# Quantum Speed Limit for States and Observables of Perturbed Open Systems

Benjamin Yadin, Satoya Imai, and Otfried Gühne
*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany*

(Received 14 August 2023; revised 1 March 2024; accepted 6 May 2024; published 7 June 2024)

Quantum speed limits provide upper bounds on the rate with which a quantum system can move away from its initial state. Here, we provide a different kind of speed limit, describing the divergence of a perturbed open system from its unperturbed trajectory. In the case of weak coupling, we show that the divergence speed is bounded by the quantum Fisher information under a perturbing Hamiltonian, up to an error which can be estimated from system and bath timescales. We give three applications of our speed limit. First, it enables experimental estimation of quantum Fisher information in the presence of decoherence that is not fully characterized. Second, it implies that large quantum work fluctuations are necessary for a thermal system to be driven quickly out of equilibrium under a quench. Moreover, it can be used to bound the response to perturbations of expectation values of observables in open systems.

DOI: 10.1103/PhysRevLett.132.230404

The quantum time-energy uncertainty relation was first put on a rigorous footing by Mandelstam and Tamm [1], who showed that the time for a quantum state to evolve to an orthogonal one is limited according to its energy uncertainty. Since then, many generalizations and variations have been derived, including the Margolus-Levitin bound [2] involving mean energy. Such inequalities are referred to as quantum speed limits. We now have extensions to mixed states and driven and open systems [3–7], and an understanding of the connection to the geometry of quantum state spaces [8–10].

Quantum speed limits imply bounds on information processing rates [11–13] and maximum physically allowable rates of communication [14]. There are also applications within quantum thermodynamics, including bounding entropy production rates [15], heat engine efficiency and power [16,17], and battery charging rates [18–21]. There is an intimate relation between speed limits and metrology, in which the best precision in estimating parameter encoded into a state depends on how quickly the state evolves [22,23].

A central quantity in metrology, the quantum Fisher information (QFI) [9,24], is interpretable as a (squared) speed in state space, and can be used to quantify many important properties of quantum states. Sufficiently large QFI demonstrates many-body entanglement [25–30] and steering [31]; similarly, QFI can be used as a measure of coherence in a given basis [32–34], of macroscopic quantumness [35] and of optical nonclassicality [36,37], and can witness general quantum resources [38]. Therefore, it is desirable to experimentally measure lower bounds to QFI. One common method, among others [39,40], is to adopt speed limits to estimate the QFI from a measure of distance between an initial state and one evolved for a short time [41,42].

In addition, it can be useful to bound the speed at which two quantum states separate when they undergo different dynamics. Examples of applications include the discrimination of unitary operations [43], the performance of adiabatic quantum computation [44], fidelity of quantum control [45], dynamics of entanglement [46], and multiparameter metrology [47].

In this work, we devise a novel type of speed limit that describes the response of a Markovian open quantum system to a perturbation to its dynamics. The inequality upper bounds the distance between the perturbed and unperturbed trajectories in state space in terms of the QFI of the system with respect to the perturbation. Importantly, this holds under minimal assumptions without detailed knowledge of the dynamics. For a system weakly coupled to its environment, the speed limit is given in terms of the QFI under a perturbing Hamiltonian, up to an error bounded in terms of relevant physical timescales. We show how this may be used for an experimental lower bound on the QFI. We then provide an application to the thermodynamics of systems perturbed out of equilibrium, showing that quantum fluctuations in the work performed during a sudden quench are required for fast departure from the initial state. Finally, we give an application to linear response by bounding changes in expectation values under perturbations.

*Preliminaries.*—The Mandelstam-Tamm bound [1] relates the energy variance $\mathrm{Var}(\psi, H) = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$ of a pure state $|\psi\rangle$ to the time $\tau$ it needs to evolve to an orthogonal one under Hamiltonian $H$:

$$\tau \geq \frac{\pi}{2\sqrt{\mathrm{Var}(\psi, H)}}. \tag{1}$$

(We work in units where $\hbar = 1$ throughout.) Therefore a large energy variance is necessary to evolve quickly to an orthogonal state. This result has since been strengthened to account for mixed states and nonorthogonality. The Uhlmann bound [8] involves the fidelity $F(\rho_0, \rho_t) \coloneqq \mathrm{tr}\sqrt{\sqrt{\rho_0}\rho_t\sqrt{\rho_0}}$

 

between the initial and final states, $\rho_0$ and $\rho_t = e^{-itH}\rho_0 e^{itH}$, recast into the Bures angle $\theta_B(\rho,\sigma) \coloneqq \arccos F(\rho,\sigma)$.

Instead of energy variance, we require the QFI of the system. Most generally, QFI measures the sensitivity of a continuously parameterized family of states to small changes in a parameter [24]. Here, we consider the time parameter, so the QFI is a function of the state $\rho_t$ and its derivative $\mathrm{d}\rho_t/\mathrm{d}t$. Throughout this Letter, we consider evolutions generated by Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) superoperators [48,49] $\mathrm{d}\rho_t/\mathrm{d}t = \mathcal{L}_t(\rho_t)$, for which the QFI $\mathcal{F}(\rho_t, \mathcal{L}_t)$ is a function of $\rho_t$ and $\mathcal{L}_t$. One definition is expressed in terms of the spectral decomposition $\rho_t = \sum_i \lambda_i(t)|\psi_i(t)\rangle\langle\psi_i(t)|$: $\mathcal{F}(\rho_t,\mathcal{L}_t) = 2\sum_{i,j:\lambda_i(t)+\lambda_j(t)>0} |\langle\psi_i(t)|\mathcal{L}_t(\rho_t)|\psi_j(t)\rangle|^2/[\lambda_i(t)+\lambda_j(t)]$. The Uhlmann bound is

$$\theta_B(\rho_0,\rho_t) \le \frac{1}{2}\int_0^t \mathrm{d}s\sqrt{\mathcal{F}(\rho_s,\mathcal{H}_s)} \quad \forall\ t \ge 0, \qquad (2)$$

where $\mathcal{H}_t(\cdot) = -i[H_t,\cdot]$ generates time evolution under the time-dependent Hamiltonian $H_t$. This bound derives ultimately from the infinitesimal expansion of the Bures angle as a metric on state space, $\theta_B(\rho_t,\rho_{t+\mathrm{d}t})^2 = \frac{1}{4}\mathcal{F}(\rho_t,\mathcal{L}_t)\mathrm{d}t^2$, with the finite Bures angle being the length of a geodesic between two points [9]. Equation (1) can be derived from Eq. (2) [15]; one sees that the square-root QFI may be interpreted as a "statistical speed" [22,24].

