Semi-Device-Independently Characterizing Quantum Temporal Correlations

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We develop a framework for characterizing quantum temporal correlations in a general temporal scenario, in which an initial quantum state is measured, sent through a quantum channel, and finally measured again. This framework does not make any assumptions on the system nor on the measurements, namely, it is device independent. It is versatile enough, however, to allow for the addition of further constraints in a semi-device-independent setting. Our framework serves as a natural tool for quantum certification in a temporal scenario when the quantum devices involved are uncharacterized or partially characterized. It can hence also be used for characterizing quantum temporal correlations when one assumes an additional constraint of no-signalling in time, there are upper bounds on the involved systems' dimensions, rank constraints—for which we prove genuine quantum separations over local hidden variable models—or further linear constraints. We present a number of applications, including bounding the maximal violation of temporal Bell inequalities, quantifying temporal steerability, and bounding the maximum successful probability in quantum randomness access codes.

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Bell's theorem [1] limits correlations that classical local hidden-variable models exhibit. This feature of quantum mechanics, referred to as nonlocality [2], is not only the defining feature that sets apart quantum from classical mechanics, but is also exploited in technological-minded applications. Notably, it can be used in new modes of quantum certification not requiring any (possibly unwarranted) assumptions on the states nor on the measurements. In such device-independent (DI) quantum certification [2–5], interestingly, data alone can be seen as being sufficient to certify properties. Along this line of thought, randomness certification [6], entanglement verification [7,8] and estimation [9], quantum state certification [10], steerability witnessing [11,12], and measurement incompatibility certification [13] have all been obtained through the observed nonlocal correlations only and no assumption is made on the shared quantum state nor the measurement involved. The Navascués-Pironio-Acín hierarchy [9,14–16]—building on earlier work [17,18]—has been a key tool in these efforts. The framework of device independence is compelling, in that one learns about properties of quantum systems without making assumptions about the devices with which these properties are being assessed.

That said, the original Bell scenario referring to spatial correlations is by no means the only setting that certifies quantum features beyond what local hidden-variable models can deliver. It has been extended to include temporal correlations, making reference to non-macro-realistic temporal correlations of single systems between two instances in time [19,20]. Leggett and Garg [21] showed that, in quantum theory, there exist non-macro-realistic temporal correlations. The original Leggett-Garg scenario is as follows: A quantum state is initially prepared and sent through a quantum channel. During the dynamics, the same measurement is performed at some, at least three, points in time. This has then been generalized to an identical preparation step, but followed by multiple choices of measurements at each point of time [22,23]. Such a setting is dubbed the temporal Bell scenario, since one may view it as a temporal analogue of the standard Bell scenario. Unlike the Leggett-Garg scenario, measurement outcomes between two points of time are sufficient to observe non-macro-realistic correlations. Like the situation in the Bell scenario, researchers are searching for a practical way to characterize quantum temporal correlations. The question is, given observed statistics in a temporal scheme, do there exist quantum states and measurements reproducing such statistics? Steps have been taken to characterize quantum temporal correlations in the standard Leggett-Garg scenario [24]. Nevertheless, characterizing quantum temporal correlations in the temporal Bell scenario remains an open problem, again with implications for device independence. Indeed, it is not even known whether such an approach can be pursued at all. From the practical point of view, temporal correlations play an essential role in modern quantum technologies. Famous instances include unitary evolution in quantum circuits and Bennett-Brassard [25] type quantum key distribution. Therefore, studying and characterizing temporal correlations advances the implementation of these cutting-edge technologies. Moreover, since many of them involve the issue of information security, providing a semi-device-independent framework renders them more practical or equips them with more stringent security promises.

