Universality Classes of Relativistic Fluid Dynamics: Foundations

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A general organizing principle is proposed that can be used to derive the equations of motion describing the near-equilibrium dynamics of causal and thermodynamically stable relativistic systems. The latter are found to display some new type of universal behavior near equilibrium that allows them to be grouped into universality classes defined by their degrees of freedom, information content, and conservation laws. The universality classes expose a number of surprising equivalences between different theories, shedding new light on the near-equilibrium behavior of relativistic systems.

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Introduction.—The construction of a relativistic hydrodynamic theory usually relies on two choices. First, one must choose the degrees of freedom that describe the hydrodynamic state. Then, a guiding principle is used to derive the dynamical equations for those fields. Such guiding principle may be, e.g., a thermodynamic principle [1–5], a variational principle [6–9], kinetic theory [10], holography [11,12], or effective theory arguments [13–18]. The level of freedom involved in these two choices is immense [19], leading to a plethora of alternative theories, which are constantly being added to the "relativistic hydrodynamics landscape" [1,10,12–16,18,20–36]. Nevertheless, it is known that some theories can become equivalent in certain regimes. For instance, second-order theories for viscous hydrodynamics arising from very different choices of both fields and dynamical equations [10,12,37–41] are mathematically equivalent near equilibrium [42–45]. Equivalency here means that, although those theories can be different in the nonlinear regime, they become indistinguishable when linearized around homogeneous equilibrium states. Demonstrating such equivalency sometimes requires making a complicated change of variables so that, almost "magically" (see, e.g., [45]), one linearized theory is transformed into the other. The fact that such transformations are possible near equilibrium cannot be some fortuitous coincidence. Rather, this should follow as a consequence of the underlying properties of equilibrium states in relativity.

Here, we present a general organizing principle that can be used to derive the equations of motion (EOM) describing the near-equilibrium dynamics of any causal and thermodynamically stable relativistic system. The method is based on a new result, rigorously proven here, which establishes that the EOM describing the system's linearized disturbances can always be obtained from a four-vector field E^μ known as the "information current" [46], which is a quadratic function of the perturbations (e.g., temperature variations). Assuming an isotropic and homogeneous equilibrium state, we show how the information current can be systematically constructed for arbitrary theories in terms of the corresponding linear-order perturbation fields, grouped according to their transformation properties under the $SO(3)$ rotation group. Entropy production follows from $\partial_\mu E^\mu \leq 0$, which describes the fact that our initial information about the microstate of the system [47] is erased as all microstates evolve towards the equilibrium state. Furthermore, we prove that, no matter how complicated a hydrodynamic theory is, one can always rearrange its equations of motion so that they resemble Israel-Stewart theory [1] near equilibrium. This is used to reveal that causal and thermodynamically stable theories of relativistic fluid dynamics possess a new type of universal behavior near equilibrium that allows them to be grouped into universality classes. Each class describes a physically different behavior, defined solely by the degrees of freedom, the corresponding information current, and the conservation laws. The existence of such universality classes unveils a number of startling equivalences between seemingly different sets of equations of motion. Finally, as an application, we use our approach to show that (in the linear regime) an isotropic solid may be viewed as a fluid with an additional conserved charge, which transforms as a symmetric (0,2)-tensor under $SO(3)$. Many more specific examples can be worked out for different universality classes (see accompanying paper). *Notation*: We use $\hbar = k_B = c = 1$, a $(-,+,+,+)$ Minkowski metric, Greek indices run from 0 to 3, lowercase Latin indices from 1 to 3. Uppercase Latin indices are multi-index labels. $M_{(AB)}$ and $M_{[AB]}$ denote symmetric and antisymmetric parts of M_{AB}.

Hyperbolicity from thermodynamics.—To understand how the universality classes come about, we need to derive some new results concerning the properties of the EOM of relativistic systems near equilibrium. We first recap some general facts about relativistic thermodynamics. All states of thermodynamic equilibrium maximize some thermodynamic potential Φ [48–53]. For equations linearized about equilibrium, the quantity $E \coloneqq \Phi_{\text{eq}} - \Phi \geq 0$ plays the role of a nonincreasing Lyapunov functional [54], which can be expressed as a volume integral $E(\Sigma) = \int_\Sigma E^\mu d\Sigma_\mu$, where Σ is a Cauchy surface, $d\Sigma_\mu$ is its normal surface element (with orientation $d\Sigma_0 > 0$ [55]), and E^μ is the information current [46] (see the Supplemental Material [56] for a brief review). For the state of thermodynamic equilibrium to be stable against perturbations in all reference frames [57,58], E^μ must be future-directed and timelike for any nonvanishing perturbation [46]. The second law of thermodynamics also requires

$$\partial_\mu E^\mu = -\sigma \leq 0, \tag{1}$$

where σ is the entropy production rate. We consider here the dynamics of linear deviations about thermodynamic equilibrium. Hence, let $\varphi^A(t, x^j)$ be some real linear-order perturbation fields that we use to characterize the system's state. Thus, $\varphi^A = 0$ in equilibrium. We now prove the following result:

Theorem 1.—Consider the following system of partial differential equations on $\mathbb{R}^{1+3}$,

$$M^\mu_{AB}\partial_\mu \varphi^B = -\Xi_{AB}\varphi^B, \tag{2}$$

where M^μ_{AB} and Ξ_{AB} are constant matrices, and M^0_{AB} is invertible. Suppose that there exist constant symmetric matrices E^μ_{AB} and σ_{AB} such that Eq. (1) holds for any smooth solution of (2), where E^μ and σ are given by

$$E^\mu = \frac{1}{2}E^\mu_{AB}\varphi^A\varphi^B, \qquad \sigma = \sigma_{AB}\varphi^A\varphi^B, \tag{3}$$

and E^μ is future-directed timelike (and hence nonvanishing) over the support of φ^A. Then, the system (2) is causal, and it can be equivalently rewritten in a symmetric hyperbolic form as follows:

$$E^\mu_{AB}\partial_\mu\varphi^B = -\sigma_{AB}\varphi^B - \Xi_{[AB]}\varphi^B. \tag{4}$$

Proof.—System (2) admits smooth solutions of the form,

$$\varphi^A(t, x^j) = (e^{-(M^0)^{-1}\Xi t})^A_B(Z^B - W^B) + (e^{-(M^0)^{-1}(\Xi + M^j a_j)t})^A_B W^B e^{a_j x^j}, \tag{5}$$

for any real $Z^A, W^A, a_j = \text{const}$. Thus, $E^\mu_{AB}Z^A Z^B = 2E^\mu(0)$ must be timelike future directed for any $Z^A \neq 0$. In particular, E^0_{AB} must be positive definite and, hence, invertible. Now, we always have the freedom to redefine the matrices M^μ_{AB} and Ξ_{AB} by contracting both sides of (2) with an invertible matrix $\mathcal{N}^A_C$. Let us then "fix" the matrix M^0_{AB} to coincide with E^0_{AB} (this is possible because both are invertible). If we contract (2) with φ^A, and we plug (3) into (1), we obtain the two equations below,

$$\varphi^A M^\mu_{AB}\partial_\mu\varphi^B + \varphi^A\Xi_{AB}\varphi^B = 0, \quad \varphi^A E^\mu_{AB}\partial_\mu\varphi^B + \varphi^A\sigma_{AB}\varphi^B = 0. \tag{6}$$

Both are respected along *all* solutions of (2). If we subtract the second equation of (6) to the first, the terms $\varphi^A M^0_{AB}\partial_t\varphi^B$ cancel out (we have fixed $M^0_{AB} = E^0_{AB}$). Evaluating the result along (5), at $x^\mu = 0$, we obtain

$$Z^A(M^j_{AB} - E^j_{AB})a_j W^B + Z^A(\Xi_{AB} - \sigma_{AB})Z^B = 0. \tag{7}$$

Since this must be true for any choice of Z^A, W^A, and a_j, we obtain $M^j_{AB} = E^j_{AB}$ and $\Xi_{(AB)} = \sigma_{AB}$. We have recovered (4). But the matrices E^μ_{AB} are symmetric [59], and E^0_{AB} is positive definite. Thus, the system (4) is symmetric hyperbolic and, since $E^\mu_{AB}Z^A Z^B$ is future-directed timelike for any $Z^A \neq 0$, it is also causal [19]. ■

This theorem shows that thermodynamic stability in relativistic systems implies not only causality [46] but also symmetric hyperbolicity in the linear regime (provided that we have an information current). This is convenient given that, in most physical systems, symmetric-hyperbolicity follows directly from Onsager symmetry [60]. Furthermore, by showing that the EOM are symmetric hyperbolic, we establish that the initial value problem of all thermodynamically consistent theories linearized about homogeneous equilibrium is well posed [61–63], i.e., given initial data, solutions to the equations always exist, are unique, and depend continuously on the data. The assumptions in Theorem 1 are quite general, encompassing an astounding number of different theories. For example, they are satisfied by all Israel-Stewart-like theories [1,12,37,64], in an arbitrary hydrodynamic frame [27,38,65], and with an arbitrary number of chemical species [66]. They are also satisfied within Carter's multifluid theory [6,40,67], GENERIC-based theories [41,45,68,69], Geroch-Lindblom theories [19,44,70,71], and divergence-type theories [39,42,72]. Theorem 1 shows that the EOM of all of those theories can be found (and written in symmetric-hyperbolic form) in terms of the information current. Later in this paper, we show how the information current can be systematically determined from symmetry arguments. In the Supplemental Material [56], we explore the origin of Theorem 1 in detail.

