Biorthogonal Dynamical Quantum Phase Transitions in Non-Hermitian Systems

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By utilizing biorthogonal bases, we develop a comprehensive framework for studying biorthogonal dynamical quantum phase transitions in non-Hermitian systems. With the help of the previously overlooked associated state, we define the automatically normalized biorthogonal Loschmidt echo. This approach is capable of handling arbitrary non-Hermitian systems with complex eigenvalues and naturally eliminates the negative value of Loschmidt rate obtained without the biorthogonal bases. Taking the non-Hermitian Su-Schrieffer-Heeger model as a concrete example, a $1/2$ change of dynamical topological order parameter in biorthogonal bases is observed which is not shown in self-normal bases. Furthermore, we discover that the periodicity of biorthogonal dynamical quantum phase transitions depends on whether the two-level subsystem at the critical momentum oscillates or reaches a steady state.

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The past decades have witnessed the flourishing of non-Hermitian physics in nonconservative systems as found in a variety of physical realms including open quantum systems [1], electronic systems with interactions [2–4], and classical systems with gain or loss [5–7]. In these systems [8–23], many novel physics and unprecedented phenomena have been explored recently, such as the exceptional points [24,25], the non-Hermitian skin effects [26,27], the bulk Fermi arcs [28], and so on. In contrast to the Hermitian systems with real eigenvalues and orthogonal eigenstates, the eigenvalues and eigenstates in a general non-Hermitian Hamiltonian are not necessarily real and orthogonal [29]. To be more precise, the orthogonality of eigenstates is replaced by the notion of biorthogonality that defines the relation between the Hilbert space of states and its dual space, leading to the so-called "biorthogonal quantum mechanics" [30–33]. A direct consequence of the biorthogonality is that the transition probability between one state $|\phi\rangle$ to its time-evolved state $|\phi(t)\rangle$ should be carefully defined. The traditional viewpoint $p = |\langle\phi(t)|\phi\rangle|^2$ used in Hermitian systems cannot be applied to a general non-Hermitian Hamiltonian. A schematic framework to deal with this becomes an urgent topic due to the rapid development of nonequilibrium studys in non-Hermitian systems [34–45].

Dynamical quantum phase transition (DQPT) is arguably one of the most important nonequilibrium phenomena in modern many-body physics and has also been extensively studied in past decades [46–50]. It was first introduced in the Hermitian transverse field Ising model [46] and was generalized to mixed state [51,52], finite temperature [53–57], Floquet systems [58–61], and slow quench process [62]. It was also observed experimentally with trapped ions [63,64], Rydberg atoms [65], ultracold atoms [66], superconducting qubits [67], nanomechanical and photonic systems [68–70]. The key quantity characterizing DQPT is the Loschmidt echo or dynamical fidelity, $\mathcal{L}(t) \equiv |\langle\Psi(0)|\Psi(t)\rangle|^2$, quantifying the time-dependent deviation from an arbitrary initial state $|\Psi(0)\rangle$. In Hermitian systems, DQPTs occur whenever the time-evolved state $|\Psi(t)\rangle$ becomes orthogonal to the initial state $|\Psi(0)\rangle$ and the critical time t_c is defined as $\mathcal{L}(t_c) = 0$. Efforts have been made to generalize this concept to non-Hermitian systems, leading to many interesting predictions such as the half-integer jumps in dynamical topological order parameter (DTOP) [71], but the special biorthogonality has been ignored and an enforced normalized factor has been used [71–74]. Very recently, it has been shown that the non-Hermitian systems should be described by the biorthogonal fidelity and the biorthogonal Loschmidt echo instead of the conventional counterparts in Hermitian systems [75,76], but it is limited to the parity-time symmetry cases [77,78]. A natural treatment for a general non-Hermitian Hamiltonian is still lacking, which severely limits our exploration of the richness of DQPTs in these systems.

