Microscopic Intervention Yields Abrupt Transition in Interdependent Ferromagnetic Networks

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The study of interdependent networks has recently experienced a boost with the development of experimentally testable materials that physically realize their novel critical behaviors, calling for systematic studies that go beyond the percolation paradigm. Here we study the critical kinetics and phase transitions of a model of interdependent spatial ferromagnetic networks where dependency couplings between networks are realized by a thermal interaction having a tunable spatial range. We show how the critical phenomena and the phase diagram of this realistic model are highly affected by the range of thermal dissipation and how the latter influences the microscopic kinetics of the model. Furthermore, we show the existence of a new phase where localized microscopic interventions by heating or magnetic fields yield a macroscopic phase transition. Our results unveil rich phenomena and realistic protocols for controlling the macroscopic phases of interdependent materials by means of microscopic interventions.

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For over a decade, interdependent networks [1,2] have provided a versatile theoretical framework to study a variety of complex systems, from man-made infrastructures [3,4] to biological [5,6], physiological [7], or ecological systems [8–10]. Dependency couplings, in particular, have become a paradigmatic abstraction to model functional interactions in multilayer systems [11] and to study the cascade of local malfunctions across scales. Percolation theory [12,13], in this regard, has provided a variety of tools to understand the spreading of cascading failures in interdependent networks, producing a large volume of theoretical predictions [14–24].

The recent development of experimentally testable materials [25] that physically realize interdependent networks, calls for systematic studies to go beyond the percolation paradigm [26,27] in order to identify and control further interdependent physical systems and study their critical behaviors and underlying kinetics. Here we introduce a model of spatial interdependent ferromagnetic networks where two 2D Ising spin lattices interact via thermal dependency links with a tunable interaction range. We characterize the collective phases of the model, its phase transitions at varying interaction ranges, and study the macroscopic effects induced by microscopic interventions. In particular, we demonstrate how to trigger specific critical cascading dynamics that yield an abrupt transition via microscopic localized heating and localized magnetic fields and characterize a phase diagram as a function of the spatial range of the thermal coupling. Altogether, our results generalize localized interventions in interdependent ferromagnetic networks and provide useful protocols to control the macroscopic phases of interdependent materials via microscopic interventions.

Model.—We consider a system composed of two 2D ferromagnetic lattices of size $N = L^2$ under the influence of a common heat bath of temperature $T = 1/\beta$. Each node models a ferromagnetic grain which is endowed with an Ising spin $\sigma = \pm 1$ so that the configuration of spins in network μ at time t is $\boldsymbol{\sigma}_\mu(t) = \{\sigma_i^\mu(t)\}_{i\leq N}$, see Fig. 1. The magnetic states of the two lattices are made mutually interdependent by turning on local thermal couplings whose strength is inversely proportional to the local ordering of spins (see Discussion). Hence, locally aligned (ordered) spins in one network create weak thermal couplings on their dependent nodes in the other network, while locally paramagnetic (disordered) neighborhoods of a spin create strong couplings. We control the range of the thermal coupling—whose value will generally depend on the thermoconductive material used to fabricate the multilayer system—by a tuneable range r (see Fig. 1). This implies that each node in one layer is thermally coupled with all the nodes in the other layer up to a distance r (in contrast to the "one-to-one" dependency in the percolation paradigm, see SM [28]) and influenced by their local order. In this case, the dependency of node i in network μ on its interdependent nodes in network μ' (and vice versa) is reflected by the change in local temperature as

$$\beta_i^\mu = \beta\Sigma_i^{\mu'}(r), \qquad \Sigma_i^\mu(r) = \frac{1}{|K_i(r)|}\sum_{|i-j|\leq r}\sum_{j\leq N} A_{ij}^\mu \sigma_j^\mu. \qquad (1)$$