Finally, we remark that the hypotheses of Theorem 1 are violated by first-order theories [13,14,17,18,73–75] because their regularized information current contains derivatives [76]. Also, we note that in most hydrodynamic theories, apart from the "holographic theories" discussed in [44,77], one finds $\Xi_{[AB]} = 0$. Consequently, in this case, the linear field equations (4) are uniquely determined by the information current and the entropy production rate. We will assume this in the following and work with theories specified by the triplet $\{\varphi^A, E^\mu, \sigma\}$.

In this context, one can find conditions under which two seemingly different theories are, in reality, different manifestations of the same near-equilibrium physics. This is the content of

Theorem 2.—Let $\{\varphi^A, E^\mu, \sigma\}$ and $\{\tilde{\varphi}^C, \tilde{E}^\mu, \tilde{\sigma}\}$ be two linear theories, for which all the hypotheses of Theorem 1 hold, and such that $\Xi_{[AB]} = \tilde{\Xi}_{[CD]} = 0$. Then, such theories are equivalent if and only if there is an invertible matrix $\mathcal{N}^A_C$ such that, for arbitrary Z^C,

$$E^\mu(\mathcal{N}^A_C Z^C) = \tilde{E}^\mu(Z^C), \qquad \sigma(\mathcal{N}^A_C Z^C) = \tilde{\sigma}(Z^C). \quad (8)$$

Proof.—Theorem 1 implies that the field equations of the two theories can be recast as

$$E^\mu_{AB}\partial_\mu \varphi^B = -\sigma_{AB}\varphi^B, \qquad \tilde{E}^\mu_{CD}\partial_\mu \tilde{\varphi}^D = -\tilde{\sigma}_{CD}\tilde{\varphi}^D. \quad (9)$$

Suppose that an invertible matrix $\mathcal{N}^A_C$ that satisfies (8) exists. Then $E^\mu_{AB}\mathcal{N}^A_C\mathcal{N}^B_D = \tilde{E}^\mu_{CD}$, and $\sigma_{AB}\mathcal{N}^A_C\mathcal{N}^B_D = \tilde{\sigma}_{CD}$. But this implies that if we contract the first equation of (9) with $\mathcal{N}^A_C$, and we make the replacement $\varphi^B = \mathcal{N}^B_D\tilde{\varphi}^D$, we obtain the second equation of (9). Hence, the equations of the two theories are the same, just written using different variables. Vice versa, suppose that the two linear theories are the same theory. Then, since the information current is unique [46], we must have that $E^\mu = \tilde{E}^\mu$, and $\sigma = \tilde{\sigma}$. But this is equivalent to saying that there is a one-to-one mapping $\varphi^A = \mathcal{N}^A_C\tilde{\varphi}^C$ for which Eq. (8) holds, with $Z^C = \tilde{\varphi}^C$. ∎

Theorem 2 can be employed to prove that many apparently different theories currently in use reduce to exactly the same theory close to equilibrium. For example, a fluid mixture of two chemical substances undergoing a chemical reaction is indistinguishable from the Israel-Stewart theory for bulk viscosity [78], close to equilibrium (see also [79–81]). Several other examples are discussed in our companion paper [82].

Israel-Stewart representation.—In classical field theory, the existence of conservation laws is associated with the presence of a collection of currents j^μ_I (where I is a new multi-index spanning the conserved quantities) with vanishing divergence $\partial_\mu j^\mu_I = 0$ [83] (e.g., baryon number conservation). In linear fluid theories characterized by $\{\varphi^A, E^\mu, \sigma\}$ such as those considered here, such currents manifest themselves through the existence of a (constant) matrix $\mathcal{N}^A_I$ such that $\mathcal{N}^A_I\sigma_{AB} = 0$. In fact, if we contract the field equations of the theory, $E^\mu_{AB}\partial_\mu\varphi^B = -\sigma_{AB}\varphi^B$, with $\mathcal{N}^A_I$, the right-hand side vanishes, and we recover the equations $\partial_\mu j^\mu_I = 0$, with

$$j^\mu_I = \mathcal{N}^A_I E^\mu_{AB}\varphi^B. \quad (10)$$

In the Supplemental Material [56], we prove the following useful result:

Theorem 3.—Let $\{\varphi^A, E^\mu, \sigma\}$ be a linear theory for which all the hypotheses of Theorem 1 hold, and $\Xi_{[AB]} = 0$. If the conservation laws $\partial_\mu(\mathcal{N}^A_I E^\mu_{AB}\varphi^B) = 0$ are all independent, then there is a one-to-one change of variables $\varphi^A \to \{\mu^I, \Pi^a\}$ such that E^μ, j^μ_I, and σ take the form (all matrices below are constant),

$$E^\mu = \frac{1}{2}E^\mu_{IJ}\mu^I\mu^J + E^\mu_{Ib}\mu^I\Pi^b + \frac{1}{2}E^\mu_{ab}\Pi^a\Pi^b,$$
$$j^\mu_I = E^\mu_{IJ}\mu^J + E^\mu_{Ib}\Pi^b, \qquad \sigma = \sigma_{ab}\Pi^a\Pi^b, \quad (11)$$

where E^μ_{IJ}, E^μ_{ab}, and σ_{ab} are symmetric matrices. If $\mathcal{N}^A_I$ accounts for all conservation laws, then σ_{ab} is invertible.