In this Letter, we address this issue by proposing a new theoretical framework to deal with these nonequilibrium phenomena in general non-Hermitian systems. Based on biorthogonal quantum mechanics, we reformulate the transition probability between $|\Psi(0)\rangle$ and $|\Psi(t)\rangle$ with the biorthogonal bases and the associated states. We introduce the concept of biorthogonal dynamical quantum phase transitions and compare it with its self-normal counterpart, utilizing the non-Hermitian Su-Schrieffer-Heeger model as a concrete example. We conduct a comprehensive investigation of the biorthogonal Loschmidt rate, biorthogonal DTOP, Fisher zeros, and the transition probability in momentum space during a sudden quench. Our calculations reveal a half-integer jump in the biorthogonal DTOP and demonstrate that the periodicity of biorthogonal DQPTs depends on whether the two-level subsystem at the critical momentum undergoes oscillation or settles into a steady state. Our theory establishes a general framework for studying nonequilibrium DQPTs in non-Hermitian systems.

We first review some basic properties of biorthogonal quantum mechanics in non-Hermitian systems and reformulate the probability assignment rules between two states with biorthogonal bases [30]. For a general non-Hermitian Hamiltonian $H \neq H^\dagger$, the eigenvalue equations of H and $H^\dagger$ are given by

$$\begin{aligned} H|u_n\rangle &= \epsilon_n|u_n\rangle, \qquad \langle u_n|H^\dagger = \epsilon_n^*\langle u_n|, \\ H^\dagger|\tilde{u}_n\rangle &= \epsilon_n^*|\tilde{u}_n\rangle, \qquad \langle\tilde{u}_n|H = \epsilon_n\langle\tilde{u}_n|, \end{aligned} \tag{1}$$

where ϵ_n is the nth eigenvalue, and $|u_n\rangle$ and $|\tilde{u}_n\rangle$ are the right and left eigenstates that satisfy the completeness relation $\sum_n |\tilde{u}_n\rangle\langle u_n| = 1$ and the biorthonormal relation $\langle\tilde{u}_m|u_n\rangle = \delta_{m,n}$. Note that under this condition $\langle\tilde{u}_n|u_n\rangle = 1$, eigenstates are no longer normalized. In particular, we have $\langle u_n|u_n\rangle \geq 1$ and $\langle u_n|u_m\rangle \neq 0$ if $n \neq m$. It challenges the traditional probabilistic interpretation used in Hermitian quantum mechanics. For instance, there cannot be a "transition" from one eigenstate $|u_m\rangle$ to another eigenstate $|u_n\rangle$ due to the orthonormal relation $\langle u_n|u_m\rangle = 0$ if $n \neq m$ in Hermitian systems. To reconcile these apparent contradictions we need the introduction of the so-called associated state and the redefinition of the inner product. For an arbitrary state $|\psi\rangle$, its associated state $|\tilde{\psi}\rangle$ is defined according to the following relation [30]:

$$|\psi\rangle = \sum_n c_n|u_n\rangle \leftrightarrow |\tilde{\psi}\rangle = \sum_n c_n|\tilde{u}_n\rangle, \tag{2}$$

while the dual state $\langle\tilde{\psi}| = \sum_n c_n^*\langle\tilde{u}_n|$ is given by the Hermitian conjugate of $|\tilde{\psi}\rangle$. The inner product between $|\psi\rangle$ and another state $|\phi\rangle = \sum_m d_m|u_m\rangle$ is thus defined as

$$\langle\phi,\psi\rangle \equiv \langle\tilde{\phi}|\psi\rangle = \sum_{m,n}\langle\tilde{u}_m|d_m^* c_n|u_n\rangle = \sum_n d_n^* c_n. \tag{3}$$

It is easy to show that the norm of a state $|\psi\rangle$ is $\sqrt{\langle\tilde{\psi}|\psi\rangle}$. With these new definitions, the transition probability between $|\psi\rangle$ and $|\phi\rangle$ for a biorthogonal system is given by

$$p = \frac{\langle\tilde{\psi}|\phi\rangle\langle\tilde{\phi}|\psi\rangle}{\langle\tilde{\psi}|\psi\rangle\langle\tilde{\phi}|\phi\rangle}, \tag{4}$$