*Perturbation speed limit.*—Here, we prove the main result for a system undergoing arbitrary Markovian dynamics with a perturbation. We take the common definition equating Markovianity with divisibility, namely, that the mapping $\mathcal{N}_{t_1,t_0}$ of states between any times $t_0 < t_1$ is completely positive and trace preserving, and satisfies $\mathcal{N}_{t_2,t_0} = \mathcal{N}_{t_2,t_1}\mathcal{N}_{t_1,t_0}$ for all $t_0 \le t_1 \le t_2$. This is equivalent to the dynamics being dictated by a GKSL generator $\mathcal{L}_t$ [50].

*Result 1.*—Consider a system starting in state $\rho_0$ which may evolve along one of two trajectories: (i) Markovian free evolution, $\mathrm{d}\rho_t/\mathrm{d}t = \mathcal{L}_t(\rho_t)$; or (ii) perturbed evolution, $\mathrm{d}\sigma_t/\mathrm{d}t = \mathcal{L}'_t(\sigma_t) = \mathcal{L}_t(\sigma_t) + \mathcal{P}_t(\sigma_t)$ (satisfying the initial condition $\sigma_0 = \rho_0$). The Bures angle between the trajectories satisfies

$$\theta_B(\rho_t,\sigma_t) \le \frac{1}{2}\int_0^t \mathrm{d}s\sqrt{\mathcal{F}(\sigma_s,\mathcal{P}_s)} \quad \forall\ t \ge 0. \qquad (3)$$

*Proof.*—Here we summarize the proof detailed in Sec. I of [51]. We use three facts about the Bures angle: (i) the triangle inequality, (ii) contractivity under quantum channels [59], and (iii) its infinitesimal expansion, stated above. At time $t$, consider $\rho_t$, $\sigma_t$ and their corresponding time-evolved states $\rho_{t+\delta t}, \sigma_{t+\delta t}$ a short time $\delta t$ later. In addition, consider instead evolving $\sigma_t$ under the *unperturbed* dynamics for time $\delta t$, giving $\sigma'_{t+\delta t}$ (see Fig. 1). To lowest order, $\rho_{t+\delta t} = \rho_t + \delta t\mathcal{L}_t(\rho_t) + \mathcal{O}(\delta t^2)$, $\quad \sigma_{t+\delta t} = \sigma_t + \delta t\mathcal{L}_t(\sigma_t) + \delta t\mathcal{P}_t(\sigma_t) + \mathcal{O}(\delta t^2)$, and $\sigma'_{t+\delta t} = \sigma_t + \delta t\mathcal{L}_t(\sigma_t) + \mathcal{O}(\delta t^2)$. The triangle inequality gives

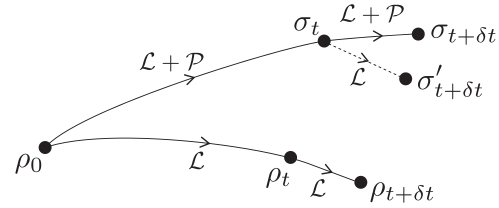


FIG. 1. Illustration of the trajectories used in the proof of the main speed limit Eq. (3).

$$\theta_B(\rho_{t+\delta t},\sigma_{t+\delta t}) \le \theta_B(\rho_{t+\delta t},\sigma'_{t+\delta t}) + \theta_B(\sigma'_{t+\delta t},\sigma_{t+\delta t}). \qquad (4)$$

For the first term on the right-hand side of Eq. (4), we use that $\rho_{t+\delta t}$ and $\sigma'_{t+\delta t}$ have been evolved for time $\delta t$ under the same dynamics comprising the channel $\mathcal{N}_{t+\delta t,t}$. Contractivity of $\theta_B$ therefore implies $\theta_B(\rho_{t+\delta t},\sigma'_{t+\delta t}) = \theta_B[\mathcal{N}_{t+\delta t,t}(\rho_t),\mathcal{N}_{t+\delta t,t}(\sigma_t)] \le \theta_B(\rho_t,\sigma_t)$. For the second term, we use the infinitesimal form of $\theta_B$, and that $\sigma_{t+\delta t} - \sigma'_{t+\delta t} = \delta t\mathcal{P}_t(\sigma_t) + \mathcal{O}(\delta t^2)$, to write $\theta_B(\sigma'_{t+\delta t},\sigma_{t+\delta t}) = (\delta t/2)\sqrt{\mathcal{F}(\sigma_{t+\delta t},\mathcal{P}_t)} + \mathcal{O}(\delta t^2)$. Putting these into Eq. (4), $\theta_B(\rho_{t+\delta t},\ \sigma_{t+\delta t}) \le \theta_B(\rho_t,\ \sigma_t) + (\delta t/2)\sqrt{\mathcal{F}(\sigma_{t+\delta t},\ \mathcal{P}_t)} + \mathcal{O}(\delta t^2)$. Subtracting the first term on the right, dividing by $\delta t$ and taking $\delta t \to 0$ gives $\mathrm{d}\theta_B(\rho_t,\sigma_t)/\mathrm{d}t \le \frac{1}{2}\sqrt{\mathcal{F}(\sigma_t,\mathcal{P}_t)}$; integrating gives the result.

