Local in Time Conservative Binary Dynamics at Fourth Post-Minkowskian Order

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Leveraging scattering information to describe binary systems in generic orbits requires identifying local and nonlocal in time tail effects. We report here the derivation of the universal (nonspinning) local in time conservative dynamics at fourth post-Minkowskian order, i.e., $\mathcal{O}(G^4)$. This is achtieved by computing the nonlocal-in-time contribution to the deflection angle, and removing it from the full conservative value in [C. Dlapa *et al.*, Phys. Rev. Lett. **128**, 161104 (2022).; C. Dlapa *et al.*, Phys. Rev. Lett. **130**, 101401 (2023).]. Unlike the total result, the integration problem involves two scales—velocity and mass ratio—and features multiple polylogarithms, complete elliptic and iterated elliptic integrals, notably in the mass ratio. We reconstruct the local radial action, center-of-mass momentum and Hamiltonian, as well as the exact logarithmic-dependent part(s), all valid for generic orbits. We incorporate the remaining nonlocal terms for ellipticlike motion to sixth post-Newtonian order. The combined Hamiltonian is in perfect agreement in the overlap with the post-Newtonian state of the art. The results presented here provide the most accurate description of gravitationally bound binaries harnessing scattering data to date, readily applicable to waveform modeling.

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Introduction.—Motivated by the impending era of high-precision gravitational-wave (GW) astronomy with observatories such as LISA [1], the Einstein telescope [2] and the Cosmic Explorer [3], and the incredibly rich amount of information expected from compact binary sources [4–9], the (long dormant [10]) post-Minkowskian (PM) expansion in general relativity—entailing a perturbative series in G (Newton's constant) but to all orders in the relative velocity—has experienced a resurgence in recent years, e.g., [11–46]. This is, in part, thanks to the repurposing of modern integration techniques from collider physics (see Refs. [42,44] and references therein), which have led to a plethora of new results. Notably, using worldline effective field theory (EFT) methodologies [47–51], the rapidly evolving state of the art includes the total relativistic impulse (yielding the scattering angle and emitted GW flux) of nonspinning [11,12] and spinning [39,40] bodies to $\mathcal{O}(G^4)$, akin of a "three-loop" calculation in particle physics, as well as partial results in the conservative sector at 5PM [46].

The derivations in [11,12], together with a (Firsov-type [52,53]) resummation scheme [21,22], have led to an unprecedented agreement between analytic results and numerical simulations [54–56], paving the way to more accurate waveform models for hyperbolic encounters. However, due to nonlocal in time effects [57,58], unbound results cannot be used to describe generic ellipticlike motion (away from the large-eccentricity limit [11]). As shown in [23], the binding energy for quasicircular orbits obtained from scattering results [via the "boundary-to-bound" (B2B) analytically continuation [21,22]] does not reproduce—other than logarithms—the known post-Newtonian (PN) values [57–61] (see also [62]). Hence, to fully harness the power of scattering calculations, a separation between local and nonlocal in time effects was thus imperative. In this Letter we report the derivation of the universal (nonspinning) local in time conservative dynamics of binary systems at $\mathcal{O}(G^4)$. This is obtained via a direct computation of the nonlocal-in-time contribution to the scattering angle. Following [23,63], the calculation entails an integral over the energy spectrum times the logarithm of the center-of-mass GW frequency. To solve the integration problem, we implement the methodology of differential equations, already used in [11,12]. However, unlike the total impulse, which obeys a simple (power-law) mass scaling [16,21], isolating (non)local effects in a gauge-invariant fashion entails dealing with two relevant scales: the velocity and mass ratio. Despite the complexity of the two-scale problem, we find that it can be factorized into solving two second-order Picard-Fuchs (PF) equations. The nonlocal part of the angle features multiple polylogarithms

(MPLs), complete elliptic integrals, and integrations thereof. We find agreement in the overlap with the 6PN values in [63].

