

Parity Quantum Computing as yz -Plane Measurement-Based Quantum Computing

Isaac D. Smith^{1,*}, Hendrik Poulsen Nautrup¹, and Hans J. Briegel^{1,2}

¹University of Innsbruck, Institute for Theoretical Physics, Technikerstr. 21A, Innsbruck A-6020, Austria

²University of Konstanz, Department of Philosophy, Fach 17, Konstanz 78457, Germany

 (Received 19 January 2024; accepted 9 April 2024; published 28 May 2024)

We show that universal parity quantum computing employing a recently introduced constant depth decoding procedure is equivalent to measurement-based quantum computation (MBQC) on a bipartite graph using only yz -plane measurements. We further show that *any* unitary MBQC using only yz -plane measurements must occur on a bipartite graph. These results have a number of consequences and open new research avenues for both frameworks.

DOI: 10.1103/PhysRevLett.132.220602

In the present era of pre-fault-tolerant quantum computation [1], there exists an array of theoretical proposals for computation that display certain advantages and differing levels of suitability for implementation on current physical devices.

Parity quantum computation [2–7] refers to one such proposal, initially based on quantum annealing [2]. The universal parity computing framework [3] leverages the properties of a certain type of quantum state encoding, the parity encoding. This encoding maps an n -qubit logical state onto $n(n+1)/2$ physical qubits, some of which obtain parity information related to subsets of logical qubits. Consequently, certain rotations acting locally on these parity qubits translate to multiqubit logical rotations on the corresponding subset [3]. The parity code is, in particular, a stabilizer code [8,9] and many of the properties of the code are well understood using the stabilizer formalism.

Stabilizer states and stabilizer codes are known to have a canonical form, namely graph states [10,11] and graph codes [12–14] respectively. Graph states form an important class of highly entangled states that support measurement-based quantum computation (MBQC) [15–20]. MBQC is a well-known alternative to the quantum circuit model driven by single qubit projective measurements instead of unitary gates.

Recently, a proposal for measurement-based encoding and decoding procedures were put forward for the parity computing regime [21], demonstrating beneficial properties in terms of their computational depth. Because of the close connection between stabilizer codes and graph codes, an investigation of the potential connections to MBQC is warranted, which we initiate in this Letter.

After presenting the required background, we demonstrate that every parity code is local Clifford equivalent to a bipartite graph code (Proposition 1). Consequently, we show that parity quantum computation with the measurement-based decoding is MBQC where all measurements are from the yz plane of the Bloch sphere, and where re-entanglement and some local operations are allowed (Theorem 1). We further show that any MBQC using only yz -plane measurements and with input and output sets of equal size must use bipartite graph states (Theorem 2). To conclude, we briefly outline some consequences of these results for both computing paradigms.

Background.—Parity quantum computing: A parity quantum computation commences by encoding the computational input state using the LHZ architecture [2] [see Fig. 1(a)]. The computation proceeds by applying unitaries from a native gate set for the parity encoding, such as that outlined in [3], which largely consists of local rotations. To finish, a decoding procedure returns the computational output. Presently, we will focus on the parity encoding procedure and universal gate set presented in [3] in combination with the measurement-based decoding procedure outlined in [21].

The parity encoding procedure maps a state on n logical qubits to a state on $n(n+1)/2$ physical qubits. Following [3], we consider an LHZ layout where n physical qubits (the “data” qubits) directly correspond to the n logical qubits. The remaining $N = n(n-1)/2$ qubits will be referred to as “parity” qubits. We denote the sets of data and parity qubits by I and $V \setminus I$, respectively.

Encoding consists of applying a sequence of CNOTs to an input state $|\psi\rangle$ and the parity qubits, which are all initialized to $|0\rangle$. Letting C represent the set of control-target pairs, the encoded state is

$$|\text{LHZ}_\psi\rangle = U_{\text{enc}}|0\rangle^{\otimes N}|\psi\rangle = \prod_{(c,t) \in C} \text{CNOT}_{(c,t)}|0\rangle^{\otimes N}|\psi\rangle. \quad (1)$$