In this Letter, we develop a framework called instrument moment matrices (IMMs) to characterize quantum temporal correlations in a temporal Bell scenario. The IMMs are matrices of expectation values of the postmeasurement states, where measurements are described by instruments. By construction, if the initial state and the measurements follow quantum theory, the IMMs are positive semidefinite. As such, quantum temporal correlations can be characterized by semidefinite programming [26]. Besides, the characterization will be more accurate when the size of the IMMs becomes larger (see Refs. [14,15] for the original idea behind such a hierarchical characterization and [9,11–13,27–30] for some variants). Our characterization is implemented both in a fully DI and semi-DI fashion that incorporates partial knowledge about the devices: We generalize the reading of semi-DI settings of Ref. [31] and advocate—complementing similarly motivated steps closer to the setting of fully specified devices of "semi-device-dependent" characterization [32]—that this intermediate regime is highly reasonable and important. By DI we mean that the results are based on the observed temporal correlations only, but no measurements and channels have to be specified *a priori*. In the temporal scenario, there is no way to rule out the possibility of sending information from an earlier time; therefore, we assume there are no side channels in our setting. However, since the space of temporal correlations is so abundant that temporal quantum correlations can, in general, be realized by classical ones [33,34], we have to add additional constraints to reveal quantum advantages. For this reason, we further consider (1) the constraint of no signaling in time, (2) the constraint on the system's dimension, and (3) the constraint on the system's rank. We show that IMMs allow us to characterize several quantum resources and tasks in DI and semi-DI scenarios. These include computing the maximal quantum violation of temporal Bell inequalities, estimating the degree of temporal steerability, computing the successful probabilities in scenarios of quantum randomness access codes, and identifying quantum state preparation. For including the rank constraint, to the best of our knowledge, this is the first work to enforce additional constraint apart from the dimensional constraint into a device-independent scenario [35].

The scenario.—First, we introduce the notion of an instrument. An instrument $\{\mathcal{J}_a^{A_1\to A_2}\}$ is a set of completely positive and trace nonincreasing maps mapping a quantum state ρ^{A_1} to a postmeasurement state $\mathcal{J}_a^{A_1\to A_2}(\rho^{A_1})$ where $a\in\mathcal{A}=\{0,1,2,\ldots\}$ can be treated as the assigned outcome associated with the state $\mathcal{J}_a^{A_1\to A_2}(\rho^{A_1})$. The probability of obtaining the outcome a, denoted by $P(a)$, can be computed via $P(a)=\mathrm{tr}[\mathcal{J}_a^{A_1\to A_2}(\rho^{A_1})]$, therefore, one has $\mathrm{tr}\sum_a\mathcal{J}_a^{A_1\to A_2}(\rho^{A_1})=\mathrm{tr}(\rho^{A_1})$ due to the normalization.

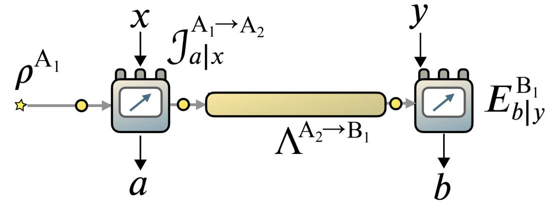

FIG. 1. The scenario considered in this Letter.

In our scenario, we can choose different instruments to measure the state. We use the notation $\{\mathcal{J}_{a|x}^{A_1\to A_2}\}$ to denote the collection of instruments, where $x\in\mathcal{X}=\{0,1,2,\ldots\}$ labels the choice of measurement settings (see Fig. 1). The postmeasurement state $\mathcal{J}_{a|x}^{A_1\to A_2}(\rho^{A_1})$ is then submitted into a quantum channel $\Lambda^{A_2\to B_1}$. Finally, the evolved state is measured by another measurement. At this stage, we only care about the outcome, and hence the measurements can be described by positive operator-valued measures (POVMs) $\{E_{b|y}^{B_1}\}$ that are positive semidefinite $E_{b|y}^{B_1}\succeq 0$ and normalized as $\sum_b E_{b|y}^{B_1}=\mathbb{1}$, where $b\in\mathcal{B}$ and $y\in\mathcal{Y}$ denote the measurement outcome and setting, respectively. By repeating the above experiment, we observe a set of probabilities $\{P(a,b|x,y)\coloneqq P(b|a,x,y)P(a|x)\}$, termed temporal correlations, which can be obtained by the Born rule