We derive the local in time contribution to the conservative scattering angle by removing the (unbound) nonlocal terms from the total result in [11,12]. The local radial action follows directly via the B2B map [21–23]. Using the relations in [21,22], we reconstruct the universal local-in-time center-of-mass momentum and Hamiltonian in isotropic gauge, together with the complete logarithmic dependence, all applicable to generic motion. We also provide—for all practical purposes—results expanded to 30 orders in the (symmetric) mass ratio and all orders in the velocity (with an error beyond 30PN). To incorporate the remaining (nonlogarithmic) nonlocal part of the bound dynamics, we adapt to our isotropic gauge the values obtained in [63] to 6PN order. The combined Hamiltonian at $\mathcal{O}(G^4)$ perfectly matches in the overlap with the state of the art in PN theory [62–64]. The results presented here can be directly inputed onto waveform models for gravitationally bound eccentric orbits, potentially increasing their accuracy by incorporating an infinite tower of (local in time) velocity corrections.

(Non)local in time tail effects.—The scattering of the emitted radiation off of the binary's gravitational potential, or "tail effect," enters in the 4PM conservative dynamics both through local as well as nonlocal in time interactions [57–61]. Because of this, although an effectively local description is possible to any order [11,12], the coefficients of the radial action (or Hamiltonian) depend on the type of motion, and therefore are not related via analytic continuation for generic orbits. Our strategy is to identify the local and nonlocal in time parts of $\mathcal{S}_r$, the total radial action. Because of the structure of tail effects [23,57,58], the nonlocal in time tail terms can be shown to take the gauge-invariant form

$$\mathcal{S}_r^{(\mathrm{nloc})} = -\frac{GE}{2\pi}\int_\omega \frac{dE}{d\omega}\log\left(\frac{4\omega^2}{\mu^2}e^{2\gamma_E}\right), \tag{1}$$

where $\int_\omega \equiv \int_{-\infty}^{+\infty}(d\omega/2\pi)$, E and $(dE/d\omega)$ are the total (incoming) energy and emitted GW spectrum in the center-of-mass frame. The "renormalization scale" μ, which cancels against a similar term in the local in time part [11,34,58], can be arbitrarily chosen. The factor of $4e^{2\gamma_E}$ (with γ_E Euler's constant) follows the PN conventions [57,58]. An explicit derivation of (1) in the context of the PM expansion can be found in [23], see also [63] for a discussion in the PN regime. For unbound motion, the scattering angle is given by $(\chi/2\pi) = -\partial_j\mathcal{I}_r$, with $\mathcal{I}_r \equiv (\mathcal{S}_r/GM^2\nu)$ and $j \equiv (J/GM^2\nu)$ the (reduced) radial action and angular momentum, and $M = m_1 + m_2$, $q = m_2/m_1$ $(m_2 \le m_1)$, $\nu = m_1 m_2/M^2$ the total mass, mass ratio, and symmetric mass ratio, respectively. We split the PM coefficients of the deflection angle in impact parameter space as

$$\frac{\chi}{2} = \sum_{n=1}\left(\chi_b^{(n)} + \chi_b^{(n)\log}\log\frac{\mu b}{\Gamma}\right)\left(\frac{GM}{b}\right)^n, \tag{2}$$

where $\gamma \equiv u_1 \cdot u_2$ (using the mostly negative metric convention), $u_{1,2}$ the incoming velocities, and $\Gamma \equiv E/M = \sqrt{1+2\nu(\gamma-1)}$. (The reader should keep in mind that logarithms of the velocity may still appear in both coefficients.) In the rest of the Letter we choose $\mu \equiv 1/GM$ for the renormalization scale.