In the case $\mathcal{L}_t = 0$, $\rho_t = \rho_0$ is stationary and the bound reduces to a previously known one [60]. The closed-system case is obtained with Hamiltonian dynamics $\mathcal{L}_t = \mathcal{H}_t$ and $\mathcal{P}_t(\cdot) = v\mathcal{V}_t(\cdot) \coloneqq -iv[V_t,\cdot]$. Note that the bound in this case is equivalent to Uhlmann's (2), as can be seen by moving to the interaction picture. Thus Eq. (3) generalizes previous speed limits. The relevant statistical speed measures sensitivity of the system to the perturbation.

Uhlmann's bound, relying on the triangle inequality, is saturated by geodesics [8]. Equation (3) additionally uses contractivity of $\theta_B$ under quantum channels, thus can only be saturated if $\mathcal{L}_t$ causes no contraction between the trajectories. This will not generally hold; however, our later example in Fig. 2 shows that the bound may be practically quite tight.

Using a similar method, we prove a related *observable* speed limit. This bounds the difference in expectation value of any observable $A$ between the trajectories, extending previous speed limits for observables [20,61] to the perturbation setting.

*Result 2.*—Using the same assumptions as Result 1, for any observable $A$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}[A(\rho_t-\sigma_t)]\right| \le 2\|\mathcal{L}_t^\dagger(A)\|D_{\mathrm{tr}}(\rho_t,\sigma_t) + \sqrt{\mathrm{Var}(\sigma_t,A)\mathcal{F}(\sigma_t,\mathcal{P}_t)}, \qquad (5)$$

where $\mathcal{L}_t^\dagger$ is the adjoint map of $\mathcal{L}_t$, $\|\cdot\|$ denotes the largest singular value, and $D_{\mathrm{tr}}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1$ is the trace distance.

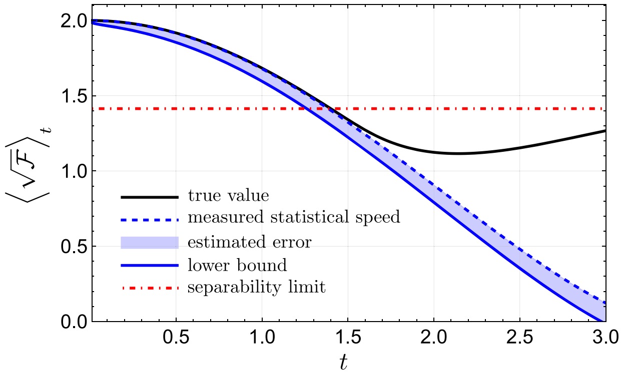


FIG. 2. Two-qubit example with local dephasing noise, showing how the time-averaged value of $\sqrt{\mathcal{F}(\eta_s,\mathcal{V})}$ from $s=0$ to $t$ can be lower-bounded using the speed limit Eq. (6). The initial state $[(|00\rangle+|11\rangle)/\sqrt{2}]$ is maximally entangled. In units of $h=1$, we take $\lambda=\gamma=v=0.1$ and $\epsilon\leq 4\lambda^2\gamma/h=0.004$. The measured statistical speed is the left-hand side of Eq. (8), taking a Bell-basis measurement $\{[(|00\rangle\pm|11\rangle)/\sqrt{2}],[(|01\rangle\pm|10\rangle)/\sqrt{2}]\}$. The estimated error $2\Delta_{\rm est}(t)/vt$ (shaded area) is subtracted to give the lower bound.

(See proof in Sec. II of [51].) A notable difference with Eq. (3) is the additional "drift" term depending on $\mathcal{L}_t$.

The power of result 1 comes from requiring no detailed information about the unperturbed dynamics (in contrast to result 2); such details only appear implicitly via the evolution of the perturbed trajectory. For the remainder of this Letter, we assume (typical in quantum control) that the perturbation comes from a controlled change to the system's Hamiltonian, $H_t\to H_t+vV_t$ (including the constant $v$ to quantify the size of the perturbation). For many applications (see later sections), one is interested in QFI with respect to $V_t$; however, the resulting perturbation to the master equation $\mathcal{P}_t$ could contain additional terms. The identification of $\mathcal{P}_t$ with $v\mathcal{V}_t$ may be justified in the singular coupling limit [62] and in collision models of open system dynamics [63]. However, in weak coupling, a change to the system's Hamiltonian generally adds an additional perturbation to the master equation. We therefore now study the error incurred by the approximation $\mathcal{P}_t\approx v\mathcal{V}_t$, and, correspondingly, the use of QFI with respect to $V_t$ in the right-hand side of Eq. (3). We consider time-independent dynamics for simplicity.

*Weak coupling.*—To address this, we now consider a system weakly coupled to a Markovian environment and derive the error incurred by approximating the true perturbed trajectory with one where the dissipative part of the dynamics is unchanged. Such situations are ubiquitious in experiments encompassing discrete-[64] and continuous-variable [65] systems. We assume a standard weak-coupling master equation with secular approximation [66], $(d\rho_t/dt)=\mathcal{L}(\rho_t)=-i[H+H_{\rm LS},\rho_t]+\mathcal{D}(\rho_t)$, where the Lamb shift Hamiltonian $H_{\rm LS}$ and dissipator $\mathcal{D}$ are given by $H_{\rm LS}=\lambda^2\sum_{\omega,\alpha,\beta}S_{\alpha\beta}(\omega)A_\alpha^\dagger(\omega)A_\beta(\omega)$ and $\mathcal{D}(\rho)=\lambda^2\sum_{\omega,\alpha,\beta}\gamma_{\alpha\beta}(\omega)[A_\beta(\omega)\rho A_\alpha^\dagger(\omega)-\frac{1}{2}\{A_\alpha^\dagger(\omega)A_\beta(\omega),\rho\}]$, involving real and imaginary parts $\gamma_{\alpha\beta},S_{\alpha\beta}$ of the bath correlation function and jump operators $A_\alpha(\omega)$ assocated with Bohr frequencies $\omega$ (see Sec. III of [51] for details). We factor out the coupling strength $\lambda$ such that $A_\alpha=\mathcal{O}(1)$ (independent of $\lambda$).