$$\begin{aligned}P(a,b|x,y)&=\mathrm{tr}\left\{E_{b|y}^{B_1}\left[\Lambda^{A_2\to B_1}\left(\mathcal{J}_{a|x}^{A_1\to A_2}(\rho^{A_1})\right)\right]\right\}\\&=\mathrm{tr}\left[E_{b|y}^{B_1}\mathcal{I}_{a|x}^{A_1\to B_1}(\rho^{A_1})\right]\end{aligned}\qquad(1)$$

where $\{\mathcal{I}_{a|x}^{A_1\to B_1}\coloneqq\Lambda^{A_2\to B_1}\circ\mathcal{J}_{a|x}^{A_1\to A_2}\}_a$ is a valid instrument for each x.

The instrument moment matrices and their DI formulation.—The IMMs are constructed by applying complete-positive maps $\mathcal{E}$ on the postmeasurement states $\mathcal{I}_{a|x}^{A_1\to B_1}(\rho^{A_1})$, i.e., $\mathcal{E}[\mathcal{I}_{a|x}^{A_1\to B_1}(\rho^{A_1})]=\sum_n K_n[\mathcal{I}_{a|x}^{A_1\to B_1}(\rho^{A_1})]K_n^\dagger$ with $K_n\coloneqq\sum_i|i\rangle_{\bar{B}_1B_1}\langle n|S_i$ being the Kraus operators. Here, $\{|i\rangle_{\bar{B}_1}\}$ and $\{|j\rangle_{B_1}\}$ are orthonormal bases for the output space and input space, respectively. Following Ref. [9], given a level ℓ we choose $\{S_i\}$ as $\mathbb{1}\cup\mathcal{S}^{(1)}\cup\mathcal{S}^{(2)}\cup\ldots\cup\mathcal{S}^{(\ell)}$, where $\mathcal{S}^{(\ell)}$ is composed of the ℓth order products of the operators in the set $\{E_{b|y}^{B_1}\}_{b=1,\ldots,|\mathcal{B}|-1}^{y=1,\ldots,|\mathcal{Y}|}$. The ℓth-level IMMs can be defined as

$$\chi_{a|x}^{(\ell)}\coloneqq\mathcal{E}[\mathcal{I}_{a|x}(\rho^{A_1})]=\sum_{i,j}|i\rangle\langle j|\mathrm{tr}\left[\mathcal{I}_{a|x}(\rho^{A_1})S_j^\dagger S_i\right].\qquad(2)$$

Therefore, the entry of the ith row and jth column of $\chi_{a|x}^{(\ell)}$ can be treated as the "expectation value" of the product of

$S_j^\dagger$ and S_i given the state $\mathcal{I}_{a|x}^{\mathrm{A_1}\to\mathrm{B_1}}(\rho^{\mathrm{A_1}})$. In the Supplemental Material, Appendix C [37], we explicitly provide an example of IMMs. Note that the IMMs are positive semidefinite whenever $\mathcal{I}_{a|x}$, ρ, $E_{b|y}^{\mathrm{B_1}}$ are quantum realizable: The constraints of positive semidefiniteness $\chi_{a|x}^{(\ell)} \succeq 0$ serve as a natural characterization of the quantum set of temporal correlations $\{P(a,b|x,y)\}$. The characterization is improved when the level ℓ increases. When the improvement is hard to be observed from a level ℓ_c, we say $\chi_{a|x}^{(\ell_c)}$ provides a proper approximation of the quantum set of temporal correlations. We will from now on use the notation $\chi_{a|x}$ to simply denote $\chi_{a|x}^{(\ell)}$.

