Hybrid Goldstone Modes from the Double Copy Bootstrap

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We perform a systematic classification of scalar field theories whose amplitudes admit a double copy formulation and identify two building blocks at four-point and 13 at five-point. Using the four-point blocks simultaneously as bootstrap seeds, this naturally leads to a single copy theory that is a gauged nonlinear sigma model. Moreover, its double copy includes a novel theory that can be written in terms of Lovelock invariants of an induced metric, and includes Dirac-Born-Infeld and the special Galileon in specific limits. The amplitudes of these Goldstone modes have two distinct soft behavior regimes, corresponding to a hybrid of nonlinear symmetries.

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Introduction.—The double copy framework manifests a remarkable connection between the unique (at lowest order in derivatives) interacting theories of spin-1 and spin-2: the amplitudes of general relativity can be written as the squares of specific color-dual kinematic numerators that define Yang-Mills (YM) [1,2]. This connection has its origin in open-closed string duality [3] and is closely related to the scattering equations approach of [4,5]. The double copy has since been extended to include supersymmetric theories and loop level, as reviewed in [6,7], as well as scalar field theories with enhanced soft limits that generalize the Adler zero and hence can be seen as Goldstone theories [8,9].

A natural question regards the uniqueness of the kinematic numerators: Are higher-derivative corrections encoded in other color-dual kinematic numerators? For the color-dual kinematic numerator of YM, there is a single additional possibility at three-point (while at four-point, there are already eight different tensorial structures [10,11]) that generates the unique F^3 correction. Using this as a seed interaction, double copy compatibility at four-point then implies the inclusion of a F^4 term. Moreover, following the same logic at five-point requires the further quartic term D^2F^4 [12]. It was conjectured to go up to all derivatives, leading to a UV complete series that is part of the bosonic open string amplitude [12,13].

A related result was found very recently for higher derivatives to a specific scalar field theory, the nonlinear sigma model (NLSM) [14]. Again, higher-derivative corrections to the four-point seed interactions were found to be constrained by higher-point consistency. The only known theory that satisfies these constraints at all order (apart from the NLSM itself) is Z theory, again with an infinite tower of derivatives [16,17].

In this Letter, we perform a related analysis for scalar field theories with Goldstone modes. The crucial difference with [14] is that we do not restrict ourselves to four-point *contact* interactions. As we will show, this allows for a unique additional *exchange* interaction. Similarly, we classify all possible double copy scalar seeds at $n = 5$ and find 29 independent structures, of which 13 generate physical amplitudes.

Using a linear combination of the two four-point seed interactions, we then employ the bootstrap procedure to construct theories for Goldstone modes with a hybrid character: while the entire theory has a particular soft degree $\sigma_{\min}$ [where the soft degree σ is defined as $A_n \sim \mathcal{O}(p^{\sigma})$ when an external momentum p becomes soft], it contains a subsector that is defined by having $\sigma_{\max} = \sigma_{\min} + 1$ instead. We present examples of a single and a double copy: a gauged version of the NLSM with $\sigma_{\min} = 0$ and a particular higher-derivative extension of Dirac-Born-Infeld (DBI) with $\sigma_{\min} = 2$. As the latter is formulated in terms of Lovelock invariants, we refer to it as DBI-Lovelock.

In contrast to the results of [12,14], we find no need for infinite sets of quartic higher-derivative corrections; in this sense, our results are more akin to the extended DBI theory [18–20]. We provide an interpretation for this difference in the conclusion, and outline further implications and generalizations.

BCJ representations.—The double copy or Bern-Carrasco-Johansson (BCJ) approach [1,2,6,7] has identified a number of field theories, famously including general relativity and YM, whose amplitudes can be rewritten in terms of a sum over $(2n-5)!!$ trivalent diagrams:

$$A_n = \sum_{\text{trivalent}} \frac{N\tilde{N}}{D}. \tag{1}$$

The denominator in the above sum consists of the propagators for every diagram, while the numerator instead is the product of two so-called BCJ numerators that encode the characteristics of the particles in the scattering process. While each trivalent diagram has an associated kinematic numerator, only $(n-2)!$ of these are independent; a convenient basis for these is provided by the Del Duca–Dixon–Maltoni basis [21].

