}A = A\wedge A$ is satisfied and that all ω_i are closed $\mathrm{d}\omega_i = 0$.

The individual differential forms can be simplified by exploiting the underlying geometry of the family of elliptic curves in (13). As an example, consider the following two functions

$$\begin{aligned} T_1(y,z) = \int \Bigg[&\mathrm{d}y \left(\frac{-z}{y} (4y^2 + 4y(z-4) + z(z-4)) \Psi_0 - 8z \frac{(y+z-4)(y+z)}{(t+2y-4)} \partial_y \Psi_0 \right) \\ &+ \mathrm{d}z \left(\frac{-z}{4-z} \frac{-48 + 4y + 2y^2 + 12z + yz}{z+y-4} \Psi_0 \right) \Bigg], \\ T_2(y,z) = \int \Bigg[&\mathrm{d}y \sqrt{4-z}\sqrt{-z} \left(\frac{z}{y} \frac{4 + 2y - y^2 - z - yt}{2(y+z-4)} \Psi_0 - \frac{1}{2} z (1+y) \partial_y \Psi_0 \right) \\ &+ \mathrm{d}z \sqrt{-z}\sqrt{4-z} \left(\Psi_0 \frac{y-4}{2(y+z-4)} + \frac{(y-4)y(1+y)}{2(-4+2y+z)} \partial_y \Psi_0 \right) \Bigg], \end{aligned} \tag{18}$$

which are among the objects required to express the matrix A in (10). Again, formulas assume $y > 4$ and $z < 0$. While the details of the construction are immaterial here and are discussed elsewhere [86,87], it suffices to say that one can parameterize the kinematical variables by

$$y = 2\frac{(1-x)(1+t_4)}{t_4 - x}, \qquad z = 4\frac{t_4(1-x^2)}{x^2 - t_4^2}, \tag{19}$$

where $\{[x:Y:1], t_4\}$ are the canonical coordinates on the moduli space of elliptic curves given by $Y^2 = (x^2-1)(x^2-t_4)$. In these coordinates, the period in (15) becomes

$$\Psi_0(x,t_4) = \frac{2(x^2-t_4)}{-Y} K(t_4). \tag{20}$$

We can identify t_4 with a Hauptmodul for the congruence subgroup $\Gamma_1(4) \subset \mathrm{SL}_2(\mathbb{Z})$ [88]. Strikingly, by using Eq. (19) the two transcendental integrals in Eq. (18) turn into combinations of simpler functions

$$\begin{aligned} T_1(x,t_4) &= 8t_4 \frac{K(t_4)}{\pi} \left[(1-t_4)\mathcal{F}(x,t_4) - \frac{x^2-1}{(1+t_4)Y} \right], \\ T_2(x,t_4) &= \frac{1}{\pi} \sqrt{\frac{t_4}{1+t_4}} \frac{t_4(3-2x) - 3x + 2}{t_4 - x} K(t_4) - \frac{f(t_4)}{2\pi}, \end{aligned} \tag{21}$$

where $f(t_4)$ is given by

$$\partial_{t_4} f = 2\frac{1-t_4}{\sqrt{t_4}(1+t_4)^{3/2}} K(t_4), \tag{22}$$

and $\mathcal{F}(x,t_4)$ is the derivative of the Abel map:

$$\mathcal{F}(x,t_4) = K(t_4)\partial_{t_4} \left[\frac{1}{K(t_4)} \int_{-1}^{x} \frac{\mathrm{d}X}{\sqrt{(X^2-1)(X^2-t_4)}} \right]. \tag{23}$$

Other differential forms in the alphabet can also be substantially simplified and all double integrals over the periods can be rewritten in terms of rational functions of T_1 and T_2. With this, all differential forms are given by combinations of five algebraic functions $\{\sqrt{x^2-1}, \sqrt{x^2-t_4}, \sqrt{1+t_4}, \sqrt{t_4}, \sqrt{1-t_4}\}$, and three transcendental functions $\{K(t_4), f(t_4), \mathcal{F}(x,t_4)\}$.

We stress that the choice of canonical coordinates in Eq. (19) is not merely an academic curiosity, and the final, simplified form is essential to implement the numerical evaluation of the iterated integrals described below. To explicitly solve the integrals, we first expand Eq. (11) in ϵ. At each order, we fix all boundary conditions imposing regularities at different phase-space points and obtain fully analytic results for the nonplanar master integrals in terms of Chen iterated integrals [89].