Integrand construction.—Because of the overall factor of G in (1), it is sufficient to construct the integrand to $\mathcal{O}(G^3)$. Using the results in [33], and multiplying by a factor of $(2e^{\gamma_E}k\cdot u_{\mathrm{com}})^{2\tilde{\epsilon}}$, with $k = (\omega, \boldsymbol{k})$ the (on-shell) radiated momentum, and $u_{\mathrm{com}} \equiv (m_1 u_1 + m_2 u_2)/E$ the center-of-mass velocity, we readily derive a covariant version which, after projecting on the center-of-mass frame, matches at $\mathcal{O}(\tilde{\epsilon})$ the expression in (1). We find it convenient to distinguish the $\tilde{\epsilon}$-expansion from the standard $D = 4 - 2\epsilon$ that we use for dimensional regularization. To the families of (two-loop) scalar integrals introduced in [33] for computing the total impulse, we add

$$I^{\pm\pm;T_5\ldots T_9}_{\nu_1\ldots\nu_{10}} = \int_{\ell_1,\ell_2}\frac{\delta^{(\nu_1-1)}(\ell_1\cdot u_1)\delta^{(\nu_2-1)}(\ell_2\cdot u_2)}{(\pm\ell_1\cdot u_2)^{\nu_3}(\pm\ell_2\cdot u_1)^{\nu_4}} \times (k\cdot u_{\mathrm{com}})^{2\tilde{\epsilon}-\nu_{10}}\prod_{j=5}^{9}\frac{1}{D_{j,T_j}^{\nu_j}}, \tag{3}$$

with a noninteger powered propagator, where we use the same notation as in [33] (see also [42]). The radiative momentum is rewritten as $k^\alpha = \ell_1^\alpha + \ell_2^\alpha - q^\alpha$, with $q^\alpha \equiv (q^0, \boldsymbol{q})$ the momentum transfer, obeying $q\cdot u_a = 0$ (not to be confused with the mass ratio). The choice of $i0^+$-prescription for the square propagators, either retarded or advanced, is encoded in $T_j \in \{\mathrm{ret}, \mathrm{adv}\}$:

$$\begin{aligned}
D_{5,\mathrm{ret/adv}} &= (\ell_1^0 \pm i0)^2 - \boldsymbol{\ell}_1^2, D_{6,\mathrm{ret/adv}} = (\ell_2^0 \pm i0)^2 - \boldsymbol{\ell}_2^2,\\
D_{7,\mathrm{ret/adv}} &= (\ell_1^0 + \ell_2^0 + q^0 \pm i0)^2 - (\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 - \boldsymbol{q})^2,\\
D_{8,\mathrm{ret/adv}} &= (\ell_1^0 - q^0 \pm i0)^2 - (\boldsymbol{\ell}_1 - \boldsymbol{q})^2,\\
D_{9,\mathrm{ret/adv}} &= (\ell_2^0 - q^0 \pm i0)^2 - (\boldsymbol{\ell}_2 - \boldsymbol{q})^2.
\end{aligned} \tag{4}$$

Using integration-by-parts (IBP) reduction techniques implemented in the packages `LiteRed` [65] and `FiniteFlow` [66], we find 17 master integrals contributing to the radiation region (where the k momentum goes on-shell), which isolates the contribution to the energy loss from the total impulse. It is possible to select integrals such that we can take $\nu_1 = \nu_2 = 1$, $\nu_{10} = 0$.

The final set, specified by $\nu_{3\ldots9}$, becomes

$$(-1,0,0,0,1,1,1),\quad (-1,0,0,0,1,1,2),\quad (-1,0,0,0,1,2,1),$$
$$(-1,0,0,0,2,1,1),\quad (0,-1,0,0,1,1,1),\quad (0,-1,0,0,1,1,2),$$
$$(-1,0,0,1,1,0,1),\quad (0,-1,0,1,1,0,1),\quad (-1,0,0,1,1,1,1),$$
$$(-1,0,1,0,1,1,0),\quad (0,-1,1,0,1,1,0),\quad (-1,0,1,0,1,1,1),$$
$$(-1,0,1,1,1,1,1),\quad (-1,0,1,1,1,1,2),\quad (-1,0,1,1,2,1,1),$$
$$(0,1,1,1,1,0,0),\quad (1,0,0,0,1,1,1),$$

modulo different choices of $i0^+$-prescriptions ($T_{5\ldots9}$) and signs in front of linear propagators.