When focusing on temporal correlations, quantum systems do not "outperform" classical systems in that a classical system with a sufficiently high dimension carries information allowing observers at later time to obtain. The simplest scheme is that an observer at earlier time can just send all the information about the measurement settings and outcomes to an observer at later time, then the correlation space will be filled by such a strategy. To let quantum systems demonstrate their superior performance, a constraint is to limit their dimension. By doing so, it has been shown that quantum systems outperform classical systems with the same dimension [43]. If we require that the entire system is embedded in dimension at most d, we have $P(a,b|x,y)=\mathrm{tr}\{E_{b|y}^{\mathrm{B_1}}[\mathcal{I}_{a|x}^{\mathrm{A_1}\to\mathrm{B_1}}(\rho^{\mathrm{A_1}})]\}$, with $\rho^{\mathrm{A_1}}\in\mathcal{L}(\mathcal{H}_d^{\mathrm{A_1}})$, $\mathcal{I}_{a|x}^{\mathrm{A_1}\to\mathrm{B_1}}$: $\mathcal{L}(\mathcal{H}_d^{\mathrm{A_1}})\to\mathcal{L}(\mathcal{H}_d^{\mathrm{B_1}})$, and $E_{b|y}^{\mathrm{B_1}}\in\mathcal{D}(\mathcal{H}_d^{\mathrm{B_1}})$. Following the idea of Ref. [36], the set of probabilities $P(a,b|x,y)$ generated by d-dimensional systems can be characterized by embedding IMMs into dimension-restricted IMMs, namely, $\{\chi_{a|x}\}_{a,x}\in\mathcal{G}_d$ where $\mathcal{G}_d$ is the set of IMMs composed of d-dimensional quantum systems.

The second kind of constraint we would like to impose is an upper bound on the rank of Bob's measurements. To this end, when generating Bob's d-dimensional POVMs $E_{b|y}^{\mathrm{B_1}}$, we generate $E_{b|y}^{\mathrm{B_1}}$ with rank k only, namely, $\mathrm{Rk}(E_{b|y}^{\mathrm{B_1}})=k$, where $\mathrm{Rk}(\cdot)$ denotes the rank. We denote with $\mathcal{G}_d^k$ the set of IMMs with such a construction, i.e., $\{\chi_{a|x}\}_{a,x}\in\mathcal{G}_d^k$. In our method, the rank constraint cannot be considered alone without the dimensional constraint. The reason is that when generating the POVM elements $E_{b|y}^{\mathrm{B_1}}$, the dimension of them is automatically defined. In the same sense, in the typical dimension-constraint scenario, one implicitly sets the upper bound on the rank of measurements to be full rank. The final constraint we would like to consider is the so-called no signaling in time (NSIT). Such a constraint states that the observer at earlier time cannot transmit information by changing the measurement settings, i.e., $\sum_a P(a,b|x,y)=\sum_a P(a,b|x',y)$, yielding $\sum_a\chi_{a|x}=\sum_a\chi_{a|x'}\ \forall\, x\neq x'$. Since no information is transmitted between two observers at different points of time, the NSIT constraint in the temporal scenario is in general the same as the typical (i.e., spatial) Bell scenario.

Now we have four types of constraints used for characterizing quantum sets of temporal correlations: the device-independent (DI), DI + dimensional, DI + rank, and NSIT constraints. They are respectively denoted as (1) DI: $\chi_{a|x}\succeq 0$; (2) DI + dim: $\chi_{a|x}\succeq 0$, $\{\chi_{a|x}\}_{a,x}\in\mathcal{G}_d$; (3) DI + dim +rank: $\chi_{a|x}\succeq 0$, $\{\chi_{a|x}\}_{a,x}\in\mathcal{G}_d^k$; (4) NSIT: $\chi_{a|x}\succeq 0$, $\sum_a\chi_{a|x}=\sum_a\chi_{a|x'}\ \forall\, x\neq x'$.