where the denominator acts as a natural normalizing factor. In this case, p is a real number ranging from 0 to 1, which meets the requirements of probability interpretation satisfactorily. If the Hamiltonian is Hermitian $H = H^\dagger$, we have $|\tilde{u}_n\rangle = |u_n\rangle$ and $|\tilde{\psi}\rangle = |\psi\rangle$. Then Eq. (4) reduces to the conventional definition of the transition probability in Hermitian quantum mechanics $p = \langle\psi|\phi\rangle\langle\phi|\psi\rangle/(\langle\psi|\psi\rangle\langle\phi|\phi\rangle)$. Another important consequence of Eq. (4) is that the projection from $|\psi\rangle = \sum_n c_n|u_n\rangle$ to $|u_n\rangle$ becomes

$$p_n = \frac{\langle\tilde{\psi}|u_n\rangle\langle\tilde{u}_n|\psi\rangle}{\langle\tilde{\psi}|\psi\rangle\langle\tilde{u}_n|u_n\rangle} = \frac{c_n^* c_n}{\sum_m c_m^* c_m}, \tag{5}$$

satisfying the normalization condition $\sum_n p_n = 1$.

By now the static aspects of non-Hermitian systems are clear, but the dynamical problems remain controversial. Although the time evolution of an arbitrary initial state $|\Psi(0)\rangle$ under a time-independent Hamiltonian H is given by the well-known Schrödinger equation $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$, the direct generalization $|\tilde{\Psi}(t)\rangle = e^{-iH^\dagger t}|\tilde{\Psi}(0)\rangle$ [78] may lead to unreasonable complex probabilities if there is no parity-time symmetry [79]. To establish a general framework for non-Hermitian Hamiltonian with complex eigenvalues, we use the associated state $|\tilde{\Psi}(t)\rangle = \sum_n c_n|\tilde{u}_n\rangle$ where $c_n = \langle\tilde{u}_n|\Psi(t)\rangle$ to deal with the dynamical problem. The overlap between $|\Psi(0)\rangle$ and $|\Psi(t)\rangle$ is characterized by the biorthogonal Loschmidt echo,

$$\mathcal{L}(t) = \frac{\langle\tilde{\Psi}(0)|\Psi(t)\rangle\langle\tilde{\Psi}(t)|\Psi(0)\rangle}{\langle\tilde{\Psi}(t)|\Psi(t)\rangle\langle\tilde{\Psi}(0)|\Psi(0)\rangle}. \tag{6}$$

As in the Hermitian case, the biorthogonal DQPTs can be defined as $\mathcal{L}(t_c) = 0$ with the critical time t_c.

We may examine the biorthogonal DQPT by considering a general non-Hermitian Hamiltonian $H = \sum_k \psi_k^\dagger H_k \psi_k$ with $H_k = \boldsymbol{d}_k \cdot \boldsymbol{\sigma}$. Here, $\boldsymbol{\sigma}$ is the vector of the Pauli matrices, $\boldsymbol{d}_k = (x_k, y_k, z_k)$ represents the vector of expansion coefficients, and at least one component of $\boldsymbol{d}_k$ is complex in non-Hermitian systems. The eigenenergies of H_k are given by $\pm\epsilon_k = \pm\sqrt{\boldsymbol{d}_k^2}$, and the corresponding eigenstates are $|u_{k\pm}\rangle$. Considering a quench process where the model Hamiltonian changes from $\boldsymbol{d}_k^i$ at time $t = 0^-$ to $\boldsymbol{d}_k^f$ at time $t = 0^+$, the initial state $|\Psi(0)\rangle = \otimes_k |u_{k-}^i\rangle$ which is defined as the tensor product of all $|u_{k-}^i\rangle$ is evolved under the postquench Hamiltonian $\boldsymbol{d}_k^f$. The biorthogonal Loschmidt echo can be expressed as $\mathcal{L}(t) = \prod_k g_k(t)$ with [79]

$$g_k(t) = \frac{|\cos(\epsilon_k^f t) - i\sin(\epsilon_k^f t)\langle\tilde{u}_{k-}^i|\frac{H_k^f}{\epsilon_k^f}|u_{k-}^i\rangle|^2}{\langle\tilde{u}_{k-}^i(t)|u_{k-}^i(t)\rangle}, \tag{7}$$

where $|u_{k-}^i(t)\rangle = \mathrm{e}^{-iH_k^f t}|u_{k-}^i\rangle$. To obtain a nonzero and well-defined quantity in the thermodynamic limit it is useful to consider the biorthogonal Loschmidt rate

$$\mathrm{LR}(t) = -\lim_{N\to\infty}\frac{1}{N}\ln\mathcal{L}(t), \tag{8}$$

where N is the system size. Zeros in $\mathcal{L}(t)$ at critical times t_c correspond to nonanalyticities (cusps or divergencies) in