Theorem 3 tells us that one can always rearrange the EOM so that their mathematical structure "resembles" Israel-Stewart theory. In fact, if σ_{ab} is invertible, with matrix inverse σ^{ab}, we can express the field Eqs. (4) in terms of the variables $\{\mu^I, \Pi^a\}$ as

$$\partial_\mu(E^\mu_{IJ}\mu^J + E^\mu_{Ib}\Pi^b) = 0,$$
$$\sigma^{ab}E^\mu_{bc}\partial_\mu\Pi^c + \Pi^a = -\sigma^{ab}E^\mu_{Jb}\partial_\mu\mu^J, \quad (12)$$

which we will refer to as the "Israel-Stewart representation" of the theory. The first set of equations in (12) is the set of all conservation laws. The second set gives relaxation-type equations [84], which describe dissipation. Stability requires that E^0_{ab} and σ_{ab} be positive definite so that the second equation in (12) contains both $\partial_t\Pi^a$ and Π^a, breaking time-reversal invariance. Thus, one can interpret Π^a as "dissipative fields" [70] (e.g., viscous stresses, diffusive currents, and reaction affinities [85]). The fields μ^I are the usual "dynamical fluid fields" [70] (e.g., temperature, chemical potential, and flow velocity). Indeed, if $\Pi^a = 0$ then $2E^\mu = \mu^I j^\mu_I$, meaning that μ^I may be interpreted as the "chemical potential" of the conserved density j^0_I [60,86]. Therefore, thermodynamically stable relativistic theories can always be expressed in this Israel-Stewart representation specified by a choice of $\{\mu^I, \Pi^a, E^\mu, \sigma\}$.

Further insight can be obtained by realizing that, due to rotational invariance of the equilibrium state, we may further decompose μ^I and Π^a into irreducible tensors of the $SO(3)$ rotation group. For example, if among the fields there is a four-current δn^μ, then δn^0 behaves as a scalar under rotations, and δn^j ($j = 1, 2, 3$) behaves as a three-vector. Hence, we can assign to a given theory a list of integers $(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots)$ that specifies the geometric character of its degrees of freedom, with $\mathfrak{g}_0$ being the number of scalars, $\mathfrak{g}_1$ the number of vectors, $\mathfrak{g}_2$ the number of symmetric traceless tensors with two indices, and so forth. One can repeat the same procedure for the conservation laws, including in the count only the fields μ^I, which transform under $SO(3)$ as the respective conserved densities j_I^0. This produces a second list of integers, $(\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_1, \bar{\mathfrak{g}}_2, \ldots)$, where $\bar{\mathfrak{g}}_n \leq \mathfrak{g}_n$. From (12), we see that the theory is nondissipative if and only if $\bar{\mathfrak{g}}_n = \mathfrak{g}_n$, $\forall\ n$, i.e., if there are as many conservation laws as degrees of freedom. Therefore, one can fully specify a given theory by saying that it is "of class $(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots) - (\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_1, \bar{\mathfrak{g}}_2, \ldots)$". For example, a perfect fluid at finite chemical potential is of class $(2,1)-(2,1)$, and it does not dissipate, whereas a bulk-viscous fluid at zero chemical potential is of class $(2,1)-(1,1)$, and it dissipates (here, it is understood that $\mathfrak{g}_n = \bar{\mathfrak{g}}_n = 0$, for $n \geq 2$).

Universality classes.—Pick a class, i.e., fix the values of $\mathfrak{g}_n$ and $\bar{\mathfrak{g}}_n$. Working in the Israel-Stewart representation, give some names to the fields μ^I and Π^a and construct the most general expressions for E^μ and σ, of the form (11), compatible with rotational invariance. This produces the most general theory of the given class. By Theorem 2, any other theory belonging to the same class must be a particular case of this general theory (for some specific choice of parameters) since a mapping of the form (8) is guaranteed to exist (E^μ and σ being the most general). This general theory is usually very complicated, as it possesses a plethora of free coefficients. Luckily, Theorem 2 comes to our aid: one can reabsorb many transport coefficients through changes of variables, as this does not modify the dynamics of the system. A useful type of field redefinition is the "change of hydrodynamic frame": $\mu^I = \tilde{\mu}^I + \mathcal{R}^I_c \tilde{\Pi}^c$ and $\Pi^a = \mathcal{R}^a_c \tilde{\Pi}^c$ ($\mathcal{R}^I_c$ and $\mathcal{R}^a_c$ being constant matrices), which preserves the structure (11), mapping Israel-Stewart representations into Israel-Stewart representations. The goal is to find a transformation that maps the general theory into an already existing theory (whose physical interpretation is known), which plays the role of a "representative" of the class. If this happens, Theorem 2 guarantees that any linear theory belonging to the class is a particular realization of the representative and exhibits the same physical behavior.