Integration.—To solve for the master integrals, we derive differential equations in x and the mass ratio, q, where x is given by $\gamma = \frac{1}{2}(x + 1/x)$. We then adopt the strategy of an ϵ- (and $\tilde{\epsilon}$-)regular basis [67], such that we can set $\epsilon = 0$, and consider the expansion of the integrand, differential equations, and boundary constants, only to $\mathcal{O}(\tilde{\epsilon})$. The latter are determined via a small-q expansion, together with the techniques described in [42] (adapted to the new factors of $\tilde{\epsilon}$). From this setting, it is then straightforward to find a solution of the differential equations through iterated integration.

For the parts containing MPLs, and similarly to the x variable, it is useful to rationalize the square root of the energy $(E/m_1) = \sqrt{1+2\gamma q+q^2} = \sqrt{(q+x)(q+1/x)}$, by introducing a new variable, y, defined through $q^{-1} = -\gamma - (v_\infty/2)(y+1/y)$, with $v_\infty \equiv \sqrt{\gamma^2-1}$. Hence, we find the traditional harmonic polylogarithms with letters $\{x, 1+x, 1-x\}$ [26,33], as well as MPLs which depend on the velocity and mass ratio via the new letters: $\{y, 1+y, 1-y, y-[(1+x)/(1-x)], y-[(1-x)/(1+x)], 1+2[(1-x)/(1+x)]y+y^2\}$. In addition to MPLs, the solution to the differential equations depend on another set of functions, through an *a priori* irreducible fourth-order PF equation, already at $\mathcal{O}(\tilde{\epsilon}^0)$. However, a Baikov representation [68,69] of the maximal cut suggests a simpler Calabi-Yau twofold as the relevant geometry. Indeed, in terms of the variables $(qx, q/x)$, the differential equations can be solved, in the first and subsequently the second variable, via two equivalent second-order PF equations (per variable). The solution can then be written in terms of products of K's [such as the f_1 in (5) below] as well as the leading three derivatives w.r.t. the mass ratio. As in previous PM computations, e.g., [11,12,34], $\mathrm{K}(z) = \int_0^1 dt/\sqrt{(1-t^2)(1-zt^2)}$, is the complete elliptic integral of the first kind.

After the leading order solution is known, it is then straightforward to obtain the $\mathcal{O}(\tilde{\epsilon})$ part. We find that it can be written in terms of (at most) twofold iterated integrals, with elliptic kernels depending on the mass ratio, q, as the integration variable. The full set is given by

$$f_1 = \frac{\mathrm{K}(-qx)\mathrm{K}(1+\frac{q}{x}) - \mathrm{K}(-\frac{q}{x})\mathrm{K}(1+qx)}{\pi},$$
$$f_2 = \frac{f_1}{q}, \quad f_3 = \partial_x f_1, \quad f_4 = \frac{\partial_x f_1}{q},$$
$$f_5 = \left[\frac{1-x^2}{x}(1+q\partial_q) - \frac{1-q^2}{q}x\partial_x\right]\frac{f_1}{\sqrt{(q+x)(q+\frac{1}{x})}}. \quad (5)$$

Remarkably, while individually this is not the case, the combination of complete elliptic integrals in f_1 has a simple power-series expansion in the PN limit ($x \to 1$). Furthermore, the f_i's are real, and have (at most) simple poles in q. Let us point out, however, that a simplified version of the iterated integrals may still be possible. In particular, upon assigning to $\mathrm{K}(z)$ a transcendental weight *one*, we notice that the iterated integrals would have up to weight *four*, in contrast to the MPL part with maximum weight *two*. Hence, we expect that either the naïve assignment is incorrect or an even simpler form exists. We leave this open for future work.