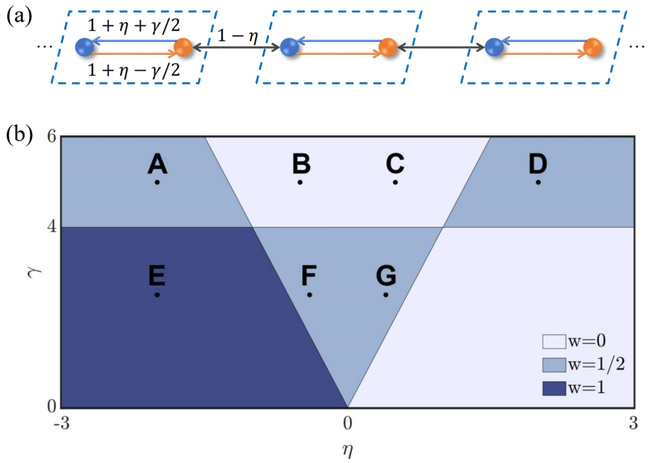

FIG. 1. (a) The sketch of non-Hermitian Su-Schrieffer-Heeger model. The box indicates the unit cell. (b) The $\gamma-\eta$ phase diagram. Each region filled with a different color corresponds to a phase with a topological winding number w. The boundaries between the regions are determined using $\epsilon_k = 0$. The black points represent different parameters (η, γ) which will be used below. Specifically, the points A to G represent $(-2, 5)$, $(-0.2, 5)$, $(0.2, 5)$, $(2, 5)$, $(-2, 1)$, $(-0.2, 1)$, and $(0.2, 1)$, respectively.

LR(t). If there is at least one pair of critical parameters k_c and t_c such that $g_{k_c}(t_c) = 0$, then $\mathcal{L}(t_c) = 0$. The solution of $g_{k_c}(t_c) = 0$ is

$$t_c = \frac{\pi}{2\epsilon^f_{k_c}}(2n+1) - \frac{i}{\epsilon^f_{k_c}}\tanh^{-1}\langle\tilde{u}^i_{k_c-}|\frac{H^f_{k_c}}{\epsilon^f_{k_c}}|u^i_{k_c-}\rangle, \tag{9}$$

where n is an integer. If we can obtain a positive real solution t_c, the system will undergo a biorthogonal DQPT. In general, it is difficult to access k_c in a finite-size system because momentum takes quantized values. Thus the divergence of LR(t) in the thermodynamic limit becomes a cusp in a finite-size system except for some fine-tuned quench parameters or twist boundary conditions [50]. For the same reason, it is also difficult to obtain a positive real solution t_c in a finite system. To address this issue, we can solve the equation $\mathcal{L}(t) = 0$ inversely and study the distribution of the roots in the complex time plane. The solution of $\mathcal{L}(t_c) = 0$ depends on the system size N. If for all $\epsilon > 0$, there exists a natural number N_0 such that $\mathrm{Im}[t_c(N)] < \epsilon$ holds for all $N > N_0$, a real solution $\lim_{N\to\infty}\mathrm{Im}[t_c(N)] = 0$ can be obtained, indicating a biorthogonal DQPT.