We denote the size of the free system Hamiltonian $H$ by $h$ (measuring the size of the smallest energy gap and not to be confused with the Planck constant) and of the perturbing Hamiltonian by $v$ (taking $V=\mathcal{O}(1)$). The important timescales are those of the intrinsic system dynamics $\tau_S\sim h^{-1}$, the perturbation $\tau_V\sim v^{-1}$, the system relaxation $\tau_R\sim\lambda^{-2}\gamma^{-1}$, and the bath correlation decay $\tau_B$. We make the following assumptions: (i) Born-Markov approximation, $\tau_B\ll\tau_R$, (ii) rotating wave approximation, $\tau_S\ll\tau_R$ [62,66], (iii) small perturbation relative to the bath, $\tau_B\ll\tau_V$, and (iv) small perturbation relative to the system, $\tau_S\ll\tau_V$.

Upon perturbing $H\to H'=H+vV$, we replace $H_{\rm LS}\to H'_{\rm LS}=H_{\rm LS}+vH^{(1)}_{\rm LS}$ and $\mathcal{D}\to\mathcal{D}'=\mathcal{D}+v\mathcal{D}^{(1)}$ to first order in $v$. This alters the Bohr frequencies $\omega$ and components $A_\alpha(\omega)$. Expressions for these are derived in Sec. III of [51], the perturbations $H^{(1)}_{\rm LS},\mathcal{D}^{(1)}$ being of size $\epsilon\coloneqq\max_\psi\|\mathcal{H}^{(1)}_{\rm LS}(\psi)+\mathcal{D}^{(1)}(\psi)\|=\mathcal{O}(\tau_S/\tau_R)+\mathcal{O}(\tau_B/\tau_R)$. It follows from assumptions (i)–(iv) that these terms are small compared with others in the master equation.

Applying bound (3) to this setting, we identify the *true* perturbed trajectory $d\eta_t/dt=\mathcal{L}'(\eta_t)$ and the *approximate* perturbed trajectory $d\sigma_t/dt=\mathcal{L}(\sigma_t)+v\mathcal{V}(\sigma_t)$. In the latter, we only perturb the Hamiltonian term and ignore additional terms of size $\epsilon$. All trajectories have the same initial state $\rho_0$.

*Result 3.*—For an open system in the weak coupling regime perturbed by the Hamiltonian $vV$,

$$\theta_B(\rho_t,\eta_t)\leq\frac{1}{2}\int_0^t ds\, v\sqrt{\mathcal{F}(\eta_s,\mathcal{V})}+\Delta(t),\qquad(6)$$

where the error term is bounded by the estimate

$$|\Delta(t)|\lesssim\Delta_{\rm est}(t)\coloneqq\frac{4\sqrt{2}}{3}\|V\|\epsilon^{\frac{1}{2}}(vt)^{\frac{3}{2}}+\epsilon vt.\qquad(7)$$

See Sec. IV of [51] for the proof. For short times, the QFI term in Eq. (6) is roughly $vt\sqrt{\mathcal{F}(\rho_0,\mathcal{V})}$—hence, the error is negligible when $\sqrt{\mathcal{F}(\rho_0,\mathcal{V})}\gg\max\{\sqrt{\epsilon vt},\epsilon\}$.

In specific cases, one can determine the error parameter $\epsilon$ more precisely. We demonstrate this for a spin-boson model of two qubits interacting with a bath of many harmonic oscillators [62]. We take $H=(h/2)(\sigma_z\otimes\mathbb{1}+\mathbb{1}\otimes\sigma_z)$, $V=\frac{1}{2}(\sigma_x\otimes\mathbb{1}+\mathbb{1}\otimes\sigma_x)$, and an independent coupling of each qubit to a bath of the form $\lambda\sigma_z\otimes\sum_k g_k(b_k+b_k^\dagger)$, $g_k$

being dimensionless coefficients. Here, $\sigma_i$ are Pauli matrices and $b_k$ is the annihilation operator for the bosonic mode $k$. This gives local dephasing dynamics $\mathcal{D}(\rho) = \lambda^2\gamma[(\sigma_z \otimes \mathbb{1})\rho(\sigma_z \otimes \mathbb{1}) + (\mathbb{1} \otimes \sigma_z)\rho(\mathbb{1} \otimes \sigma_z) - 2\rho]$, writing $\gamma = \gamma(0)$. Then we find $\epsilon \leq 4\lambda^2\gamma/h$ (see Sec. V of [51]). In this case, the component of order $\tau_B/\tau_R$ vanishes. A numerical demonstration of the tightness of the speed limit for this example is shown in Fig. 2.

*Witnessing large QFI.*—Quantum correlations and other resources may be witnessed experimentally by showing that the QFI under some Hamiltonian $H$ exceeds a given threshold $\mathcal{F}_*$. For closed systems, a standard method [42] obtains a lower bound to $\theta_B(\rho_0, \rho_t)$ with evolution under $V$, by measuring the system at either time 0 or $t$. Each measurement has probability distribution $p_i(0), p_i(t)$; their similarity is quantified by the Bhattacharyya coefficient [67] $B[\mathbf{p}(0), \mathbf{p}(t)] \coloneqq \sum_i \sqrt{p_i(0)p_i(t)}$, which satisfies $\arccos B[\mathbf{p}(0), \mathbf{p}(t)] \leq \theta_B(\rho_0, \rho_t)$. This bound holds for any measurement and can be saturated. Thanks to Eq. (2), the resource is thus witnessed when $(2/tv)\arccos B[\mathbf{p}(0), \mathbf{p}(t)] > \sqrt{\mathcal{F}_*}$.