Scattering angle.—After solving for the master integrals and plugging them back into the integrand, we arrive at the radial action, and from there to the nonlocal in time contribution to the deflection angle, $\chi^{(4)}_{b(\mathrm{nloc})}$ and $\chi^{(4)\log}_{b(\mathrm{nloc})}$, at 4PM order. As anticipated in [34], the logarithmic part takes on a simple closed form,

$$\frac{1}{\pi\Gamma}\chi^{(4)\log}_{b(\mathrm{nloc})} = -2\nu\chi_{2\epsilon}(\gamma) = \frac{-2\nu}{(\gamma^2-1)^2}\left(h_5 + h_9\log\left(\frac{\gamma+1}{2}\right) + \frac{h_{10}\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right), \quad (6)$$

with $\chi_{2\epsilon}$ introduced in [34], and the $h_{5,9,10}$ are polynomials depending only on γ, which also enter the nonlogarithmic part (see below). The latter, on the other hand, also involves a set of iterated integrals in the mass ratio. Despite its complexity, it is straightforward to construct a "self-force" (SF) expanded version, for which we find the generic form, valid to any nSF order,

$$\frac{1}{\pi\Gamma}\chi^{(4)(n\mathrm{SF})}_{b(\mathrm{nloc})} = \frac{\nu}{(\gamma^2-1)^2}\left\{h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3\log\left(\frac{\gamma+1}{2}\right) + \frac{h_4\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5\log\left(\frac{\gamma-1}{8}\right) + h_6\log^2\left(\frac{\gamma+1}{2}\right)\right.$$
$$+ h_7\mathrm{arccosh}(\gamma)^2 + \frac{h_8\log(2)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_9\log\left(\frac{\gamma-1}{8}\right)\log\left(\frac{\gamma+1}{2}\right)$$
$$\left. + \frac{h_{10}\log(\frac{\gamma^2-1}{16})\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_{11}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right) + \frac{h_{12}\left[\mathrm{arccosh}(\gamma)^2 + 4\mathrm{Li}_2\left(\sqrt{\gamma^2-1}-\gamma\right)\right]}{\sqrt{\gamma^2-1}}\right\}. \tag{7}$$

Because of the structure of the full solution, except for the h_1, h_3, and h_4 carrying information from the (iterated) elliptic sector ($h_{1,4}$) and the new letters in the MPLs depending on the mass ratio (h_3), the remaining h_i's are SF exact. We find the nSF coefficients may be split as

$$h_i = h_i^{(0)}(\gamma) + \sqrt{1-4\nu}h_i^{(1)}(\gamma) + \Delta h_i(\gamma,\nu), \tag{8}$$

where the $h_i^{(0)}(\gamma)$, $h_i^{(1)}(\gamma)$, are polynomials in γ only. The $\Delta h_i(\gamma,\nu)$ vanish except when $i = 1, 3, 4$, for which they become polynomials both in γ and ν, up to $\mathcal{O}(\nu^n)$. We provide in [70] their values up to $n = 30$. The 30SF result (with an error beyond 30PN) is in perfect agreement in the overlap with the 6PN values in [63]. Let us emphasize that the definition of nonlocal-in-time in [63,64] includes not only the expression in (1) (W_1 in [63]), but also an extra contribution (W_2 in [63]). Because of the local in time (and gauge-dependent) nature of W_2, we do not add it to (1). Therefore, (7) agrees (in the overlapping realm of validity) with the scattering angle obtained from the W_1-only terms in Eq. (3.14) of [63]. After subtracting from the total conservative angle in [11,12], we arrive at the local in time counterpart, [Although amenable to a conservativelike description of the relative dynamics, we keep the other (time-symmetric) radiation-reaction corrections, i.e., “2 rad” in [12], in the dissipative part.]

$$\chi^{(4)}_{b(\mathrm{loc})} = \chi^{(4)\mathrm{cons}}_{b(\mathrm{tot})} - \chi^{(4)}_{b(\mathrm{nloc})}, \quad \chi^{(4)\log}_{b(\mathrm{loc})} = -\chi^{(4)\log}_{b(\mathrm{nloc})}, \tag{9}$$

where we used the fact that the $\log(\mu b/\Gamma)$ cancels out in the total value. The result in (9) can now be used to describe generic bound orbits, as we discuss next.