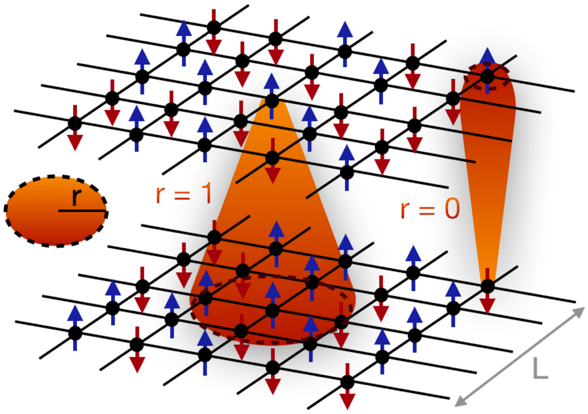

FIG. 1. Illustration of the model. Two 2D ferromagnetic layers (lattices) of size $N = L^2$ are interdependent on each other via thermal couplings. The magnetic state of each network μ is described by its spin configuration $\boldsymbol{\sigma}_\mu = \{\sigma_1^\mu, \ldots, \sigma_N^\mu\}$ where each node of the network is an Ising spin pointing up (blue arrow) or down (red arrow) having $\sigma_i^\mu = \pm 1$. Interdependence is realized via a thermal coupling where each node in network μ affects all nodes up to a distance r in the other network μ' and vice versa. In simulations, we adopt periodic boundary conditions to avoid boundary effects.

Here, $\beta_i^\mu = 1/T_i^\mu$ is the inverse local temperature around spin i in layer μ in Boltzmann units, $\Sigma_i^\mu(r)$ is the average magnetization of nodes within a distance r from node i in network μ, A_{ij}^μ is the adjacency matrix of network μ, and $K_i(r)$ is the set of all nodes up to a distance r from node i. Thus, while a ferromagnetic neighborhood of a spin ($\Sigma_i^{\mu'}(r) \simeq 1$) yields $\beta_i^\mu \simeq \beta$, i.e., the local temperature is weakly affected by the local ordering, paramagnetic neighborhoods [i.e., $\Sigma_i^{\mu'}(r) \to 0$] yield instead the local temperatures increase $\beta_i^\mu \to 0$, inducing a strong overheating effect. We show below that the dependency interaction range, r, plays a critical role and controls a magnetothermal runaway triggering a self-amplifying (degaussing) propagation of spin flips. For short-range dependencies, i.e., small r, the effect of thermal fluctuations remains local and the ferroparamagnetic transition remains continuous. On the other hand, long-range thermal dependencies (sufficiently large r) strongly amplify local thermal fluctuations, resulting in spontaneous first-order transitions and hysteretic behaviors.

Global kinetics and transitions.—We model the temporal evolution of the magnetic state of the system initiated from a given initial condition sigma $\boldsymbol{\sigma}_\mu(0)$ via two thermally coupled Glauber kinetics [29]. Given the thermal dependency between layers, we flip a randomly chosen spin σ_i^μ in network μ with probability [27]

$$\omega_i^\mu(\sigma_i^\mu) = \left(1 + \exp\left(2J\beta\sigma_i^\mu\Sigma_i^{\mu'}(r)\sum_{j\leq N} A_{ij}^\mu \sigma_j^\mu\right)\right)^{-1} \quad (2)$$

and continue the process until equilibrium. Here, a single Monte Carlo step (MCS) corresponds to $2N$ attempts to flip randomly chosen spins in the two networks and the number of MCSs represents the time t when a measurement of the magnetization state is performed. This way, the magnetic evolution of the system from the initial conditions $\boldsymbol{\sigma}_\mu(0)$ can be tracked by measuring the instantaneous average magnetization of network μ, $M_\mu(t) = (1/N)\sum_i \sigma_i^\mu(t)$ at time t using the instantaneous spin configuration $\boldsymbol{\sigma}_\mu(t)$ after t Monte Carlo steps.

Next, we measure the steady-state magnetization as a function of temperature for different dependency ranges r both for mutually ordered initial conditions ($\sigma_i = +1$ for all spins in both networks) and for disordered ones ($\sigma_i = \pm 1$ randomly), see Fig. 2(a). Interestingly, we find a critical dependency range, $r_c \simeq 2$, below which the transition is continuous but smeared out, with a transition, from the ferromagnetic to the paramagnetic phase, due to the coupling between the network, at temperatures much lower than the classical Onsager's threshold, $T_c \simeq 2.7$, known for the isolated case [30–34] [see Fig. 2(a) and SM for the different behavior of the percolation paradigm in the $r = 0$ limit].

When increasing the value of r above r_c, the phase transition becomes immediately abrupt, the smearing-out character of the weak ferromagnetic branch disappears, and hysteretic behaviors come into play (in contrast to the one-sided transition observed in the percolation paradigm, see SM). This is apparent when measuring the critical temperatures, i.e., $T_{c,>}(r)$ (heating direction) and $T_{c,<}(r)$ (cooling direction) from the ordered phase ($M \simeq 1$) and disordered phase ($M \simeq 0$) respectively, see Fig. 2(b). While the smeared out continuous transition at $r < r_c$ shows no sign of hysteresis, i.e., $T_{c,<}(r) = T_{c,>}(r)$, the abrupt transitions reported at $r > r_c$ feature clear hysteresis behaviors, where $T_{c,>}(r) > T_{c,<}(r)$.