We have applied this method to some selected (parity invariant [26]) classes. The representatives are reported in Table I (we have fixed $\mathfrak{g}_n = \bar{\mathfrak{g}}_n = 0$, for $n > 2$). An interesting pattern can be recognized. In the absence of vector conservation laws, the currents do not have inertia, and they can only diffuse. If we include one vector conservation law, which plays the role of the linear momentum, only then will the system behave like an actual fluid. If we include more vector conservation laws, the fluid can sustain multiple nondiffusive relative flows, and it behaves like a superfluid. If we include a tensor conservation law, the system can conserve the "memory" of the deformations it experiences, and it becomes elastic. Combine two vector conservation laws with one tensor conservation law, and the result is a superfluid-elastic system, i.e., a supersolid. We provide the information currents and entropy production rates of the theories listed in Table I in the Supplemental Material. In our companion

TABLE I. Universality classes near equilibrium.

$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\bar{\mathfrak{g}}_0$	$\bar{\mathfrak{g}}_1$	$\bar{\mathfrak{g}}_2$	Representatives
a	0	0	$\leq a$	0	0	Chemistry
a	1	0	$\leq a$	1	0	Fluid mixture [6,79]; models for bulk viscosity [78]
a	a	0	$\leq a$	$\leq a$	0	Carter multifluids [67,90]
1	1	0	0	0	0	Diffusion of a nonconserved density
			1	0	0	Cattaneo model of diffusion [84,91]
			1	1	0	Perfect fluid at $\mu = 0$; barotropic perfect fluid
2	1	0	1	1	0	Bulk viscous fluid at $\mu = 0$
			2	1	0	Perfect fluid
2	2	0	2	0	0	Coupled diffusion of two conserved densities [92]
			1	1	0	Heat conductive bulk viscous fluid at $\mu = 0$
			2	1	0	Heat conductive fluid at $\mu \neq 0$ [93]
			2	2	0	Relativistic superfluid [94]
1	1	1	1	1	0	Maxwell material [45,95] at $\mu = 0$
			1	1	1	Elastic material at $\mu = 0$ or at $T = 0$ [88,89]
1	1	2	1	1	0	Burgers material [96] at $\mu = 0$; MIS* [30]
3	2	1	1	1	0	Israel-Stewart theory in a "general frame" [27] at $\mu = 0$
			2	1	0	Israel-Stewart theory [1,37] at $\mu \neq 0$
			3	1	1	Elastic heat conducting material
			3	2	1	Supersolid [97,98]; Inner crust of neutron stars [99]

paper [82], we use those expressions to discuss in detail the equivalence between the theories in the most relevant classes. We consider here a system of class $(1,1,1)-(1,1,1)$. Its fields are $\mu^I = \{\delta\mu, \delta u^k, \delta\Pi^{kl}\}$, which may be interpreted as the perturbations to the chemical potential, flow velocity, and shear stress tensor (which is symmetric and traceless). Since we have as many degrees of freedom as conservation laws (i.e., there are no fields of "Π^a type"), the entropy production rate vanishes. Thus, the most general theory is

$$
\begin{aligned}
TE^0 &= \frac{1}{2}\frac{dn}{d\mu}(\delta\mu)^2 + \frac{1}{2}(\rho + P)\delta u^k \delta u_k + \frac{\delta\Pi^{kl}\delta\Pi_{kl}}{4G}, \\
TE^j &= n\delta\mu\delta u^j + \delta\Pi^{jk}\delta u_k, \qquad T\sigma = 0. \qquad (13)
\end{aligned}
$$