Furthermore, similar to the DTOP in Hermitian systems [80], we can introduce a biorthogonal DTOP to describe biorthogonal DQPT. The biorthogonal DTOP is defined as

$$\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} dk\partial_k\phi^G_k(t), \tag{10}$$

where the biorthogonal geometrical phase is $\phi^G_k(t) = \phi_k(t) - \phi^{\mathrm{dyn}}_k(t)$, with $\phi_k(t)$ being the phase of $g_k(t)$, and the biorthogonal dynamical phase is given by [79]

$$\phi^{\mathrm{dyn}}_k(t) = -\int_0^t ds\frac{\langle\tilde{u}^i_{k-}(s)|H^f_k|u^i_{k-}(s)\rangle}{\langle\tilde{u}^i_{k-}(s)|u^i_{k-}(s)\rangle} + \frac{\mathrm{i}}{2}\ln\langle\tilde{u}^i_{k-}(t)|u^i_{k-}(t)\rangle. \tag{11}$$

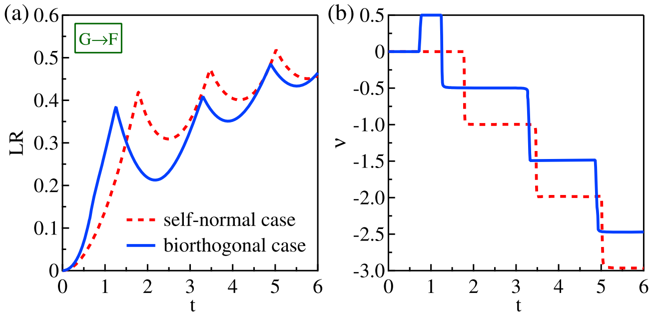

FIG. 2. (a) The Loschmidt rate and (b) the DTOP with self-normal and biorthogonal bases for a typical quench process (from G to F).

To demonstrate the efficacy of our novel theoretical framework in dealing with non-Hermitian Hamiltonians featuring complex eigenvalues, we present a comprehensive examination of the non-Hermitian Su-Schrieffer-Heeger model as outlined below. The Hamiltonian is

$$H = \sum_j\left[\left(1+\eta+\frac{\gamma}{2}\right)c^\dagger_{j,b}c_{j,a} + \left(1+\eta-\frac{\gamma}{2}\right)c^\dagger_{j,a}c_{j,b} + (1-\eta)c^\dagger_{j,a}c_{j+1,b} + (1-\eta)c^\dagger_{j+1,b}c_{j,a}\right], \tag{12}$$

where η determines the strength of intracell and intercell hopping and γ controls the degree of non-Hermiticity, as shown in Fig. 1(a). For the periodic boundary condition, the bulk Hamiltonian gets the standard bilinear form $H = \sum_k \psi^\dagger_k(\boldsymbol{d}_k\cdot\boldsymbol{\sigma})\psi_k$ in momentum space, where $\psi^\dagger_k = (c^\dagger_{k,a}, c^\dagger_{k,b})$ and

$$\boldsymbol{d}_k = \left((1+\eta)+(1-\eta)\cos k, (1-\eta)\sin k - i\frac{\gamma}{2}, 0\right). \tag{13}$$

The dispersion is $\pm\epsilon_k = \pm\sqrt{x_k^2 + y_k^2}$. It becomes gapless at the exceptional points. Thus the solution of $\epsilon_k = 0$ determines the phase boundary, i.e., $k_c = 0, \gamma = \pm 4$ and $k_c = \pi, \gamma = \pm 4\eta$. Combining with the winding number $w = (1/2\pi)\int_0^{2\pi} dk\partial_k(\arctan(y_k/x_k))$ [81], we present the phase diagram in Fig. 1(b) for convenience, where we only consider the case $\gamma > 0$ due to the symmetry of the phase diagram with respect to γ.

For comparison, we also calculate the traditional DQPT based on self-normal Loschmidt echo $\mathcal{L}(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2$,