An important and arguably the simplest example is given by the color factors that consist of products of structure constants f_{abc}. At multiplicity n, these are given by a product of $n-2$ structure constants:

$$N_{abc\ldots} = f_{ab}^{x_1} f_{x_1 c}^{x_2} f_{x_2 \ldots} \ldots. \tag{2}$$

When viewed as a representation of the permutation group S_n, the above numerators satisfy the nested commutator structure [22]

$$-N_{abcd\ldots} = N_{bacd\ldots} = N_{c[ab]d\ldots} = N_{d[[ab]c]\ldots} = \ldots, \tag{3}$$

which will be referred to as generalized Jacobi identities. Moreover, the color factors are even or odd under reflection,

$$N_{abcd\ldots} = (-)^n N_{\ldots dcba}. \tag{4}$$

We have identified which irreducible representations (irreps) of S_n the above constraints correspond to, with the dimensions of these irreps adding up to $(n-2)!$ in every case; see Table I.

Instead of structure constants, we will be interested in color-dual kinematic numerators that only contain Mandelstam variables (scalar numerators for short). These are relevant for scalar field theories: in single scalar field theories, the particles only carry momentum information and thus Mandelstam invariants. Moreover, in multi-scalar field theories such as the NLSM with multiple flavors plus higher-derivative corrections, the color information factorizes and thus one of the two BCJ numerators again only involves Mandelstam variables.

The possibilities can be phrased in terms of S_n representations. The set of n external momenta forms the so-called standard irrep $[n-1, 1]$ with dimension $n-1$. Lorentz invariants then consist of inner products of momenta and live in the irrep $[n-2, 2]$ with dimension $\frac{1}{2}n(n-3)$; these correspond to the Mandelstam invariants. Moreover, we work in general dimensions and hence are not affected by Gram determinant considerations that reduce the number of independent Mandelstam variables.

TABLE I. BCJ-compatible numerators as irreps of S_n.

n	Kinematic numerators
4	[2, 2]
5	[3,1,1]
6	[4, 2], [3,1,1,1], [2,2,2]
7	[5,1,1], [4,2,1], [3,3,1], [3,2,1,1], [2,2,1,1,1]
8	[6, 2], [5,2,1], [5,1,1,1], [4, 4], [4,3,1], $2\times$[4, 2, 2], [4,2,1,1], [4,1,1,1,1], $2\times$[3, 3, 1, 1], [3,2,2,1], [3,2,1,1,1], [2,2,2,2], [2,2,1,1,1,1]

The above approach reduces the classification of scalar numerators to a representation theory problem [24] of S_n: for the number of scalar numerators at a given multiplicity n and at a given order p in Mandelstam variables, one simply calculates the symmetric product of p irreps $[n-2, 2]$ and decomposes this into S_n irreps. A comparison with the BCJ-required irreps of Table I then directly gives the number of possible scalar numerators at this order.

Not all scalar numerators will contribute to the amplitude; some solutions N will give a vanishing contribution to (1), independent of the choice for $\tilde{N}$. Interestingly, these gauge solutions can also be characterized by representation theory: all scalar numerators that can be written as the product of Mandelstam variables with a specific S_n irrep drop out of the amplitude. The first example surfaces at four-point and reads

$$N_{abcd} = s_{ab} G_{abcd}, \tag{5}$$

where the convention $s_{i\ldots j} = (p_i + \cdots + p_j)^2$ is adopted, and G is a fully antisymmetric tensor and hence lives in the [1,1,1,1]. We have listed the analogous irrep requirements at higher multiplicities in Table II. For further details, see Supplemental Material [25].

BCJ seed classification.—We will now proceed to systematically classify all scalar numerators at lower multiplicities at four and five-point [33] using representation theory.