This standard method neglects decoherence, so we propose a protocol for open systems. The idea is to measure the system, with or without perturbation, for a known time, and use the distinguishability of the trajectories to lower bound the average speed of response via results 1 and 3. In two types of experimental runs, either the system evolves under the free dynamics, or one adds the perturbation $v\mathcal{V}$. In each case, the same measurement is performed at a known time $t$, giving statistics $p_i(t)$ and $q_i(t)$, respectively. First assuming the perturbation is exactly $\mathcal{P} = v\mathcal{V}$, the right-hand side of Eq. (3) is $(tv/2)$ times the time-averaged value of $\sqrt{\mathcal{F}(\sigma_s, \mathcal{V})}$, which quantifies the average speed of response to the perturbation. It follows that the resource must be present at some time $s \in [0, t]$ along the perturbed trajectory whenever

$$\frac{2\arccos B[\mathbf{p}(t), \mathbf{q}(t)]}{tv} > \sqrt{\mathcal{F}_*}, \tag{8}$$

as the threshold $\mathcal{F}_*$ is exceeded. In Sec. I of [51] we generalize this to a time-varying coefficient $v_t$. In weak coupling, the error $\Delta_{\mathrm{est}}(t)$ from Eq. (6) increases the threshold in Eq. (8) to $\sqrt{\mathcal{F}_*} + [2\Delta_{\mathrm{est}}(t)/vt]$. The change in this threshold is $\mathcal{O}(\sqrt{\epsilon v t}) + \mathcal{O}(\epsilon)$.

A demonstration for witnessing entanglement is shown in Fig. 2 for the two-qubit dephasing model described above, taking a Bell-basis measurement for the Bhattacharrya coefficient. For any two-qubit separable state $\rho_{\mathrm{sep}}$ with the chosen local $V$, we have $\mathcal{F}(\rho_{\mathrm{sep}}, \mathcal{V}) \leq \mathcal{F}_* = 2$ [26,27,29]. Here, this threshold is broken by the exact QFI for $t \lessapprox 1.41$, while entanglement is witnessed taking into account the error estimate for $t \lessapprox 1.26$.

*Quantum work fluctuations.*—Here, we show implications for driving a system out of equilibrium. Consider a system with Hamiltonian $H$ initially in thermal equilibrium at inverse temperature $\beta$, in the Gibbs state $\rho_{\mathrm{th}} = e^{-\beta H}/\mathrm{tr}e^{-\beta H}$. At time 0, $H$ is quickly changed to $H'$, involving fluctuating work $w$ done on the system. The mean and variance of work are computed from $\Delta H \coloneqq H' - H$: $\langle w \rangle = \mathrm{tr}[\rho_{\mathrm{th}}\Delta H]$, $\mathrm{Var}_w = \mathrm{Var}(\rho_{\mathrm{th}}, \Delta H)$. If the system is left to thermalize to the new Gibbs state $\rho'_{\mathrm{th}} = e^{-\beta H'}/\mathrm{tr}e^{-\beta H'}$, then its Helmholtz free energy $F_{H,\beta}$ decreases. This is defined by $F_{H,\beta} = \mathrm{tr}[\rho_{\mathrm{th}}H] - \beta^{-1}S(\rho_{\mathrm{th}})$, where $S(\rho_{\mathrm{th}}) = -\mathrm{tr}[\rho_{\mathrm{th}}\ln\rho_{\mathrm{th}}]$ is the von Neumann entropy [68]. The second law of thermodynamics implies that the change $\Delta F \coloneqq F_{H',\beta} - F_{H,\beta} \leq \langle w \rangle$—this is equivalent to saying that the dissipated work $W_{\mathrm{diss}} \coloneqq \langle w \rangle - \Delta F \geq 0$ [68]. $W_{\mathrm{diss}}$ is thus associated with nonequilibrium entropy production.

In order to study small deviations from equilibrium, we follow the paradigm of Refs. [69,70], where $\Delta H$ is assumed small. One finds a fluctuation-dissipation relation [70]

$$\frac{\beta}{2}\mathrm{Var}_w = W_{\mathrm{diss}} + Q_w. \tag{9}$$

Here, $Q_w \geq 0$ is a quantum correction to the usual classical relation [71,72], thus Eq. (9) represents a modification of a classical statistical law near equilibrium that takes into account quantum effects. It also limits coherent protocols that aim to simultaneously minimize work fluctuations and dissipation [70].

Equation (9) holds for various slow driving settings; in our case with a single small quench, $Q_w$ is determined by a quantity closely related to QFI:

$$Q_w = \frac{\beta}{2}\bar{I}(\rho_{\mathrm{th}}, \Delta H), \tag{10}$$

where $\bar{I}(\rho, A) \coloneqq \int_0^1 \mathrm{d}k \frac{1}{2}\mathrm{tr}([\rho^k, A][A, \rho^{1-k}])$. The details of this result [70] are recalled in Sec. IV of [51]. $\bar{I}$ belongs to a family of generalized QFI quantities [73] whose members are interpreted as measures of quantum coherence (also known as asymmetry in this context): $\bar{I}(\rho, H)$ and $\mathcal{F}(\rho, \mathcal{H})$, among others, quantify the coherence of a state $\rho$ with respect to a Hamiltonian $H$ [23,32,34,39,74]. Moreover, they can be regarded as quantum contributions to the variance of $H$ [75–77]. Some key properties justifying this interpretation are $\bar{I}(\rho, H) \leq \mathrm{Var}(\rho, H)$, with equality for pure states, and $\bar{I}(\rho, H) = 0$ when $\rho$ commutes with $H$. Therefore, as required of a measure of quantum work fluctuations, $2Q_w/\beta = \bar{I}(\rho_{\mathrm{th}}, \Delta H)$ vanishes exactly when $[H, \Delta H] = 0$.