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

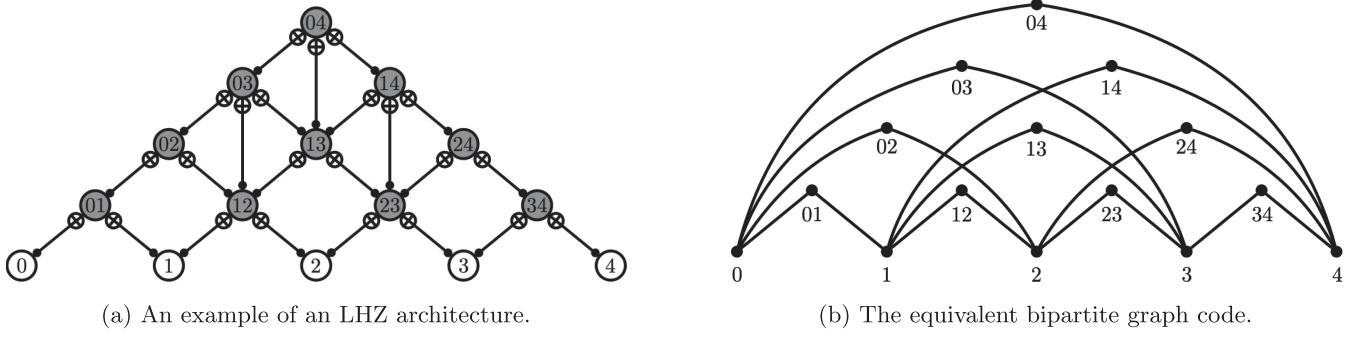


FIG. 1. (a) The parity encoding encodes an input state prepared on the data qubits (white circles). The gray circles denote parity qubits prepared in $|0\rangle$ and CNOT gates are applied to data and parity according to the layout as shown. This parity code is equivalent to a graph code for the bipartite graph shown in (b).

Different constraint sets C can produce the same encoded state. For example, C could contain only pairs where every control is a data qubit and every target a parity qubit, which may involve non-nearest neighbor interactions for a given physical layout. Equivalently, it is possible to take C to contain only nearest-neighbor CNOTs, where now some control qubits are parity qubits [see, e.g., Fig. 1(a)]. The compilation of a given parity code into a nearest-neighbor layout is an interesting optimization problem and a topic of ongoing research [5–7].

For each $|\psi\rangle$, $|\text{LHZ}_\psi\rangle$ is a state in the parity codespace for the given architecture. The stabilizer of the parity code is generated by operators of the form

$$K'_{(ij\dots k)} := Z_{(ij\dots k)} \otimes Z_i \otimes Z_j \otimes \dots \otimes Z_k, \quad (2)$$

where the single subscripts i, j , etc., indicate data qubits and the subscript $(ij\dots k)$ indicates the parity qubit that encodes the parity information of data qubits i, j and so on. The operators $K'_{(ij\dots k)}$ for all parity qubits are mutually independent and generate the codespace. Note that often each parity qubit is taken to encode the parity of just two data qubits.

A benefit of this encoding consists in the ability to implement diagonal multiqubit logical operations via single qubit physical rotations. For example, applying a local Z rotation to a parity qubit (ij) effectively applies a logical $Z_i \otimes Z_j$ -rotation, from which a controlled-phase gate between logical qubits i and j can be obtained via local Z rotations on the corresponding data qubits [3]. For full universal quantum computation, in conjunction with Z rotations and controlled-phase gates, it suffices to be able to implement a logical X rotation. For a data qubit i , this can be done via a decoding sequence of CNOTs along all parity qubits containing parity information about i , a local X -rotation at data qubit i , and a re-encoding sequence of CNOTs (see Ref. [3] for more details).

Until recently, the typical parity decoding procedures involved applying the encoding sequence of CNOT gates in

reverse. In [21], an equivalent decoding procedure was proposed involving local X measurements on parity qubits followed by local Z operations conditional on measurement outcomes. One benefit of this approach is that full and partial decoding can be performed in constant-depth regardless of the size of the architecture.

For this gate set and measurement-based decoding, a unitary U applied to input state $|\psi\rangle$ in the parity regime can be decomposed into a series of layers, where each layer involves parity qubit rotations followed by decoding. For notational simplicity, we consider full decoding in each layer. Denoting the set of layers by L , the set of data qubits by I and the set of parity qubits by $V \setminus I$, the computation can be written as

$$U|\psi\rangle = \prod_{l=1}^L \left(U_{\text{data}}^{(l)}(\boldsymbol{\alpha}^{(l)}, \boldsymbol{\phi}^{(l)}) D_{\text{dec}}^{(l)}(\boldsymbol{\theta}^{(l)}) U_{\text{enc}} |0\rangle^{\otimes N} \right) |\psi\rangle, \quad (3)$$