Quantum upper bounds on temporal Bell inequalities.— To demonstrate that the IMMs provide a proper characterization, we first show that the IMMs can be used to compute an upper bound on the maximal quantum violation of a temporal Bell inequality. This result is also crucial from the practical point of view since we have to make sure that the temporal Bell inequality used for certifying nonclassicality (i.e., a non-macro-realistic dynamics [19]) provides different bounds for quantum and classical models. To simplify the problem, we consider the temporal Clauser-Horne-Shimony-Holt (CHSH) scenario [22,23,44,45], i.e., the scenario with binary settings and outcomes. The generalization to arbitrary scenarios can be straightforwardly obtained. The temporal CHSH inequality is written as

$$K_{\mathrm{CHSH}} \coloneqq \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \leq 2, \quad (3)$$

where $\langle A_xB_y\rangle \coloneqq P(a=b|x,y)-P(a\neq b|x,y)$. The bound with the value of 2 is obtained from the so-called macrorealistic model [19,20]. As has been known, the inequality can be violated since quantum physics does not admit a macrorealistic model. An quantum upper bound on the inequality can be computed via semidefinite programming [26]

$$\max\{K_{\mathrm{CHSH}}|\chi_{a|x}\succeq 0,\ \forall\, a,x\}. \quad (4)$$

The solution gives us the value of 4, the maximal algebraic value. This coincides with one of the results in [46], which states that any correlation admitting the arrow of time can always be realized by quantum theory [47]. Even when we consider the dimensional constraint, i.e., the DI + dim constraint with $d=2$, the tight quantum upper bound on K_{CHSH} is still 4. The bound is tight since there exists a quantum realization to achieve the bound. It is interesting to note that if we further restrict Bob's POVMs to be rank 1, i.e., the DI + dim +rank constraint with $(d,k)=(2,1)$, the upper bound on K_{CHSH} will be within the numerical precision with $2\sqrt{2}$, same with the Tsirelson bound [48] in the spatial CHSH scenario. Finally, if we consider the NSIT constraint, the scenario will be the same as that of the spatial CHSH; that is, two-way communication is forbidden. The upper bound on K_{CHSH} we obtain is within the numerical precision with the Tsirelson bound [48], $2\sqrt{2}$. The different quantum bounds for the latter two with the former two schemes provide an important application: Exceeding the value of $2\sqrt{2}$ sufficiently identifies at least one of the following three facts: (1) the underlying qubit measurements are not one rank (i.e., full rank), (2) the dimension of the system is beyond qubit, and (3) there exists one-way communication.

Bounding the degree of temporal steerability.—The idea of temporal steerability was first proposed in Ref. [49]. The works of Refs. [50–52] have reformulated the classical model in [49] by introducing the hidden-state model [53]. In our formulation, the hidden-state model is described by (see also Ref. [54]) $\mathcal{I}_{a|x}(\rho) = \sum_\lambda P(\lambda)P(a|x,\lambda)\sigma_\lambda$, where $P(\lambda)$, $P(a|x,\lambda)$ are probabilities and σ_λ are quantum states. The equation above tells us that the postmeasurement states $\mathcal{I}_{a|x}(\rho)$ are simply a classical postprocessing of the set of fixed states σ_λ. In quantum theory, there exist instruments $\mathcal{I}_{a|x}$ such that the postmeasurement states $\mathcal{I}_{a|x}(\rho)$ do not admit a hidden-state model. The incompatibility with a hidden-state model is called temporal steering. Here, we show that by observing the statistics $P(a,b|x,y)$, we are still capable of bounding the degree of temporal steerability in DI and semi-DI scenarios. See Appendix D [37] for the detailed derivation and computational results. Very recently, it has been shown that temporal steerability has a physical meaning: it is equivalent to the time a thermodynamic bath requires to bring the states $\mathcal{I}_{a|x}(\rho)$ to the thermal state [55]. Therefore, our method can also be used for device-independently estimating the thermalization time.