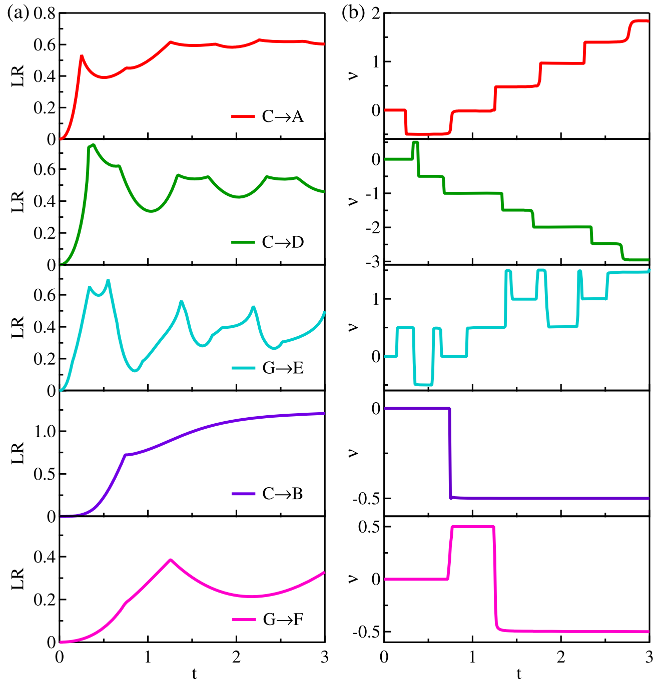

FIG. 3. Five different types of quenching processes showing a $\frac{1}{2}$ change in the biorthogonal DTOP. (a) and (b) illustrate the biorthogonal Loschmidt rate and biorthogonal DTOP, respectively. The $\frac{1}{2}$ change in the biorthogonal DTOP appears if and only if the prequench phases are in the middle of the phase diagram, with winding numbers of $\frac{1}{2}$ or 0.

with an enforced normalized factor to avoid the negative value of Loschmidt rate [72]. Figure 2(a) presents the self-normal and biorthogonal Loschmidt rate from the same quenching process. The first feature is that the critical time at which the cusp appears is different from each other. A similar phenomenon has been observed in equilibrium quantum phase transitions, where the biorthogonal and self-normal fidelity predict different quantum critical points [77]. It turns out that the biorthogonal fidelity captures the correct critical point due to the special biorthoganality in non-Hermitian systems [77]. Similarly, the nonequilibrium quantum phase transitions should be described by the biorthogonal time-dependent version of the fidelity, i.e., biorthogonal Loschmidt echo. The second feature is that there is an additional critical time $t_c \approx 0.74$ in biorthogonal bases. This can be seen more clearly in the DTOP as shown in Fig. 2(b). A half-integer jump of $\nu(t)$ can be observed in the biorthogonal bases while there is no such change in the self-normal bases.

To show that the half-integer jump of $\nu(t)$ is not a fine-tune result [82], we study different quenching processes by changing η but with γ fixed. Figures 3(a) and 3(b) present five typical behaviors of the biorthogonal Loschmidt rate $\mathrm{LR}(t)$ and biorthogonal DTOP $\nu(t)$, respectively. And more detailed information can be found in the Supplemental Material [79]. The half-integer jump of $\nu(t)$ can appear alone, periodically, or accompanied by an integer jump, exhibiting rich behavior in a single non-Hermitian Hamiltonian. We also find that the half-integer jump phenomenon occurs if and only if the prequench phases are in the middle of the phase diagram with 0 or 1/2 winding number.

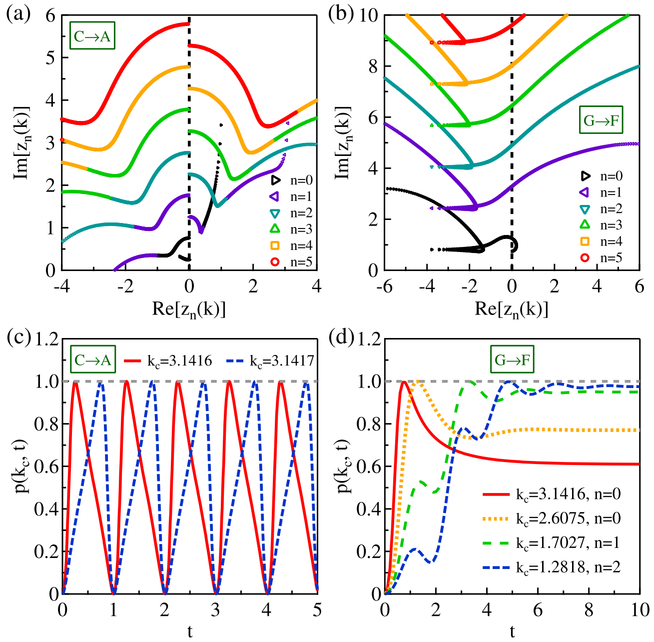

FIG. 4. (a),(b) The dynamical Fisher zeros in the complex plane $z_n(k) = \mathrm{i}t_n(k)$ with k lies in the first Brillouin zone for two typical quench processes. (c),(d) The corresponding transition probability $p(k_c, t)$ between $|u^i_{k_c-}(t)\rangle$ and $|u^i_{k_c+}\rangle$ at the critical momentum k_c.