*Result 4.*—Quantum work fluctuations are necessary for fast departure from equilibrium. For a system weakly coupled to a thermal environment, at all times $t > 0$ following the quench $H \to H'$, the distance between the initial state $\rho_{\mathrm{th}}$ and the system's state $\rho_t$ obeys

$$\theta_B(\rho_{\mathrm{th}}, \rho_t) \leq t\sqrt{3\bar{I}(\rho_{\mathrm{th}}, \Delta H)} + \Delta(t), \tag{11}$$

where $\Delta(t)$ is the weak coupling error from Eq. (7).

The proof is given in Sec. VI of [51]. Fast departure from $\rho_{\rm th}$ thus requires large quantum work fluctuations as measured by $\bar{I}(\rho_{\rm th}, \Delta H)$—equivalently, $\rho_{\rm th}$ must have a high degree of quantum coherence with respect to $\Delta H$.

The physical importance of the correction $\Delta(t)$ is seen in the "classical" case where $[H, \Delta H] = 0$ (i.e., the energy levels change but not the eigenstates). Then $\bar{I} = 0$, but the system must deviate from $\rho_{\rm th}$ in order to reach the new steady state $\rho'_{\rm th}$. From our earlier discussion of the weak coupling error, by identifying $v$ with $\|\Delta H\|$, we therefore see that the *quantum driving regime*—when the $\bar{I}$ terms dominates on the right-hand side of Eq. (11)—corresponds to $\sqrt{\bar{I}(\rho_{\rm th}, \Delta H)}/\|\Delta H\| \gg \max\{\sqrt{\epsilon \|\Delta H\| t}, \epsilon\}$. The left-hand side of this inequality measures quantum work fluctuations relative to the size of $\Delta H$. In the quantum driving regime, coherent evolution resulting from the changed Hamiltonian happens faster than thermalization.

*Linear response.*—Our results can also be applied to linear response theory, bounding the size of a response to a perturbation. Following, for example, Ref. [78], assume the system begins at $t = 0$ in a (possibly nonequilibrium) stationary state $\pi$, satisfying $\mathcal{L}(\pi) = 0$. The Hamiltonian is then perturbed by $v_t V$, giving the trajectory $\rho_t$. The change in mean of $A$, $\delta A_t \coloneqq \mathrm{tr}[A(\rho_t - \pi)]$, can be bounded from Eq. (3) by $|\delta A_t| \le \|A\| \int_0^t \mathrm{d}s \sqrt{\mathcal{F}(\rho_s, \mathcal{P}_s)}$ (Sec. VII of [51]). In a weak coupling setting, approximating the QFI to lowest order in $v_s$, we have $|\delta A_t| \lessapprox \|A\| \sqrt{\mathcal{F}(\pi, \mathcal{V})}[\int_0^t \mathrm{d}s |v_s| + \Delta(t)]$. Note the term $\Delta(t)$ from result 3 reflecting the change in the dissipator [79]. Alternatively, the observable speed limit Eq. (5) similarly implies $|\delta A_t| \lessapprox \sqrt{\mathrm{Var}(\pi, A)\mathcal{F}(\pi, \mathcal{V})}[\int_0^t \mathrm{d}s |v_s| + \Delta(t)]$. This requires the additional approximation that $D_{\rm tr}(\rho_t, \pi)$ is small for short times, but may give a tighter bound in replacing $||A||$ by the generally smaller $\sqrt{\mathrm{Var}(\pi, A)}$.

*Outlook.*—In summary, we have shown that the Mandelstam-Tamm speed limit can be extended to describe the response of an open system to a perturbation, with applications to quantum resource witnessing and thermodynamics. There are several possible future directions. First, note that we can derive a similar speed limit replacing the Bures angle by the quantity $\tilde{\theta}(\rho, \sigma) = \arccos \mathrm{tr}(\sqrt{\rho}\sqrt{\sigma})$ and the QFI by (four times) the Wigner-Yanase skew information [80] $I^{\rm WY}(\rho, \mathcal{V}) = -\frac{1}{2}\mathrm{tr}([\sqrt{\rho}, V]^2)$—is this possible for generalized QFI quantities [73]? Second, for weak coupling, it would be interesting to consider slowly varying perturbations with adiabatic master equations [81], and implications for thermodynamic uncertainty relations which relate current fluctuations to entropy production [82]. Additionally, since our speed limit holds under Markovian dynamics, a violation might be used as a witness of non-Markovianity. This would add to a library of existing witnesses, including those based on monotonic decrease of QFI [50,69]. A range of other interesting applications involves studying the rate at which a perturbation can generate resources, generalizing past approaches by allowing for background decoherence. This covers charging quantum batteries [21,83], where one could include thermalization during charging, and the production of quantum resources such as coherence and entanglement [84].

We thank Florian Fröwis, George Knee, and Bill Munro for discussions that inspired this work, as well as Stefan Nimmrichter for further input. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 945422, the DAAD, the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation, Projects No. 447948357 and No. 440958198), the Sino-German Center for Research Promotion (Project No. M-0294), the ERC (Consolidator Grant No. 683107/TempoQ), and the German Ministry of Education and Research (Project QuKuK, BMBF Grant No. 16KIS1618K).

[1] L. Mandelstam and I. Tamm, The uncertainty relation between energy and time in non-relativistic quantum mechanics, in *Selected Papers* (Springer, Berlin, Heidelberg, 1991), 10.1007/978-3-642-74626-0_8.

[2] N. Margolus and L. B. Levitin, The maximum speed of dynamical evolution, Physica (Amsterdam) **120D**, 188 (1998).

[3] P. J. Jones and P. Kok, Geometric derivation of the quantum speed limit, Phys. Rev. A **82**, 022107 (2010).

[4] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Quantum speed limits in open system dynamics, Phys. Rev. Lett. **110**, 050403 (2013).

[5] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, Generalized geometric quantum speed limits, Phys. Rev. X **6**, 021031 (2016).