Characterization of quantum randomness access codes.—In the $n \to 1$ random access code (RAC) scenario, an observer (Alice) has n bits of information, denoted by $\vec{x} = (x_0, x_1, \ldots x_y, \ldots, x_{n-1})$ with $x_i \in \{0,1\}$. She then encodes them into a single bit and sends it to the other observer (Bob) who is queried for guessing Alice's yth bit. Their goal is to maximize Bob's guessing probability, i.e., $P(b = x_y|\vec{x}, y)$, where b is Bob's guess (see Fig. 2). We denote with $\mathcal{P}^{\mathrm{C}}_{n\to1}$ the maximum average (over all x_y and y) successful probability by a classical strategy. It has been shown that $\mathcal{P}^{\mathrm{C}}_{2\to1} = \mathcal{P}^{\mathrm{C}}_{3\to1} = 3/4$. In quantum theory, Alice's n bits of information are encoded in the way of

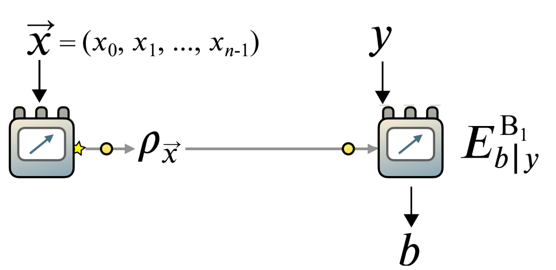

FIG. 2. The $n \to 1$ quantum randomness access codes (QRACs).

quantum state preparation, i.e., for each given $\vec{x}$, she sends the associated quantum state $\rho_{\vec{x}}$ to Bob. Bob then performs his yth quantum measurement, described by a POVM $\{E_{b|y}\}_b$, on the state. The quantum realization of the guessing probability will be $P(b = x_y|\vec{x}, y) = \mathrm{tr}(E_{b|y}\rho_{\vec{x}})$. Denoting $\mathcal{P}^{\mathrm{Q}}_{n\to1}$ as the maximum average successful probability by a quantum strategy, it has been shown that $\mathcal{P}^{\mathrm{Q}}_{2\to1} = \frac{1}{2}(1 + 1/\sqrt{2}) \approx 0.8536$ and $\mathcal{P}^{\mathrm{Q}}_{3\to1} = \frac{1}{2}(1+1/\sqrt{3}) \approx 0.7887$. In Appendix E [37], we show how to use IMMs to recover these quantum bounds.

Self-testing quantum states in a prepare-and-measure scenario.—Finally, we show that the IMMs can be used to verify a set of quantum states in a semi-DI way. More explicitly, we consider the QRAC scenario in the last section and uniquely (up to some isometries) identify the underlying set of states $\rho_{\vec{x}}$ by the observed probabilities $P(b|\vec{x}, y)$ only. Such identification, called self-testing in a prepare-and-measure scenario, has been proposed in Refs. [56–58]. We here provide an alternative approach to achieve the task. A robust self-testing can be defined as follows [56,59]). Given an upper bound d on the dimension of the systems involved, we say that the observed correlation $\vec{P} := \{P(b|\vec{x}, y)\}_{b,\vec{x},y}$ robustly self-tests, in a prepare-and-measure scenario, the reference set of states $\vec{\rho}_{\mathrm{ref}} := \{\rho^{\mathrm{ref}}_{\vec{x}}\}_{\vec{x}}$ at least with a fidelity f if for each set of states $\vec{\rho} := \{\rho_{\vec{x}} \in \mathcal{H}_d\}_{\vec{x}}$ compatible with $\vec{P}$ there exists a completely positive and trace-preserving map Λ, such that $F[\vec{\rho}_{\mathrm{ref}}, \Lambda(\vec{\rho})] \geq f$. Here, $\Lambda(\vec{\rho})$ represents $\Lambda(\rho_{\vec{x}})$ for all $\vec{x}$ and $F(\vec{\rho}, \vec{\sigma})$ is the fidelity between two sets of states $\vec{\rho}$ and $\vec{\sigma}$.