where U_{enc} acts on both the ancilla $|0\rangle^{\otimes N}$ as well as the qubits in I , $D_{\text{dec}}^{(l)}(\boldsymbol{\theta}^{(l)})$ is the operator involving parity qubit rotations and decoding for layer l given by

$$D_{\text{dec}}^{(l)}(\boldsymbol{\theta}^{(l)}) := \bigotimes_{(ij\dots k) \in V \setminus I} \langle +_{(ij\dots k)} | R_{Z_{(ij\dots k)}}(\theta_{(ij\dots k)}^{(l)}) \quad (4)$$

and $U_{\text{data}}^{(l)}(\boldsymbol{\alpha}^{(l)}, \boldsymbol{\phi}^{(l)})$ is the product of local rotation on data qubits for layer l given by

$$U_{\text{data}}^{(l)}(\boldsymbol{\alpha}^{(l)}, \boldsymbol{\phi}^{(l)}) := \bigotimes_{i \in I} R_{X_i}(\alpha_i^{(l)}) R_{Z_i}(\phi_i^{(l)}). \quad (5)$$

Note that the only distinction in the case of partial decoding is that $D_{\text{dec}}^{(l)}$ contains measurements in some subset of $V \setminus I$, the ancilla prepared in $|0\rangle$ in the subsequent layer correspond to the same subset, and the relevant U_{enc} is replaced by $U_{\text{enc}}^{(l+1)}$ which applies only the appropriate CNOTs to re-encode back to the full LHZ state.

Measurement-based quantum computing: Measurement-based quantum computing (MBQC) [15–18] consists of

three things: (i) a highly entangled graph state [10,11], (ii) a sequence of single qubit projective measurements in certain planes of the Bloch sphere, and (iii) classical, adaptive corrections of future measurements conditioned on prior measurement outcomes. Despite the indeterminacy of quantum measurements, deterministic computation can be performed, provided the sequence of measurements and underlying graph state satisfy certain properties [22].

Graph states take their name from their connection to mathematical graphs, where vertices correspond to qubits and edges correspond to two-qubit gates. We consider here graph states where a computational input state $|\psi\rangle$ can be prepared on a selected subset of vertices, denoted I .

Let G be a graph with vertex set V and edge set \tilde{E} . Let $I \subset V$ be a set of distinguished vertices such that $|I| = n$ and $|V \setminus I| = N$. Let $|\psi\rangle$ be a state in the Hilbert space associated to the input vertices, \mathcal{H}_I . Let E denote the set of edges that are not entirely contained in I . The graph state with input is then

$$|G_\psi\rangle := \prod_{\{v,v'\} \in E} CZ_{v,v'}|\psi\rangle_I |+\rangle^{\otimes N}. \quad (6)$$

For any input state $|\psi\rangle$, the graph state with input is invariant under the application of any operation in the set $\{K_v : v \in V \setminus I\}$, where

$$K_v := X_v \otimes Z_{N_v^G}, \quad (7)$$

with N_v^G denoting the set of neighbors of vertex v in G and $Z_{N_v^G} := \otimes_{v' \in N_v^G} Z_{v'}$. The K_v are all mutually independent and the set $\{K_v : v \in V \setminus I\}$ generates a 2^n -dimensional subspace of \mathcal{H}_v , the graph codespace corresponding to G (see, e.g., [12,14] for further details on graph codes [23]).

In the measurement-based regime, computation is driven by single-qubit projective measurements restricted to the xy , xz , and yz planes of the Bloch sphere. A given computation is defined by one specific outcome for each measurement, and the restriction to the given planes allows for the correction of undesired outcomes via an effective application of an appropriate stabilizer element (or products thereof). However, even with these restrictions not every sequence of measurements for a given graph state is possible. The combination of graph state and measurements that do allow for computation are well characterized by a property called gflow which is known to be a necessary and sufficient condition for deterministic MBQC [22] (see Ref. [24] for the definition).

Results.—It is known that every stabilizer code is equivalent to a graph code [13] (see also [25]). The following is an instance of this result using the specific properties exhibited by parity codes.

Proposition 1.—Every parity code is local Clifford equivalent to a bipartite graph code, where all data qubits are contained in one partition.