Since r_c is the lowest dependency range for which nucleation transition is observed [17] its value can be obtained analytically. The nucleation condition for the propagation of spontaneously disordered droplets is $\xi_{\text{down}}[T_c(r_c)] = r_c$ where $\xi_{\text{down}}(T)$ is the correlation length of a single network representing the linear size of the finite clusters formed by down-oriented spins (see SM). Figure 2(c) shows the condition satisfied with $r_c \simeq 2$ which is lower than its percolation analog ($r_c \simeq 8$) due to the different characteristics of the correlation lengths in both systems (see SM).

Transition types.—For $r > r_c$, the ferroparamagnetic phase transitions found in the coupled system are always of the first-order type. However, the character of these abrupt transitions depends on the interaction range, r. For values of r slightly above r_c, a thermal fluctuation in the local magnetization spontaneously nucleates and drives the ferromagnetic phase into the paramagnetic one. In this case, a disordered droplet is created and propagates radially in

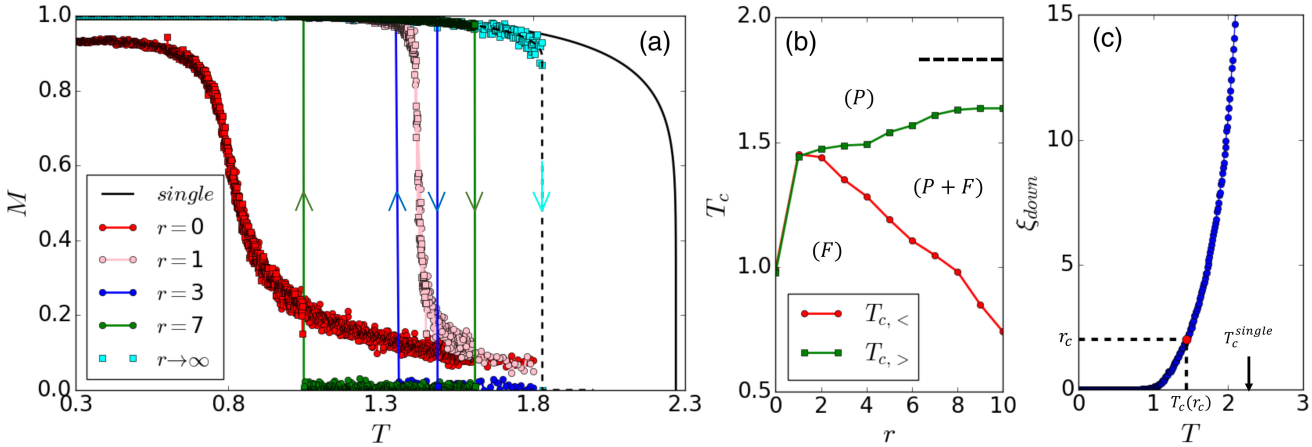

FIG. 2. Interdependent ferromagnetic phase transitions. (a) Magnetization M as a function of temperature T is shown for different values of the dependency interaction range r. A smeared-out continuous transition is observed at short interaction ranges, $r < r_c \simeq 2$, while abrupt transitions take place for $r > r_c$. The analytical solution for the long-range limit $r \to \infty$ in Eq. (3) is shown in a black dashed line and the classic Onsager solution for a single 2D lattice is shown in a continuous black line for comparison. (b) The critical temperatures, $T_{c,>}(r)$, when increasing temperature (heating) from the ordered state ($M \simeq 1$) to the disordered state ($M \simeq 0$), and $T_{c,<}(r)$ when decreasing temperature (cooling) from the disordered state to the ordered state, are the same in the continuous transition regime $r < r_c$ but are different for the abrupt regime $r > r_c$ and show hysteresis. In the long-range limit, $T_c \simeq 1.831$ is shown in a black dashed line. (c) The correlation length $\xi_{\rm down}$ of a single network is measured and the nucleation condition $\xi_{\rm down}[T_c(r_c)] = r_c$ is satisfied for $r_c \simeq 2$. Here $J = 1$, $L = 200$, and the values of M are measured after 10^4 MCSs.