To compute $F[\vec{\rho}_{\mathrm{ref}}, \Lambda(\vec{\rho})]$ in a DI way, we use a method similar to that of Ref. [60]. The fidelity can then be written as a polynomial where each monomial is of the form $\mathrm{tr}(\rho_{\vec{x}} S_j^\dagger S_i)$ with S_i being Bob's observables or their products (see Appendix F [37]). Given the observed correlation $\vec{P}$, a DI bound on $F[\vec{\rho}_{\mathrm{ref}}, \Lambda(\vec{\rho})]$, denoted as F^{DI}, can be computed as

$$\min\{F^{\mathrm{DI}}[\vec{\rho}_{\mathrm{ref}}, \Lambda(\vec{\rho})] | \chi_{\vec{x}} \succeq 0, \chi_{\vec{x}} \in \mathcal{G}^k_d\}. \quad (5)$$

We consider the example of a $2 \to 1$ scenario, where the reference preparation is chosen as a unitary equivalent to $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$, implying $d = 2$. We assume the measurement to be projective (as most works do), so that $k = 1$. The result is presented by the blue solid line in Fig. 3. The observed correlation $\vec{P}$ is represented by the average successful probability $\mathcal{P}_{2\to1} := \frac{1}{8}\sum_{x_0,x_1,y} P(b = x_y|x_0, x_1, y)$. Given the maximal quantum value of $\mathcal{P}_{2\to1} = \mathcal{P}^{Q}_{2\to1}$, we perfectly self-test the reference set of states with fidelity equal to 1. When $\mathcal{P}_{2\to1}$ is below around 0.8232, we no longer have a self-testing statement, since the fidelity is below the classical fidelity 0.8536 (see Appendix G [37]) We also compare our result with the optimal bounds proposed by Tavakoli *et al.* [56].

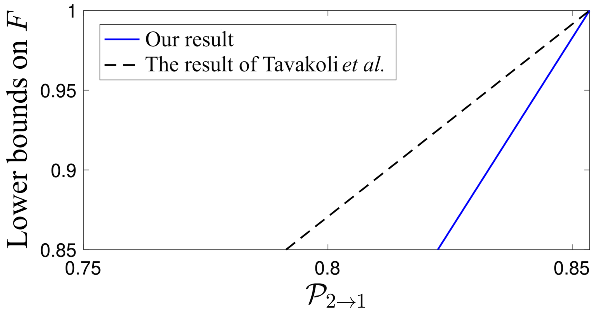

FIG. 3. Robust self-testing the reference set of states in the prepare-and-measure scenario.

Generalization to multiple-time and many-body scenarios.—Our method can be straightforwardly generalized to two scenarios. The first is considering multiple measurements acting on a single system at different time. For the second scenario, if we are interested in the time evolution of a many-body system, we can go back to our original setting depicted in Fig. 1 and replace the initial state with a many-body state. For both generalizations, the mathematical constructions of the IMMs are similar to the standard one (see Appendix H [37] for details). The difference is that the Hilbert spaces involved are much larger, yielding large-size IMMs. To reduce the consumption of computational resources, one may introduce techniques used in some previous works [61]. The comprehensive investigation is beyond the scope of this Letter, and we leave the deeper exploration of these issues as future research.

Summary and discussion.—In this Letter, we have established a general temporal scenario and develop a method, dubbed as instrument moment matrices (IMMs), to characterize quantum temporal correlations generated by such a scenario. The method of IMMs can be implemented in a fully DI scenario, but we can also include additional constraints (such as the dimension and rank of the system) when this information is accessible. Along the side, we contribute to advocating to explore the "room in the middle" between the (precise, but very restrictive) DI and device-specific scenarios: In contrast to Ref. [32] which is close to device dependence and is hence dubbed semidevice dependent, we are here close to the DI regime, in the semi-device-independent setting. We explicitly provide several DI and semi-DI examples.

Regarding implementing our protocol experimentally, there may be some loopholes we have to take care of. For instance, if the detection efficiency of the detectors is too low, there exists a classical model which can be used for reproducing the observed data [2]. Besides, if the temporal scenario is set for testing local realism (i.e., the Leggett-Garg test), one may meet the clumsiness loophole [19] issue. In that case, one can use other types of temporal Bell inequalities, such as those proposed in Ref. [64].

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