[6] S. Deffner and E. Lutz, Quantum speed limit for non-Markovian dynamics, Phys. Rev. Lett. **111**, 010402 (2013).

[7] I. Marvian, R. W. Spekkens, and P. Zanardi, Quantum speed limits, coherence, and asymmetry, Phys. Rev. A **93**, 052331 (2016).

[8] A. Uhlmann, An energy dispersion estimate, Phys. Lett. A **161**, 329 (1992).

[9] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, Phys. Rev. Lett. **72**, 3439 (1994).

[10] F. Fröwis, Kind of entanglement that speeds up quantum evolution, Phys. Rev. A **85**, 052127 (2012).

[11] H. J. Bremermann, Quantum noise and information, *Proc. Fifth Berkeley Symposium Math. Stat. Probab.* 222, 15 (1965).

[12] S. Lloyd, Ultimate physical limits to computation, Nature (London) **406**, 1047 (2000).

[13] S. Deffner, Quantum speed limits and the maximal rate of information production, Phys. Rev. Res. **2**, 013161 (2020).

[14] J. D. Bekenstein, Generalized second law of thermodynamics in black-hole physics, Phys. Rev. D **9**, 3292 (1974).

[15] S. Deffner and E. Lutz, Generalized clausius inequality for nonequilibrium quantum processes, Phys. Rev. Lett. **105**, 170402 (2010).

[16] O. Abah and E. Lutz, Energy efficient quantum machines, Europhys. Lett. **118**, 40005 (2017).

[17] V. Mukherjee, W. Niedenzu, A. G. Kofman, and G. Kurizki, Speed and efficiency limits of multilevel incoherent heat engines, Phys. Rev. E **94**, 062109 (2016).

[18] F. C. Binder, S. Vinjanampathy, K. Modi, and J. Goold, Quantacell: Powerful charging of quantum batteries, New J. Phys. **17**, 075015 (2015).

[19] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the charging power of quantum batteries, Phys. Rev. Lett. **118**, 150601 (2017).

[20] B. Mohan and A. K. Pati, Quantum speed limits for observables, Phys. Rev. A **106**, 042436 (2022).

[21] S. Julià-Farré, T. Salamon, A. Riera, M. N. Bera, and M. Lewenstein, Bounds on the capacity and power of quantum batteries, Phys. Rev. Res. **2**, 023113 (2020).

[22] M. Gessner and A. Smerzi, Statistical speed of quantum states: Generalized quantum Fisher information and Schatten speed, Phys. Rev. A **97**, 022109 (2018).

[23] C. Zhang, B. Yadin, Z.-B. Hou, H. Cao, B.-H. Liu, Y.-F. Huang, R. Maity, V. Vedral, C.-F. Li, G.-C. Guo, and D. Girolami, Detecting metrologically useful asymmetry and entanglement by a few local measurements, Phys. Rev. A **96**, 042327 (2017).

[24] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum. Inform. **07**, 125 (2009).

[25] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, Rev. Mod. Phys. **80**, 517 (2008).

[26] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Fisher information and multiparticle entanglement, Phys. Rev. A **85**, 022321 (2012).

[27] G. Tóth, Multipartite entanglement and high-precision metrology, Phys. Rev. A **85**, 022322 (2012).

[28] B. Morris, B. Yadin, M. Fadel, T. Zibold, P. Treutlein, and G. Adesso, Entanglement between identical particles is a useful and consistent resource, Phys. Rev. X **10**, 041012 (2020).

[29] L. Pezzé and A. Smerzi, Entanglement, nonlinear dynamics, and the Heisenberg limit, Phys. Rev. Lett. **102**, 100401 (2009).

[30] L. Pezzè, Y. Li, W. Li, and A. Smerzi, Witnessing entanglement without entanglement witness operators, Proc. Natl. Acad. Sci. U.S.A. **113**, 11459 (2016).

[31] B. Yadin, M. Fadel, and M. Gessner, Metrological complementarity reveals the Einstein-Podolsky-Rosen paradox, Nat. Commun. **12**, 2410 (2021).

[32] B. Yadin and V. Vedral, General framework for quantum macroscopicity in terms of coherence, Phys. Rev. A **93**, 022122 (2016).

[33] I. Marvian, Coherence distillation machines are impossible in quantum thermodynamics, Nat. Commun. **11**, 25 (2020).

[34] I. Marvian, Operational interpretation of quantum Fisher information in quantum thermodynamics, Phys. Rev. Lett. **129**, 190502 (2022).

[35] F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, Macroscopic quantum states: Measures, fragility, and implementations, Rev. Mod. Phys. **90**, 025004 (2018).

[36] B. Yadin, F. C. Binder, J. Thompson, V. Narasimhachar, M. Gu, and M. S. Kim, Operational resource theory of continuous-variable nonclassicality, Phys. Rev. X **8**, 041038 (2018).

[37] H. Kwon, K. C. Tan, T. Volkoff, and H. Jeong, Nonclassicality as a quantifiable resource for quantum metrology, Phys. Rev. Lett. **122**, 040503 (2019).

[38] K. C. Tan, V. Narasimhachar, and B. Regula, Fisher information universally identifies quantum resources, Phys. Rev. Lett. **127**, 200402 (2021).

[39] D. Girolami, Observable measure of quantum coherence in finite dimensional systems, Phys. Rev. Lett. **113**, 170401 (2014).

[40] A. Rath, C. Branciard, A. Minguzzi, and B. Vermersch, Quantum Fisher information from randomized measurements, Phys. Rev. Lett. **127**, 260501 (2021).

[41] H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezze, A. Smerzi, and M. K. Oberthaler, Fisher information and entanglement of non-Gaussian spin states, Science **345**, 424 (2014).

[42] F. Fröwis, Lower bounds on the size of general Schrödinger-cat states from experimental data, J. Phys. A **50**, 114003 (2017).