Local in time conservative dynamics.—Following the B2B dictionary, the local in time (reduced) bound radial action takes the form [21]

$$i^{4\mathrm{PM}}_{r(\mathrm{loc})} = \frac{2v_\infty^4}{3(\Gamma j)^3}\left(\frac{\chi^{(4)}_{b(\mathrm{loc})}}{\pi\Gamma} + \frac{\chi^{(4)\log}_{b(\mathrm{loc})}}{2\pi\Gamma}\log\frac{j^2}{v_\infty^2}\right). \tag{10}$$

Using the expressions in [21,22], and a dimensionally rescaled distance $\hat{r} = r/(GM)$, we can also reconstruct the center-of-mass momentum (notice we use different conventions with respect to [21,22])

$$\hat{\boldsymbol{p}}^2 = \frac{v_\infty^2}{\Gamma^2}\left(1 + \sum_{n=1}\frac{1}{\hat{r}^n}(f_n + f_n^{\log}\log\hat{r})\right), \tag{11}$$

with $\hat{\boldsymbol{p}} \equiv \boldsymbol{p}/(M\nu)$, and Hamiltonian, $\hat{H} \equiv H/(M\nu)$,

$$\hat{H} = \hat{E} + \sum_{i=1}\frac{1}{\hat{r}^i}(\hat{c}_i + \hat{c}_i^{\log}\log\hat{r}), \tag{12}$$

where $\hat{E} = \sum_a \hat{E}_a$, $\hat{E}_a = \sqrt{\hat{\boldsymbol{p}}^2 + (m_a/M\nu)^2}$. The coefficients $(\hat{c}_{4(\mathrm{loc})}, \hat{c}^{\log}_{4(\mathrm{loc})})$ are displayed in [70].

Universal logarithms.—Nonlocal in time tail effects also contribute with a $\log\hat{r}$ term in the bound dynamics. Performing a small-eccentricity expansion of (1), and using Kepler's law ($\log\Omega = -\frac{3}{2}\log\hat{r} + \cdots$, with Ω the 1PM orbital frequency), we find

$$\frac{\hat{c}^{\log}_{4(\mathrm{nloc})}}{\hat{r}^4} = -\frac{3}{2}\frac{\hat{c}^{\log}_{4(\mathrm{loc})}}{\hat{r}^4} = -3G\frac{\Gamma}{\nu}\frac{dE}{dt}\bigg|_{3\mathrm{PM}}, \tag{13}$$

where $(\xi \equiv \{[\hat{E}_1\hat{E}_2]/[(\hat{E}_1+\hat{E}_2)^2]\}, \gamma = \nu(\hat{E}_1\hat{E}_2 + \hat{\boldsymbol{p}}^2))$

$$G\frac{dE}{dt}\bigg|_{3\mathrm{PM}}(\hat{r},\hat{\boldsymbol{p}}^2) = -\frac{4\nu^3}{3\hat{r}^4}\frac{\gamma^2-1}{\Gamma^3\xi}\chi_{2\epsilon}(\gamma), \tag{14}$$

is the energy flux at 3PM order [23,34]. Similarly,

$$f^{\log}_{4(\mathrm{nloc})} = -\frac{3}{2}f^{\log}_{4(\mathrm{loc})} = -8\Gamma\nu\chi_{2\epsilon}, \tag{15}$$

consistently with (6). Hence, adding both terms,

$$\hat{H}^{\mathrm{ell}(\log)}_{4\mathrm{PM}} = \frac{4\nu^2}{3\hat{r}^4}\frac{(\gamma^2-1)}{\Gamma^2\xi}\chi_{2\epsilon}\log\hat{r}, \tag{16}$$

and likewise,

$$(\hat{\boldsymbol{p}}^2)^{\mathrm{ell}(\log)}_{4\mathrm{PM}} = -\frac{8\nu v_\infty^2}{3\Gamma\hat{r}^4}\chi_{2\epsilon}\log\hat{r}, \tag{17}$$

for the full logarithmic dependence of the bound Hamiltonian and center-of-mass momentum at 4PM.