Proof.—A parity code is defined by a stabilizer generated by the operators $K'_{(ij\dots k)} = Z_{(ij\dots k)} \otimes Z_i \otimes Z_j \otimes \dots \otimes Z_k$ for each parity qubit $(ij\dots k)$. The set of qubits upon which these operators act includes only a single parity qubit. Via conjugation by Hadamards on each parity qubit, we obtain the local Clifford equivalent stabilizer generated by $K_{(ij\dots k)} = X_{(ij\dots k)} \otimes Z_i \otimes Z_j \otimes \dots \otimes Z_k$. Since the $K_{(ij\dots k)}$ are of the form $X_{(ij\dots k)} \otimes Z_{N_{(ij\dots k)}^G}$ which generate a graph code for a graph G with edges between parity and data qubits. That is, each neighborhood $N_{(ij\dots k)}^G$ contains only vertices corresponding to data qubits, which enforces a parity qubit-data qubit bipartition. ■

An example of this correspondence is shown in Figs. 1(a) and 1(b). Note that there exist graph codes that are not local Clifford equivalent to bipartite graph codes, and hence are inequivalent to any parity code.

An immediate consequence of Proposition 1 is that, for any $|\psi\rangle$, we have

$$|\text{LHZ}_\psi\rangle = \otimes_{v \in V \setminus I} H_v |G_\psi\rangle, \quad (8)$$

where G denotes the bipartite graph corresponding to the parity code, I denotes the set of vertices corresponding to data qubits and V is the set of all qubits. We will use $V \setminus I$ to denote the set of parity qubits forthwith.

Consider the parity computation U described in Eq. (3). For simplicity, let us first consider only the initial layer $l = 1$ and drop the parameters α , ϕ , and θ from the notation since the following holds for all parameter values. Using Eq. (8), we get that

$$U_{\text{data}}^{(1)} D_{\text{dec}}^{(1)} U_{\text{enc}} |0\rangle^{\otimes N} |\psi\rangle = U_{\text{data}}^{(1)} D_{\text{dec}}^{(1)} H_{V \setminus I} |G_\psi\rangle, \quad (9)$$

where $H_{V \setminus I}$ is shorthand for $\otimes_{v \in V \setminus I} H_v$. Both $D_{\text{dec}}^{(1)}$ and $H_{V \setminus I}$ act on the same qubits and can be simplified as

$$D_{\text{dec}}^{(1)} H_{V \setminus I} = \otimes_{v \in V \setminus I} \langle 0_v | R_{X_v}(\theta_v^{(1)}). \quad (10)$$

The operator $\langle 0_v | R_{X_v}(\theta_v^{(1)})$ is a measurement in the yz plane of the Bloch sphere, and hence $D_{\text{dec}}^{(1)} H_{V \setminus I} |G_\psi\rangle$ is precisely a measurement-based computation where all measurements are restricted to that plane (the issue of measurement corrections is covered below). Denote the output of the first layer as $|\psi^{(1)}\rangle$, which is the resultant state of applying $U_{\text{data}}^{(1)}$ to the output state of the MBQC. The remaining computation is then given by

$$\prod_{l=2}^L \left(U_{\text{data}}^{(l)} D_{\text{dec}}^{(l)} U_{\text{enc}} |0\rangle^{\otimes N} \right) |\psi^{(1)}\rangle \quad (11)$$

for which the above process can be repeated. We have thus shown the following:

Theorem 1.—Universal parity quantum computing is repeated measurement-based quantum computation using yz -plane measurements, interleaved with local rotations.

It should be noted that typically, MBQC is done on a fully pre-prepared graph state where input I and output O are distinct. However, proposals for repeated MBQC which de- and re-encode graph codes have been considered previously [26]; see also [20] for a recent perspective.

In light of the above, it is prudent to demarcate the parity computing regime with respect to the MBQC regime. One could reasonably ask if there exist yz -only measurement-based computations on graphs that are not bipartite. However, the following theorem demonstrates that this in fact not the case. The theorem also takes care of any issues regarding correction of measurements (see Ref. [24] for more details).

As mentioned above, MBQC on a graph state G typically includes specifying an input and output set of vertices, denoted by I and O , respectively. For a deterministic MBQC to produce a unitary transformation (as opposed to an isometry), we require $|I| = |O|$.

Theorem 2.—MBQC on a (simple, connected) graph G with $|I| = |O|$ and using only yz -plane measurements is deterministic if and only if G is bipartite with I forming one partition.

The proof makes use of technical lemmas related to gflow which are proved along with the theorem in the Supplemental Material [24].

Discussion.—This Letter has demonstrated that (i) parity codes are local Clifford equivalent to bipartite graph codes, (ii) as a consequence, parity quantum computing can be understood as repeated MBQC where all measurements are made in the yz plane, supplemented by local rotations, and (iii) MBQC with equivalent input and output qubits and using only yz -plane measurements must use a bipartite graph state.