[43] S. Becker, N. Datta, L. Lami, and C. Rouzé, Energy-constrained discrimination of unitaries, quantum speed limits, and a Gaussian Solovay-Kitaev theorem, Phys. Rev. Lett. **126**, 190504 (2021).

[44] K. Suzuki and K. Takahashi, Performance evaluation of adiabatic quantum computation via quantum speed limits and possible applications to many-body systems, Phys. Rev. Res. **2**, 032016(R) (2020).

[45] T. Hatomura, Performance evaluation of invariant-based inverse engineering by quantum speed limits, Phys. Rev. A **106**, L040401 (2022).

[46] V. Pandey, S. Bhowmick, B. Mohan, Sohail, and U. Sen, Fundamental speed limits on entanglement dynamics of bipartite quantum systems, arXiv:2303.07415.

[47] F. Albarelli and R. Demkowicz-Dobrzański, Probe incompatibility in multiparameter noisy quantum metrology, Phys. Rev. X **12**, 011039 (2022).

[48] V. Gorini, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. (N.Y.) **17**, 821 (1976).

[49] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[50] Á. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: Characterization, quantification and detection, Rep. Prog. Phys. **77**, 094001 (2014).

[51] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.230404, which inludes Refs. [52–58], for additional details and proofs.

[52] M. Hübner, Explicit computation of the Bures distance for density matrices, Phys. Lett. **163A**, 239 (1992).

[53] R. Augusiak, J. Kołodyński, A. Streltsov, M. N. Bera, A. Acín, and M. Lewenstein, Asymptotic role of entanglement in quantum metrology, Phys. Rev. A **94**, 012339 (2016).

[54] R. A. Horn and C. R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, England, 1985), Vol. 72, pp. 692–692.

[55] B. Buča and T. Prosen, A note on symmetry reductions of the Lindblad equation: Transport in constrained open spin chains, New J. Phys. **14**, 073007 (2012).

[56] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1995), 10.1017/9781108587280.

[57] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Theory of decoherence-free fault-tolerant universal quantum computation, Phys. Rev. A **63**, 042307 (2001).

[58] R. Kubo, Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, J. Phys. Soc. Jpn. **12**, 570 (1957).

[59] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010), Chap. 9.

[60] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum speed limit for physical processes, Phys. Rev. Lett. **110**, 050402 (2013).

[61] L. P. García-Pintos, S. B. Nicholson, J. R. Green, A. del Campo, and A. V. Gorshkov, Unifying quantum and classical speed limits on observables, Phys. Rev. X **12**, 011038 (2022).

[62] D. A. Lidar, Lecture notes on the theory of open quantum systems, arXiv:1902.00967.

[63] F. Ciccarello, S. Lorenzo, V. Giovannetti, and G. M. Palma, Quantum collision models: Open system dynamics from repeated interactions, Phys. Rep. **954**, 1 (2022).

[64] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Quantum dynamics of single trapped ions, Rev. Mod. Phys. **75**, 281 (2003).

[65] Á. Rivas, A. D. K Plato, S. F. Huelga, and M. B Plenio, Markovian master equations: A critical study, New J. Phys. **12**, 113032 (2010).

[66] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2007).

[67] A. Bhattacharyya, On a measure of divergence between two statistical populations defined by their probability distribution, Bull. Calcutta Math. Soc. **35**, 99 (1943).

[68] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics–a topical review, J. Phys. A **49**, 143001 (2016).

[69] M. Scandi and M. Perarnau-Llobet, Thermodynamic length in open quantum systems, Quantum **3**, 197 (2019).

[70] H. J. D. Miller, M. Scandi, J. Anders, and M. Perarnau-Llobet, Work fluctuations in slow processes: Quantum signatures and optimal control, Phys. Rev. Lett. **123**, 230603 (2019).

[71] J. Hermans, Simple analysis of noise and hysteresis in (slow-growth) free energy simulations, J. Phys. Chem. **95**, 9029 (1991).

[72] C. Jarzynski, Nonequilibrium equality for free energy differences, Phys. Rev. Lett. **78**, 2690 (1997).

[73] D. Petz, Covariance and Fisher information in quantum mechanics, J. Phys. A **35**, 929 (2002).

[74] H. J. D. Miller and J. Anders, Energy-temperature uncertainty relation in quantum thermodynamics, Nat. Commun. **9**, 2203 (2018).

[75] S. L. Luo, Quantum versus classical uncertainty, Theor. Math. Phys. **143**, 681 (2005).

[76] P. Gibilisco, D. Imparato, and T. Isola, Inequalities for quantum Fisher information, Proc. Am. Math. Soc. **137**, 317 (2009).

[77] I. Frérot and T. Roscilde, Quantum variance: A measure of quantum coherence and quantum correlations for many-body systems, Phys. Rev. B **94**, 075121 (2016).

[78] M. Konopik and E. Lutz, Quantum response theory for nonequilibrium steady states, Phys. Rev. Res. **1**, 033156 (2019).

[79] A. Levy, E. Rabani, and D. T. Limmer, Response theory for nonequilibrium steady states of open quantum systems, Phys. Rev. Res. **3**, 023252 (2021).

[80] P. Gibilisco and T. Isola, Wigner–Yanase information on quantum state space: The geometric approach, J. Math. Phys. (N.Y.) **44**, 3752 (2003).

[81] T. Albash, S. Boixo, D. A. Lidar, and P. Zanardi, Quantum adiabatic Markovian master equations, New J. Phys. **14**, 123016 (2012).

[82] J. M. Horowitz and T. R. Gingrich, Thermodynamic uncertainty relations constrain non-equilibrium fluctuations, Nat. Phys. **16**, 15 (2020).

[83] J.-Y. Gyhm, D. Šafránek, and D. Rosa, Quantum charging advantage cannot be extensive without global operations, Phys. Rev. Lett. **128**, 140501 (2022).

[84] B. Mohan, S. Das, and A. K. Pati, Quantum speed limits for information and coherence, New J. Phys. **24**, 065003 (2022).