Currently, there are no public numerical routines to evaluate the special functions that appear in the nonplanar sector. We therefore obtain generalized series expansions for all master integrals. More precisely, we start from the differential equations in ϵ-factorized form in order to algorithmically obtain a small mass expansion for the individual master integrals. In particular, we obtain a generalized power series (including logarithms of the mass), whose coefficients can be expressed in terms of harmonic polylogarithms [90]. We obtain results that are valid both for the kinematics relevant for Bhabha and Møller scattering. As a cross check, we compared individual master integrals against a direct numerical evaluation with AMFlow [91], both for Bhabha and Møller scattering kinematics, and found agreement to high precision. Our series expansions allow for fast numerical evaluation, appropriate for phenomenological studies. A precise description of the numerical implementations can be obtained with the ancillary files provided in [92]. We stress that in principle, starting from our analytic result, series expansions in other physically relevant limits can be obtained to complement the convergence of the small mass expansion.

UV renormalization and IR factorization.—Using the master integrals calculated above, as well as the planar integrals from [37,38], we can obtain an analytic result for the bare amplitude for both polarized and unpolarized scattering. We renormalize UV divergences according to

$$\mathcal{A}^{\mathrm{r}}(\alpha_e, m, s, t, \epsilon) = Z_2^2 \mathcal{A}(\alpha_e^0, m_b, s, t, \epsilon), \qquad (24)$$

with the relation between bare and physical quantities

$$\frac{e^2}{4\pi} = \alpha_e^0 = \left(\frac{e^{\gamma_E}}{4\pi}\mu^2\right)^{\epsilon} Z_e \alpha_e(\mu), \qquad m_b = Z_m m. \qquad (25)$$

Here Z_2 and Z_m are on-shell wave function and mass renormalization constants, and Z_e refers to coupling constant renormalization either in the $\overline{\mathrm{MS}}$ or on-shell (OS) scheme. The relevant quantities are taken from [36,93–97] and collected in the Supplemental Material. As expected [98], after UV renormalization we are left with one-loop exact IR poles

$$\mathcal{A}^{\mathrm{OS}}(\alpha, m, s, t, \epsilon) = e^{\frac{\frac{\alpha}{4\pi} Z_1^{\mathrm{IR}}}{\epsilon}} \mathcal{C}(\alpha, m, s, t, \epsilon), \qquad (26)$$

where $\mathcal{C}$ is the finite remainder function, α is the on-shell electromagnetic coupling, and Z_1^{IR} is the anomalous dimension which controls the soft singularities of the amplitude to all-orders through exponentiation [99,100]. The exact form of Z^{IR} is immaterial and we report it for completeness in the Supplemental Material.

We performed several checks on our results. First of all, we verified that our two-loop amplitudes have the correct UV and IR behavior, as illustrated above. In addition, we compared both bare and renormalized one-loop amplitudes against OpenLoops [101,102] and found perfect agreement. We stress here that the unpolarized finite remainders in Conventional Dimensional Regularization equal those in the tHV scheme, while the bare and UV-renormalized amplitudes in general differ. The equality of the finite remainders provides another check of our calculation.

Discussion and conclusions.—Our results for the two-loop amplitudes for Bhabha and Møller scattering are given as generalized series expansion in $x = m/E_{\mathrm{CM}}$. They are provided as computer-readable files in [92] for both the polarized and unpolarized scattering amplitudes. We provide sufficiently high orders to obtain reliable predictions for the low-energy experiments mentioned in the introduction, where we expect the mass effects to be the largest. In the following we discuss some of the phenomenological implications of our results focusing on unpolarized Møller scattering.

Let us start by assessing the accuracy of the small-mass expansion. We begin by noticing that we expect the expansion to become unreliable in the extreme forward or backward regions, where the coefficients of the series in x develop large logarithms in $(-t)/s$ which can invalidate the convergence of the expansion [103]. To quantify the region of convergence, we compare the exact results for the one-loop amplitude A_{exact}^{1l} with the corresponding expansion A_{20}^{1l} to $\mathcal{O}(x^{20})$ and study the ratio $\delta_{\mathrm{exact},20}^{1l} = (A_{\mathrm{exact}}^{1l} - A_{20}^{1l})/A_{\mathrm{exact}}^{1l}$. Depending on the scattering energy $E_{\mathrm{CM}} = \sqrt{s}$, we find that $\delta_{\mathrm{exact},20}^{1l} \leq 1\%$ for different ranges of the scattering angle θ:

$$\begin{aligned} E_{\mathrm{CM}} &= 150m \to 2^\circ < \theta < 179^\circ, \\ E_{\mathrm{CM}} &= 32m \to 9^\circ < \theta < 174^\circ, \\ E_{\mathrm{CM}} &= 5m \to 70^\circ < \theta < 130^\circ, \end{aligned} \qquad (27)$$

where the energy values match those probed at present and future experiments. This shows that at very low energies the expansions must be interpreted with care outside of the central region. To extend the analysis to the two-loop amplitudes, we compare the series expanded to order 20 with the one expanded to order 18. We find that the same applies: for $L = 1, 2$ $(A_{20}^{Ll} - A_{18}^{Ll})/A_{20}^{Ll} \leq 1\%$ for the same values of θ as in (27). In Fig. 2 we display the various orders of the series for the two-loop amplitude, for different values of the scattering at the intermediate energy of $E_{\mathrm{CM}} = 32m$. We highlight the lack of convergence for θ not in the range $[9^\circ, 174^\circ]$ in the two subplots.