both layers, see Figs. 3(a)–3(c). By measuring the nonequilibrium magnetization as a function of time during this transition, we find a parabolic shape [Figs. 3(a) and 3(b)] indicating the classical scenario expected in homogeneous nucleation theory [35]. This is further manifested in the linear increase of the droplet radius with time $R \sim t$ [Fig. 3(c)] producing the scaling of the droplet mass $M_d = \pi R^2 \sim t^2$. In contrast to the abrupt nucleation transitions at low values of r, in the other extreme case characterized by $r \gg r_c$ (in practise, $r \propto L$), fluctuations have a systemic effect and influence all scales. In this case, the first-order transition is of mixed-order type [36–38] and the nonequilibrium behavior of the magnetization at the critical point develops a long-lived plateau relaxation to the paramagnetic phase analogous to the one found in interdependent percolation or the k core [39,40,44]. During this plateau, the magnetization fluctuates around a nearly constant value of the magnetization for relatively long times before rapidly converging into the disordered phase [see Figs. 3(d)–3(f)], showing resemblances with quasistationary states observed in nonequilibrium long-range systems [41,42].

Interestingly, the steady-state magnetization can be derived analytically in the long-range limit, $r \to \infty$. By including the thermal coupling between two layers, $\Sigma_i^\mu(r \to \infty) = M_\mu$ given in Eq. (1) in the Onsager-Kauffmann-Yang (OKY) exact solution for the spontaneous magnetization of a single layer 2D square lattice Ising model [30–33], a single self-consistent equation can be obtained (see SM for details) in the limit of symmetric (i.e., having the same ferromagnetic coupling strengths $J_A = J_B \equiv J$) Ising 2D lattices:

$$M_\mu(T) = (1 - \sinh^{-4}(2\beta J M_\mu(T)))^{1/8}. \qquad (3)$$

Figure 2 shows excellent agreement between the analytical solution, Eq. (3), and simulations for $r \to \infty$, both experiencing a spontaneous first-order ferroparamagnetic transition at the critical temperature $T_{c,>}/J \simeq 1.831$. The critical exponent $\beta = 1/2$, just above $T_{c,>}$ can be analytically validated by expanding Eq. (3) around the critical point, corroborating the mixed-order nature of the transition (details in SM).

Macroscopic transition due to microscopic intervention.— Interdependent percolation on spatial networks undergoes macroscopic transitions induced by microscopic interventions. Strategies such as localized attacks, for example, have been shown to produce novel metastable phases where, e.g., planting a few microscopic droplets nucleates the system towards its complete dismantling [23,43,45]. Thus, we focus here on realistic protocols that will induce in our physical system a desired cascade kinetics via localized microscopic interventions. In particular, we investigate the effects of localized heating and refer the interested reader to the SM for similar protocols exploiting microscopic magnetic fields and the different characteristics of the phase diagram of localized attack in the percolation paradigm.

Microscopic interventions can be performed via perturbation of physical quantities, like temperature, within a circle having a finite radius r_h in one of the layers of the system.