This information current contains all possible terms allowed by symmetry, making this theory a valid representative of its class. The transport coefficients have been given a physically-motivated name to ease their interpretation: $\rho + P$ may be interpreted as the enthalpy density, G as the shear modulus, and n as the background density. The prefactor of $\delta\Pi^{jk}\delta u_k$ could be set to unity by appropriately rescaling $\delta\Pi^{jk}$ with a field redefinition of the form $\delta\Pi^{jk} \to a\delta\Pi^{jk}$ [87]. If we compute the field equations from Eq. (4), we obtain

$$\begin{aligned} \partial_t \delta n + \partial_j (n\delta u^j) &= 0, \\ (\rho + P)\partial_t \delta u_k + \partial_k (n\delta\mu) + \partial_j \delta\Pi^j_k &= 0, \\ \frac{\partial_t \delta\Pi_{kl}}{2G} + \langle \partial_k \delta u_l \rangle &= 0, \end{aligned} \tag{14}$$

where $\langle \ldots \rangle$ extracts the symmetric traceless part. To obtain the last equation, one needs to account for the constraints on $\delta\Pi_{kl}$ (see [60] for a detailed discussion). Equations (14) describe the dynamics of an isotropic elastic material at zero temperature. The first equation is the continuity equation for particles, the second is the conservation of momentum, the third incorporates shear-stress dynamics in the Hookean approximation [88]. Combining all three equations, and using $dP/d\rho = n^2 d\mu/(\rho + P)dn$, we obtain

$$\partial_t^2 \delta u_k - \frac{G}{\rho + P}\partial^j \partial_j \delta u_k - \left[\frac{dP}{d\rho} + \frac{G}{3(\rho + P)}\right]\partial_k \partial_j \delta u^j = 0, \tag{15}$$

which is consistent with standard formulas from the theory of elasticity [89]. This describes an elastic material that can sustain both longitudinal and transversal propagating waves with speeds $c_L = \sqrt{(dP/d\rho) + [4G/3(\rho + P)]}$ and $c_T = \sqrt{[G/(\rho + P)]}$, respectively [89].

Conclusions.—Our results establish, for the first time, connections between elasticity, viscosity, superfluidity, supersolidity, diffusion, and chemistry within a systematic, fully relativistic formalism. Using this paper, one can easily write down the (linearized) EOM of an arbitrary relativistic hydrodynamic system with very limited knowledge about its behavior: one only needs to know the relevant degrees of freedom and conservation laws [100]. Then, the most general information current can be constructed, from which the EOM can be derived in full analogy with action principles. Our method's three main advantages are (a) The resulting EOMs are always causal, stable, thermodynamically consistent, and uniquely solvable for smooth initial data. (b) Given the degrees of freedom and the conservation laws, the associated theory is unique. (c) The information current uniquely determines also the fluctuating generalization of the theory, giving rise to well posed stochastic dynamics in terms of a path integral [102].

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[1] W. Israel and J. Stewart, Ann. Phys. (Leipzig) **118**, 341 (1979).
[2] I. Müller and T. Ruggeri, *Rational Extended Thermodynamics* (Springer, New York, 1993).
[3] H. C. Öttinger, Physica (Amsterdam) **259A**, 24 (1998).
[4] J. F. Salazar and T. Zannias, Int. J. Mod. Phys. D **29**, 2030010 (2020).
[5] L. Gavassino and M. Antonelli, Front. Astron. Space Sci. **8**, 686344 (2021).
[6] B. Carter, *Covariant Theory of Conductivity in Ideal Fluid or Solid Media* (Springer-Verlag, Berlin, 1989), Vol. 1385, p. 1.
[7] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Phys. Rev. D **85**, 085029 (2012).
[8] D. Montenegro and G. Torrieri, Phys. Rev. D **94**, 065042 (2016).
[9] L. Gavassino, M. Antonelli, and B. Haskell, Symmetry **12**, 1543 (2020).
[10] G. S. Denicol, H. Niemi, E. Molnár, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012).
[11] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, J. High Energy Phys. 02 (2008) 045.
[12] R. Baier, P. Romatschke, D. Thanh Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 04 (2008) 100.
[13] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **98**, 104064 (2018).
[14] P. Kovtun, J. High Energy Phys. 10 (2019) 034.
[15] F. S. Bemfica, F. S. Bemfica, M. M. Disconzi, M. M. Disconzi, J. Noronha, and J. Noronha, Phys. Rev. D **100**, 104020 (2019); **105**, 069902(E) (2022).
[16] R. E. Hoult and P. Kovtun, J. High Energy Phys. 06 (2020) 067.
[17] F. S. Bemfica, M. M. Disconzi, and P. J. Graber, Commun. Pure Appl. Anal. **20**, 2885 (2021).
[18] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. X **12**, 021044 (2022).
[19] R. Geroch and L. Lindblom, Ann. Phys. (N.Y.) **207**, 394 (1991).
[20] W. Florkowski and R. Ryblewski, Phys. Rev. C **83**, 034907 (2011).
[21] M. Martinez and M. Strickland, Nucl. Phys. **A848**, 183 (2010).
[22] A. Jaiswal, Phys. Rev. C **88**, 021903(R) (2013).
[23] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys. Rev. C **97**, 041901 (2018).
[24] P. B. Rau and I. Wasserman, Phys. Rev. D **102**, 063011 (2020).
[25] M. Kiamari, M. Rahbardar, M. Shokri, and N. Sadooghi, Phys. Rev. D **104**, 076023 (2021).
[26] E. Speranza, F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **107**, 054029 (2023).
[27] J. Noronha, M. Spaliński, and E. Speranza, Phys. Rev. Lett. **128**, 252302 (2022).