Interestingly, these results demonstrate that the universal parity computing regime has effectively singled out yz -plane unitary MBQC exactly. To the best of our knowledge, the restriction to having equivalent input and output and only yz measurements has not been considered before in the MBQC literature. On the other hand, this *is* a restriction of the full MBQC framework, which means that there is ample scope for future investigation into what other aspects of MBQC could be brought to bear on the parity computing regime, and vice versa.

As this Letter connects two previously distinct bodies of research, there are a number of consequences worth mentioning here. First, our results provide insight into recent research in the parity framework. In [21], it was noted that the parity measurement-based encoding and decoding procedures can be implemented in constant depth regardless of architecture size. Since the decoding

procedure corresponds to measuring vertices in one partition of a bipartite graph, it is clear that all measurements can be done simultaneously and corrected for in the other partition. The encoding procedure can be understood as measuring ancilla vertices of a larger graph state in the X basis, which in particular produces the required bipartite graph (see, e.g., [10,11] for a characterization of graph state deformations under Pauli measurements). It is known that all Pauli-measurements can be performed simultaneously in MBQC [18].

Second, there are a number of potential avenues for future research enabled by the results presented here. In the MBQC literature, there exist multiparticle entanglement purification protocols for bipartite graph states which exhibit favorable error thresholds for realistic scenarios [27–29]. Having demonstrated the prevalence of bipartite graph states in the parity framework, similar techniques may be of benefit for error mitigation in near-term implementations of parity quantum computations. Furthermore, proposals for fault-tolerant MBQC [30–33] and universal blind quantum computation [34] could provide the foundation for fault-tolerant and cryptographic implementations of the parity framework. Conversely, developments related to quantum optimization within the parity framework [35–37] could inspire similar developments in MBQC where application to optimization problems remains relatively unexplored. A number of these avenues are already being pursued in separate work.

We would like to thank B. Klaver and A. Messinger for useful discussions. This research was funded in whole or in part by the Austrian Science Fund (FWF) through the DK-ALM W1259-N27 and the SFB BeyondC F7102 [Grant DOI: 10.55776/F71]. This work was also co-funded by the European Union (ERC, QuantAI, Project No. 101055129).

Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

*isaac.smith@uibk.ac.at

- [1] J. Preskill, Quantum computing in the NISQ era and beyond, *Quantum* **2**, 79 (2018).
- [2] W. Lechner, P. Hauke, and P. Zoller, A quantum annealing architecture with all-to-all connectivity from local interactions, *Sci. Adv.* **1**, e1500838 (2015).
- [3] M. Fellner, A. Messinger, K. Ender, and W. Lechner, Universal parity quantum computing, *Phys. Rev. Lett.* **129**, 180503 (2022).
- [4] M. Fellner, A. Messinger, K. Ender, and W. Lechner, Applications of universal parity quantum computation, *Phys. Rev. A* **106**, 042442 (2022).