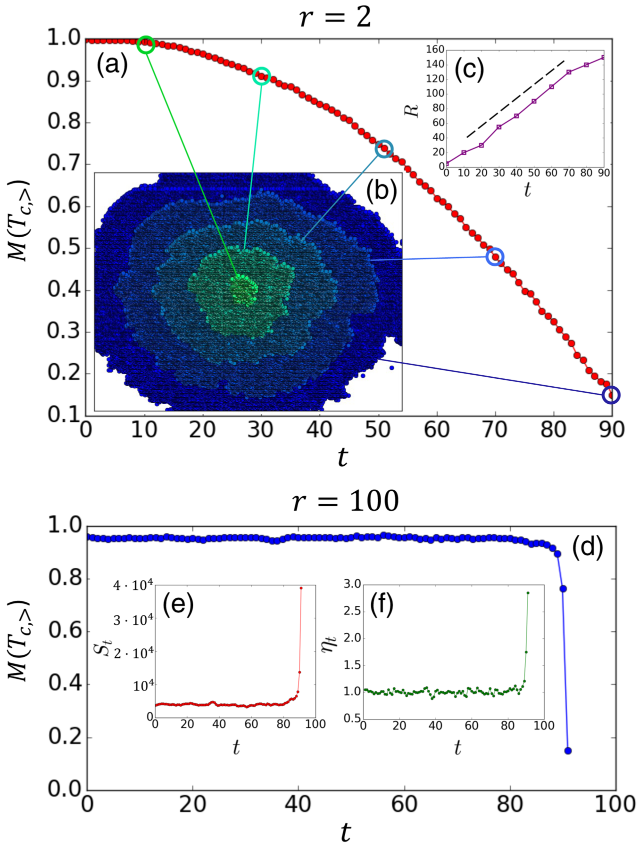

FIG. 3. Critical dynamics at $T_{c,>}$. (a) For low values of r above r_c, nucleation transition is observed with a parabolic shape decrease of the magnetization associated with (b) a circular area of disordered spins that spontaneously appears at $T_{c,>}$ and increases radially in time due to the dependency heat interactions between the layers. (c) The radius of the circle R increases linearly with time. (d) For large values of r, such as $r = 100$ shown here, a plateau is observed where the magnetization remains nearly constant for a long time. At the end of the plateau, the system converges to the disordered phase exponentially fast. (e) The number of flipped spins as a function of time, S_t, is constant during the plateau showing (f) a critical branching factor, $\eta_t = S_t/S_{t-1} \simeq 1$. The analogy to percolation of abstract interdependent networks can be seen in Berezin *et al.* [43] and Zhou *et al.* [44].

For localized heating, the local temperatures in Eq. (1) are higher for spins within the heated circle. While spins outside the circle are not affected and experience the system temperature $T = 1/\beta$, spins within the circle also experience an overheating term ΔT, with the total bath temperature $T + \Delta T = 1/\beta_\Delta$. Thus, according to Eq. (1) the inverse temperature β_i^μ felt by spin σ_i^μ can be summarized as

$$\beta_i^\mu = \begin{cases} \beta \Sigma_i^{\mu'}(r), & \text{if } d_i > r_h \\ \beta_\Delta \Sigma_i^{\mu'}(r), & \text{if } d_i \leq r_h \end{cases} \tag{4}$$

where d_i is the distance of spin σ_i from the center of the heated circle. Since the inverse local temperature, β_i^μ, influences the flipping probability of spin i, Eq. (2), spins falling within the heated circle are likely to be more disordered. We find that for a given ΔT and the fitting parameters of T and r, a finite critical radius r_h^c exists where for $r_h < r_h^c$ the disorder will not spread and the system will remain in the ordered phase while for $r_h > r_h^c$ a disordered droplet forms and the system will experience a nucleation transition induced by homogeneous nucleation kinetics into the disordered state, as demonstrated in Figs. 4(a)–4(f). We stress that r_h^c does not depend on L, i.e., it constitutes a vanishing fraction of controlled spins over the whole network and, therefore, can be regarded as a microscopic intervention (see SM for finite size analysis). Thus, a 3D phase diagram of $r_h^c(T, r, \Delta T)$ can be constructed and slices of 2D phase diagrams $r_h^c(T, r)$ can be analyzed for different heating intensities ΔT shown in Figs. 4(g)–4(j). The null case of no localized heating, i.e., $\Delta T = 0$ shown in Fig. 4(g), displays only two phases, the ordered phase for $T < T_{c,>}(r)$ and the disordered phase for $T > T_{c,>}(r)$. The border between these phases is exactly the transition line of the spontaneous transition without intervention from the heating direction $T_{c,>}(r)$ as shown in Fig. 2(b). However, when the system is locally heated, i.e., $\Delta T > 0$, a new metastable regime appears where localized microscopic heating of a radius larger than r_h^c propagates and induces a macroscopic transition. Figures 4(h)–4(j) show how the metastable regime expands toward the ordered phase as the circular heating intensity increases. The metastable regime characterizes the conditions for controlling the system state by microscopic interventions and opens an avenue toward microscopic controllable interdependent materials.

Discussion.—The recent physical realization of interdependent superconductors [25] opens new frontiers and challenges toward the development of microscopically controllable complex materials. In this Letter, we have studied, theoretically, a system of two thermally coupled Ising 2D lattices which provides a realistic benchmark to experimentally control the conditions yielding macroscopic transitions and metastable phases in interdependent ferromagnetic networks via microscopic interventions. Experimental realizations of the latter could be attained, for example, by interpreting the thermal coupling between layers in our spin model as the result of local Joule dissipation effects due to the electron scattering resistance, e.g., in networks of Ni granular ferromagnets [46]. In this light, the interdependent interaction paradigm provides a powerful yet unfamiliar framework and prototype for the study of other out-of-equilibrium cascade kinetics, like underlying giant-magnetoresistance phenomena [47,48] and other thermoelectric runaway processes [49], opening exciting venues for potential future technologies. Our results are not only applicable for experiments of coupled magnetic systems but also provide a framework for microscopic localized interventions in other spin network modeling of real-world networks including the impact of