Towards the complete bound dynamics.—Putting together the local in time coefficient plus exact logarithmic part, the total bound Hamiltonian up to 4PM order may be written as

$$\hat{H}^{\rm ell}_{\rm 4PM} = \sum_{i=1}^{i=4} \frac{\hat{c}_{i({\rm loc})}}{\hat{r}^i} + \sum_{i=1}^{i=4} \frac{\hat{c}_{i({\rm nloc})}}{\hat{r}^i} + \frac{4\nu^2}{3\hat{r}^4}\frac{(\gamma^2-1)}{\Gamma^2\xi}\chi_{2\epsilon}\log\left(\frac{\hat{r}}{e^{2\gamma_E}}\right), \qquad (18)$$

where we have absorbed the factor of $e^{2\gamma_E}$ that arises from (1) into the logarithm. The $\hat{c}_{1|2|3({\rm loc})}$ are the known local-in-time PM coefficients up to 3PM order [18,20,25,26], and $\hat{c}_{4({\rm loc})}$ is reported here for the first time. To complete the knowledge of the bound dynamics, we are still missing the [non-log $(\hat{r}/e^{2\gamma_E})$] nonlocal in time contributions, $\hat{c}_{i({\rm nloc})}$, which depend on the trajectory. These are more difficult to compute in a PM scheme, since they are often needed in the opposite limit of quasi-circular orbits, thus entering at all PM orders. Yet, they can be readily obtained within the PN approximation by evaluating the radial action in (1) in a small-eccentricity expansion. Adapting the (W_1-only) results in [63] to the isotropic gauge, we quote their values in the Supplemental Material to 6PN and eight order in the eccentricity [70]. The combined Hamiltonian in (18) is in perfect agreement to $\mathcal{O}(G^4\hat{\boldsymbol{p}}^6)$ with the $\hat{H}^{\rm ell}_{\rm 6PN(4PM)}$ derived in [62] using the state of the art in PN theory, while at the same time it incorporates all-order-in-velocity corrections. Ready-to-use expressions for the full results and 30SF-approximate are collected in [70].

Conclusions.—Novel integration techniques in combination with EFT methodologies have been extremely successful in reaching the very state of the art in our understanding of scattering dynamics in general relativity, including conservative and dissipative effects [42,46]. However, as illustrated in [23,62], although local in time and logarithms are universal, the full hyperbolic results fail to describe quasicircular binaries. This is due to the presence of orbit-dependent (nonlogarithmic) nonlocal in time effects, which preclude a smooth analytic continuation via the B2B map [21,22]. Hence, up until now, we were lacking a direct correspondence to generic bound motion, notably for the conservative sector. We have computed the nonlocal-in-time contribution to the deflection angle, and removed it from the total conservative value in [11,12], thus yielding the local in time counterpart. We then derived the radial action, center-of-mass (isotropic-gauge) momentum and Hamiltonian, as well as the total logarithmic-dependent part(s), all applicable to generic motion. Upon adapting the (nonlogarithmic) nonlocal in time effects for ellipticlike orbits computed in the PN expansion [63], the combined total Hamiltonian becomes the most accurate description of gravitationally bound binary systems obtained from PN or PM data to date, readily applicable to waveform modeling. Studies assessing the implications of our results towards constructing high-precision GW templates, as well as the derivation of a PM version of nonlocal in time effects for bound orbits, are underway.

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[70] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.221401 for two computer-readable ancillary files containing the main results reported in this letter, as well as a pdf file with the nonlocal in time coefficients of the bound Hamiltonian computed within the PN expansion.