- [5] M. Fellner, K. Ender, R. ter Hoeven, and W. Lechner, Parity quantum optimization: Benchmarks, *Quantum* **7**, 952 (2023).
- [6] M. Drieb-Schön, K. Ender, Y. Javanmard, and W. Lechner, Parity quantum optimization: Encoding constraints, *Quantum* **7**, 951 (2023).
- [7] K. Ender, R. ter Hoeven, B. E. Niehoff, M. Drieb-Schön, and W. Lechner, Parity quantum optimization: Compiler, *Quantum* **7**, 950 (2023).
- [8] D. Gottesman, *Stabilizer Codes and Quantum Error Correction* (California Institute of Technology, Pasadena, California, 1997).
- [9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, England, 2010), 10.1017/CBO9780511976667.
- [10] M. Hein, J. Eisert, and H. J. Briegel, Multiparty entanglement in graph states, *Phys. Rev. A* **69**, 062311 (2004).
- [11] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H. J. Briegel, Entanglement in graph states and its applications, in *Volume 162: Quantum Computers, Algorithms and Chaos*, Proceedings of the International School of Physics “Enrico Fermi” (IOS Press Ebooks, Amsterdam, The Netherlands, 2006), pp. 115–218, 10.3254/978-1-61499-018-5-115.
- [12] D. Schlingemann, Logical network implementation for cluster states and graph codes, [arXiv:quant-ph/0202007](https://arxiv.org/abs/quant-ph/0202007).
- [13] D. Schlingemann, Stabilizer codes can be realized as graph codes, [arXiv:quant-ph/0111080](https://arxiv.org/abs/quant-ph/0111080).
- [14] D. Schlingemann and R. F. Werner, Quantum error-correcting codes associated with graphs, *Phys. Rev. A* **65**, 012308 (2001).
- [15] R. Raussendorf and H. J. Briegel, A one-way quantum computer, *Phys. Rev. Lett.* **86**, 5188 (2001).
- [16] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, *Nat. Phys.* **5**, 19 (2009).
- [17] R. Raussendorf and H. Briegel, Computational model underlying the one-way quantum computer, [arXiv:quant-ph/0108067](https://arxiv.org/abs/quant-ph/0108067).
- [18] R. Raussendorf, D. E. Browne, and H. J. Briegel, Measurement-based quantum computation on cluster states, *Phys. Rev. A* **68**, 022312 (2003).
- [19] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, Experimental one-way quantum computing, *Nature (London)* **434**, 169 (2005).
- [20] H. Poulsen Nautrup and H. J. Briegel, Measurement-based quantum computation from clifford quantum cellular automata, [arXiv:2312.13185](https://arxiv.org/abs/2312.13185).
- [21] A. Messinger, M. Fellner, and W. Lechner, Constant depth code deformations in the parity architecture, in *Proceedings of the 2023 IEEE International Conference on Quantum Computing and Engineering (QCE)* (2023), Vol. 01, pp. 120–130, 10.1109/QCE57702.2023.00022.
- [22] D. E. Browne, E. Kashefi, M. Mhalla, and S. Perdrix, Generalized flow and determinism in measurement-based quantum computation, *New J. Phys.* **9**, 250 (2007).
- [23] Just as [3] modified the original LHZ architecture [2] to contain data qubits, we are considering graph codes where the input qubits remain unmeasured, a modification to be removed in future work.
- [24] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.132.220602> for the definition of gflow and the proof of Theorem 2.
- [25] M. Van den Nest, J. Dehaene, and B. De Moor, Graphical description of the action of local clifford transformations on graph states, *Phys. Rev. A* **69**, 022316 (2004).
- [26] M. Zwerger, H. Briegel, and W. Dür, Hybrid architecture for encoded measurement-based quantum computation, *Sci. Rep.* **4**, 5364 (2014).
- [27] W. Dür, H. Aschauer, and H.-J. Briegel, Multiparticle entanglement purification for graph states, *Phys. Rev. Lett.* **91**, 107903 (2003).
- [28] H. Aschauer, W. Dür, and H.-J. Briegel, Multiparticle entanglement purification for two-colorable graph states, *Phys. Rev. A* **71**, 012319 (2005).
- [29] M. Zwerger, H. J. Briegel, and W. Dür, Universal and optimal error thresholds for measurement-based entanglement purification, *Phys. Rev. Lett.* **110**, 260503 (2013).
- [30] A. Bolt, G. Duclos-Cianci, D. Poulin, and T. M. Stace, Foliated quantum error-correcting codes, *Phys. Rev. Lett.* **117**, 070501 (2016).
- [31] N. Nickerson and H. Bombín, Measurement based fault tolerance beyond foliation, [arXiv:1810.09621](https://arxiv.org/abs/1810.09621).
- [32] R. Raussendorf and J. Harrington, Fault-tolerant quantum computation with high threshold in two dimensions, *Phys. Rev. Lett.* **98**, 190504 (2007).
- [33] R. Raussendorf, J. Harrington, and K. Goyal, A fault-tolerant one-way quantum computer, *Ann. Phys. (Amsterdam)* **321**, 2242 (2006).
- [34] A. Broadbent, J. Fitzsimons, and E. Kashefi, Universal blind quantum computation, in *Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science* (IEEE, New York, 2009), pp. 517–526.
- [35] M. Lanthaler, C. Dłaska, K. Ender, and W. Lechner, Rydberg-blockade-based parity quantum optimization, *Phys. Rev. Lett.* **130**, 220601 (2023).
- [36] C. Dłaska, K. Ender, G. B. Mbeng, A. Kruckenhauser, W. Lechner, and R. van Bijnen, Quantum optimization via four-body rydberg gates, *Phys. Rev. Lett.* **128**, 120503 (2022).
- [37] K. Ender, A. Messinger, M. Fellner, C. Dłaska, and W. Lechner, Modular parity quantum approximate optimization, *PRX Quantum* **3**, 030304 (2022).