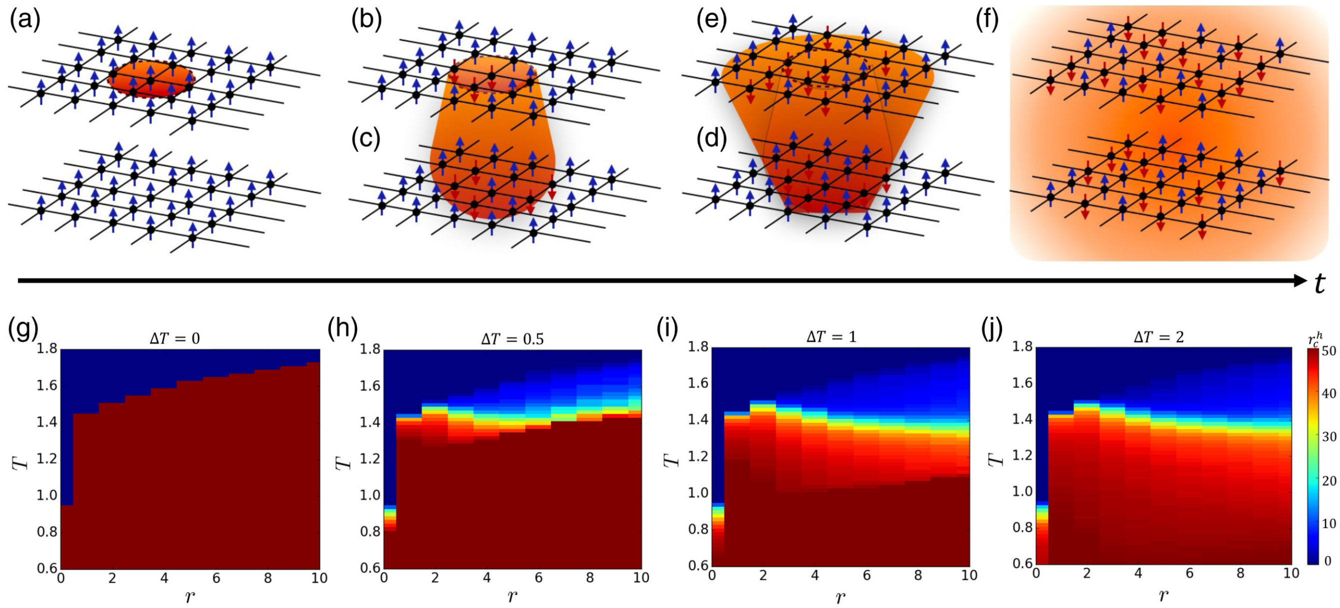

FIG. 4. Localized heating. (a) The upper layer is heated locally within a microscopic radius r_h, to temperature $T + \Delta T$. (b) This heating creates a disordered droplet which starts to dissipate heat to the bottom layer. (c) The localized regime in the bottom layer is heated up and becomes disordered as well. (d) The disordered regime in the bottom layer starts to dissipate heat back to the top layer broadening the circle of disorder. (e) The disordered droplet in the top layer extends due to the dissipation from the bottom layer. (f) This nucleation process continues until the disordered droplet takes over the system. Phase diagrams. While for low values of r, but above r_c, a spontaneous nucleation transition is observed at $T_{c,>}$, an induced nucleation macroscopic phase transition is seen for $T < T_{c,>}$ by an external microscopically localized heating. The phase diagrams of the critical heating radius r_c^h for different heating intensity ΔT are shown. (g) For $\Delta T = 0$ there are no induced nucleation transitions and only two phases separated by $T_{c,>}(r)$ appear, ordered phase (brown) and disordered phase (blue), see Fig. 2(b). (h)–(j) Once the system is locally heated, a metastable regime appears where localized heating with a radius above a critical r_c^h will induce a nucleation transition. The metastable regime expands with the localized heat intensity ΔT towards lower temperatures, shrinking the ordered phase. Here $J = 1$ and $L = 100$.

specific drug targets in a biological network [6,50] or targeted opinion shifting in social networks [51,52].

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