[28] G. Perna and E. Calzetta, Phys. Rev. D **104**, 096005 (2021).
[29] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D **105**, 045011 (2022).
[30] W. Ke and Y. Yin, Phys. Rev. Lett. **130**, 212303 (2023).
[31] R. Singh, M. Shokri, and S. M. A. Tabatabaee Mehr, Nucl. Phys. **A1035**, 122656 (2023).
[32] E. R. Most, J. Noronha, and A. A. Philippov, Mon. Not. R. Astron. Soc. **514**, 4989 (2022).
[33] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, Phys. Rev. X **12**, 041010 (2022).
[34] C. Brito and G. Denicol, Phys. Rev. D **105**, 096026 (2022).
[35] D. Wagner, A. Palermo, and V. E. Ambruş, Phys. Rev. D **106**, 016013 (2022).
[36] J. Félix Salazar and T. Zannias, Phys. Rev. D **106**, 103004 (2022).
[37] W. A. Hiscock and L. Lindblom, Ann. Phys. (Leipzig) **151**, 466 (1983).
[38] T. S. Olson, Ann. Phys. (N.Y.) **199**, 18 (1990).
[39] R. Geroch and L. Lindblom, Phys. Rev. D **41**, 1855 (1990).
[40] B. Carter, Proc. R. Soc. A **433**, 45 (1991).
[41] L. Stricker and H. C. Öttinger, Phys. Rev. E **99**, 013105 (2019).
[42] I. S. Liu, I. Müller, and T. Ruggeri, Ann. Phys. (Leipzig) **169**, 191 (1986).
[43] D. Priou, Phys. Rev. D **43**, 1223 (1991).
[44] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D **106**, 056010 (2022).
[45] L. Gavassino and M. Antonelli, Classical Quantum Gravity **40**, 075012 (2023).
[46] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. **128**, 010606 (2022).
[47] E. T. Jaynes, Am. J. Phys. **33**, 391 (1965).
[48] E. Stueckelberg, Helv. Phys. Acta **35**, 568 (1962).
[49] W. Israel, Relativistic thermodynamics, in *E.C.G. Stueckelberg, An Unconventional Figure of Twentieth Century Physics: Selected Scientific Papers with Commentaries*, edited by J. Lacki, H. Ruegg, and G. Wanders (Birkhäuser Basel, Basel, 2009), pp. 101–113.
[50] M. Grmela and H. C. Öttinger, Phys. Rev. E **56**, 6620 (1997).
[51] K. Huang, *Statistical Mechanics*, 2nd ed. (John Wiley & Sons, New York, 1987).
[52] L. Landau and E. Lifshitz, *Statistical Physics* (Elsevier Science, New York, 2013), Vol. 5.
[53] L. Gavassino, Found. Phys. **50**, 1554 (2020).
[54] L. Gavassino, Classical Quantum Gravity **38**, 21LT02 (2021).
[55] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W.H. Freeman and Co., San Francisco, CA, 1973).
[56] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.222302 for all technical proofs that were omitted from the main text. It also contains the complete atlas of representative information currents belonging to the universality classes listed in Table I.
[57] L. Gavassino, Phys. Rev. X **12**, 041001 (2022).
[58] L. Gavassino, Phys. Lett. B **840**, 137854 (2023).
[59] Note that, if in (3) we did not define E^{μ}_{AB} to be symmetric, in the second equation of (6) there would be $E^{\mu}_{(AB)}$ in place of E^{μ}_{AB}, and we would eventually get $M^{\mu}_{AB} = E^{\mu}_{(AB)}$. Hence, the system would anyway be symmetric.
[60] L. Gavassino, Phys. Rev. D **107**, 065013 (2023).
[61] R. Geroch, in *General Relativity*, edited by G. S. Hall, J. R. Pulham, and P. Osborne (1996) p. 19, arXiv:gr-qc/9602055.
[62] T. Kato, Arch. Ration. Mech. Anal. **58**, 181 (1975).
[63] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. 2: Partial Differential Equations* (John Wiley and Sons, New York, NY, 1989).
[64] G. S. Denicol, E. Molnár, H. Niemi, and D. H. Rischke, Eur. Phys. J. A **48**, 170 (2012).
[65] C. V. Brito and G. S. Denicol, Phys. Rev. D **102**, 116009 (2020).
[66] D. Almaalol, T. Dore, and J. Noronha-Hostler, arXiv:2209.11210.
[67] L. Gavassino, Classical Quantum Gravity **39**, 185008 (2022).
[68] H. C. Öttinger, Phys. Rev. D **60**, 103507 (1999).
[69] P. Ilg and H. C. Öttinger, Phys. Rev. D **61**, 023510 (1999).
[70] L. Lindblom, Ann. Phys. (N.Y.) **247**, 1 (1996).
[71] R. Geroch, J. Math. Phys. (N.Y.) **36**, 4226 (1995).
[72] T. Zannias and J. F. Salazar, Classical Quantum Gravity **40**, 087002 (2023).
[73] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **100**, 104020 (2019).
[74] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D **102**, 043018 (2020).
[75] T. Dore, L. Gavassino, D. Montenegro, M. Shokri, and G. Torrieri, Ann. Phys. (Amsterdam) **442**, 168902 (2022).
[76] L. Gavassino, N. Abboud, E. Speranza, and J. Noronha, Phys. Rev. D **109**, 085013 (2024).
[77] M. P. Heller, R. A. Janik, M. Spaliński, and P. Witaszczyk, Phys. Rev. Lett. **113**, 261601 (2014).
[78] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. Lett. **122**, 221602 (2019).
[79] L. Gavassino, M. Antonelli, and B. Haskell, Classical Quantum Gravity **38**, 075001 (2021).
[80] G. Camelio, L. Gavassino, M. Antonelli, S. Bernuzzi, and B. Haskell, Phys. Rev. D **107**, 103031 (2023).
[81] G. Camelio, L. Gavassino, M. Antonelli, S. Bernuzzi, and B. Haskell, Phys. Rev. D **107**, 103032 (2023).
[82] L. Gavassino, M. Disconzi, and J. Noronha, companion paper, Phys. Rev. D **109**, 096041 (2024).
[83] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, USA, 1995).
[84] D. Jou, J. Casas-Vázquez, and G. Lebon, Rep. Prog. Phys. **51**, 1105 (1999).
[85] D. Kondepudi and I. Prigogine, *Modern Thermodynamics* (John Wiley and Sons, Ltd., New York, 2014).
[86] R. Pathria and P. D. Beale, in *Statistical Mechanics (Third Edition)*, edited by R. Pathria and P. D. Beale (Academic Press, Boston, 2011), pp. 583–635.
[87] The case in which the prefactor of $\delta\Pi^{jk}\delta u_k$ equals zero is trivial since it would produce the field equation $\partial_t \delta\Pi_{kl} = 0$.