To better understand the biorthogonal DQPTs, we study the dynamical counterpart of Fisher zeros [83–87] in the complex time plane $z_n(k) = \mathrm{i}t_n(k)$. The lines $z_n(k)$ cross the imaginary time axis at a critical momentum k_c and yield a critical time $t_{c,n}$, as shown in Figs. 4(a) and 4(b). The discontinuity of the dynamical Fisher zeros over the imaginary axis is associated with the half-integer jump of $\nu(t)$ [71]. Figures 4(a) and 4(b) also represent two typical types of biorthogonal DQPTs, depending on whether the lines $z_n(k)$ cut the time axis periodically or nonperiodically. In order to further study these two types of biorthogonal DQPTs, we also investigate the transition probability $p(k,t)$ between the time-evolved state $|u^i_{k-}(t)\rangle = \mathrm{e}^{-\mathrm{i}H^f_k t}|u^i_{k-}\rangle$ and another initial eigenstate $|u^i_{k+}\rangle$. By expanding $|u^i_{k-}(t)\rangle$ as $|u^i_{k-}(t)\rangle = c_1|u^i_{k-}\rangle + c_2|u^i_{k+}\rangle$, we have $p(k,t) = c_2^* c_2/(c_1^* c_1 + c_2^* c_2)$ from Eq. (5). If there exists $p(k,t) = 1$, the time-evolved state $|u^i_{k-}(t)\rangle$ is biorthogonal with $|u^i_{k-}\rangle$. Then the biorthogonal Loschmidt echo equals to zero because we can rewrite $\mathcal{L}(t)$ as

$$\mathcal{L}(t) = \prod_k \frac{\langle \tilde{u}^i_{k-}|u^i_{k-}(t)\rangle\langle \tilde{u}^i_{k-}(t)|u^i_{k-}\rangle}{\langle \tilde{u}^i_{k-}(t)|u^i_{k-}(t)\rangle}. \tag{14}$$

As shown in Figs. 4(c) and 4(d), the two types of biorthogonal DQPTs exhibit two distinct behaviors of $p(k,t)$. For the periodic biorthogonal DQPTs, $p(k_c,t)$ oscillate periodically between 0 and 1 for fixed critical momenta k_c. In this situation, the two-level systems of k_c dominate. Thus the periodicity of biorthogonal DQPTs may be related to the oscillations of the two-level system [88,89]. On the other hand, $p(k_c,t)$ exhibit very interesting behavior when the biorthogonal DQPTs are no longer periodic. There are many critical momenta k_c, and each k_c corresponds to only one $t_{c,n}$. Before $t_{c,n} = -\mathrm{i}z_n(k_c)$, there are n local maximum in $p(k_c,t)$, while for $t \gg t_{c,n}$, $p(k_c,t)$ tends to a fixed value, indicating a steady state in contrast to the oscillation behavior.

In summary, we propose a new theoretical framework to study the biorthogonal DQPTs in non-Hermitian systems based on the biorthogonal quantum mechanics. We reformulate the transition probability between one state $|\Psi(0)\rangle$ and its time-evolved state $|\Psi(t)\rangle$ with the concept of the associated state. Our scheme can handle a general non-Hermitian Hamiltonian with complex eigenvalues, and the normalization factors can be introduced naturally. To demonstrate our approach, we use the non-Hermitian Su-Schrieffer-Heeger model as a concrete example. Comparing with the self-normal cases, we observe a clear $1/2$ change in the biorthogonal DTOP. Furthermore, our results show that the periodicity of biorthogonal DQPTs depends on whether the two-level subsystem at the critical momentum oscillates or reaches a steady state. Our Letter opens up avenues for exploring the rich biorthogonal DQPTs in non-Hermitian systems. An interesting topic for further investigation would be the study of the mechanism behind the $1/2$ change in the biorthogonal DTOP.

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