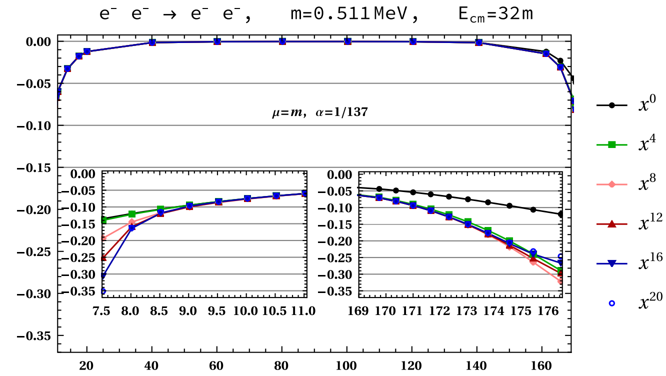

FIG. 2. Convergence of the mass expansion. Plotted are the two-loop finite remainders $\mathcal{C}^{\dagger(2)}\mathcal{C}^{(0)}$ as functions of scattering angle in degrees, at various truncation orders.

Having assessed the validity of our small-mass expansions, let us comment on the phenomenological relevance of the mass effects. So far two-loop mass effects had only been included to leading power, $\mathcal{O}(x^0)$. We expect that the finite-mass effects are more pronounced for small values of E_{CM}. In Fig. 2 we see that, for $E_{\mathrm{CM}} = 32m$, the two-loop leading-power approximation does not capture the full extend of the mass effects for $\theta \gtrsim 150^\circ$ [for small angles, we are outside the region of (27)]. We therefore expect that in that region precise NNLO results can only be obtained by including the subleading terms we have computed. The effect is even more pronounced for $E_{\mathrm{CM}} = 5m$: in Fig. 3 we show that, even in the range of intermediate angles in Eq. (27), the leading-power approximation does not provide a reliable prediction of the finite-mass effects. At the same time, we observe a very nice convergence of the mass expansion, corroborating that we can provide reliable and precise predictions for the two-loop corrections even at

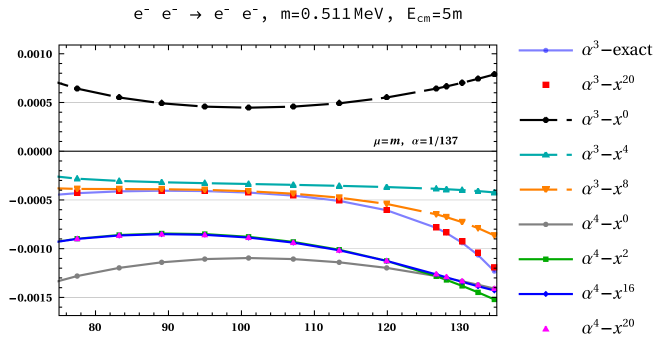

FIG. 3. Mass effects at low energies. Plotted are the one-loop $\mathcal{C}^{\dagger(1)}\mathcal{C}^{(0)}$ and two-loop $\mathcal{C}^{\dagger(2)}\mathcal{C}^{(0)}$ finite remainders as functions of scattering angle in degrees. The two-loop amplitudes are rescaled by a factor of 25.

such low energies. A full discussion of the size of the NNLO QED corrections will be presented elsewhere.

To conclude, in this Letter we have addressed the calculation of the two-loop QED corrections to the scattering of four identical massive leptons, retaining full dependence on the lepton mass. This constitutes the last outstanding ingredient necessary to perform NNLO QED phenomenological studies for standard processes as Bhabha and Møller scattering. In addition to the phenomenological interest behind these calculations, the scattering amplitudes computed in this Letter are an important example of physical processes that receive nontrivial contribution from elliptic Feynman integrals. We presented a strategy to compute these amplitudes analytically through the differential equations method and provided a robust numerical implementation. We demonstrated that for low values of E_{CM}, the mass effect can be sizable and is not captured by the leading-power approximation. We therefore expect that our results will play an important role in making precise predictions for low-energy lepton collider experiments possible.

We thank Federico Buccioni for providing numerical results for the one-loop amplitudes with OpenLoops and Christoph Nega for useful comments on the manuscript. We are also indebted to Vladimir Smirnov for collaboration in the initial stages of this project. This work was supported in part by the Excellence Cluster ORIGINS funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy—EXC-2094-390783311 and by the European Research Council (ERC) under the European Union's research and innovation programme Grant Agreement No. 949279 (ERC Starting Grant HighPHun) and No. 101043686 (ERC Consolidator Grant LoCoMotive). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

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