[88] B. L. Schumaker and K. S. Thorne, Mon. Not. R. Astron. Soc. **203**, 457 (1983).

[89] L. Landau and E. Lifshitz, *Theory of Elasticity* (Pergamon Press, New York, 1970), Vol. 7.

[90] B. Carter and I. M. Khalatnikov, Phys. Rev. D **45**, 4536 (1992).

[91] C. Cattaneo, Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* (Gauthier-Villars, Paris, 1958).

[92] L. Onsager, Phys. Rev. **37**, 405 (1931).

[93] T. S. Olson and W. A. Hiscock, Phys. Rev. D **41**, 3687 (1990).

[94] B. Carter and D. Langlois, Phys. Rev. D **51**, 5855 (1995).

[95] T. Andrade, M. Baggioli, and O. Pujolàs, Phys. Rev. D **100**, 106014 (2019).

[96] J. Málek, K. R. Rajagopal, and K. Tuma, Fluids **3**, 69 (2018).

[97] A. F. Andreev and I. M. Lifshitz, Sov. J. Exp. Theor. Phys. **29**, 1107 (1969), https://ui.adsabs.harvard.edu/abs/1969JETP...29.1107A/abstract.

[98] M. R. Sears and W. M. Saslow, Phys. Rev. B **82**, 134523 (2010).

[99] B. Carter and L. Samuelsson, Classical Quantum Gravity **23**, 5367 (2006).

[100] We note that, in general, knowledge about dispersion relations is not enough to specify the dynamics of a system uniquely. In fact, causality cannot solely be determined from dispersion relations, see [101] and our companion paper [82]. On the other hand, within our approach, given the couple $\{E^\mu, \sigma\}$, the equations of motion are unique, and causal by construction. Furthermore, when one specifies the numbers $(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots) - (\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_1, \bar{\mathfrak{g}}_2, \ldots)$, the transformation laws of the fields and possible constraints (such as $\Pi_{jk} = \Pi_{kj}$) are implemented from the start. Hence, our method, in general, provides more information about the system's dynamics than a pure spectral analysis.

[101] L. Gavassino, M. M. Disconzi, and J. Noronha, Phys. Rev. Lett. **132**, 162301 (2024).

[102] N. Mullins, M. Hippert, and J. Noronha, Phys. Rev. D **108**, 076013 (2023).