At *four-point*, the required BCJ irrep is the window of S_4 with dimension 2. The Mandelstam invariants in this case live in the same irrep. Therefore there is naturally a linear combination of Mandelstam invariants that satisfies the BCJ constraints. An explicit construction shows that it is

TABLE II. BCJ-compatible gauge parameters as irreps of S_n.

n	Gauge parameters
4	[1,1,1,1]
5	[2,2,1]
6	[3,2,1], [3,1,1,1]
7	[4, 3], [4,2,1], [4,1,1,1], [3,2,2], [3,2,1,1], [3,1,1,1,1], [2,2,2,1]

given by

$$N_4^{(1)} = s_{bc} - s_{ac}. \tag{6}$$

Moreover, at quadratic order in Mandelstam, the symmetric product of two window irreps decomposes into $[2,2] + [4] + [1,1,1,1]$ and hence there is another scalar numerator for four-point at this order. It takes the form

$$N_4^{(2)} = s_{ab}(s_{bc} - s_{ac}). \tag{7}$$

The expression (6) corresponds to the four-scalar scattering with an exchanged gluon, and (7) to the four-point contact interaction of the NLSM. We will refer to the linear and quadratic solutions as exchange and contact scalar numerators, respectively.

At higher orders, there are new solutions to the generalized Jacobi. However, it follows from representation theory that these are always of the form of one of the two above building blocks, multiplied by Mandelstam expressions that are separately invariant (and hence can be used to construct additional solutions to the BCJ conditions). To see this, note that the number of invariants at every order is given by the Taylor coefficients of the Molien series (this coincides with the Hilbert series for the case of invariant polynomial rings) [34–37]:

$$H_4^{\mathrm{Inv}}(x) = \frac{1}{(1-x^2)(1-x^3)}. \tag{8}$$

This amounts to the statement that all invariants can be written as arbitrary powers of two primary invariants:

$$I_4^{(2)} = s_{ab}s_{bc} + s_{ac}s_{bc} + s_{ab}s_{ac}, \qquad I_4^{(3)} = s_{ab}s_{ad}s_{ac}. \tag{9}$$

Moreover, the number of window irreps at every order in Mandelstam is generated by

$$H_4^{\mathrm{BCJ}}(x) = (x + x^2)H_4^{\mathrm{Inv}}(x). \tag{10}$$

All window solutions are therefore either (6) or (7) multiplied by an invariant, as also found in [11].

Turning to gauge parameters, the Molien series for the relevant irrep [1,1,1,1] is given by

$$H_4^{\mathrm{Gauge}}(x) = x^3 H_4^{\mathrm{Inv}}(x), \tag{11}$$

which generates the number of gauge parameters at 1 order higher. Indeed, it turns out that the combination

$$2N_4^{(2)}I_4^{(2)} - 3N_4^{(1)}I_4^{(3)} \tag{12}$$

is of the form (5) and drops out of the amplitude (1) for any scalar numerator $\tilde{N}$. The number of physical BCJ parameters is therefore given by

$$H_4^{\mathrm{BCJ}}(x) - xH_4^{\mathrm{Gauge}}(x) = \frac{x}{1-x^2} + \frac{x^2}{(1-x^2)(1-x^3)}, \tag{13}$$

generated by the linear or quadratic seed solutions (6) and (7) multiplied by quadratic and/or cubic invariants [38].

At *five-point*, the story is similar but more complicated. The five-point Hilbert series for invariants is given by

$$H_5^{\mathrm{Inv}}(x) = (1 + x^6 + x^7 + x^8 + x^9 + x^{15})/D_5(x), \tag{14}$$

with the denominator given by

$$D_5(x) = (1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6). \tag{15}$$

Each factor in the denominator corresponds to a primary invariant, and each term in the numerator to a secondary invariant; for example, $(1-x^2)$ represents the contribution from a quadratic primary invariant, whereas x^6 represents a sextic secondary invariant. The difference is that primary invariants can appear at any power to form new invariants, while there can only be a single secondary invariant. The latter restriction is due to relations between products of invariants, referred to as syzygies [34,36].

The BCJ irreps, instead, are given by [3,1,1] corresponding to the "hook" Young tableau. The Molien series for this is [40]

$$H_5^{\mathrm{BCJ}}(x) = (x^3 + 2x^4 + 4x^5 + 5x^6 + 6x^7 + 6x^8 + 5x^9 + 4x^{10} + 2x^{11} + x^{12})/D_5(x) \tag{16}$$

and are thus given by the numerator structures multiplied by primary invariants.

The gauge parameters in this case are generated by the irrep [2,2,1]; the number of such irreps at every order is generated by

$$H_5^{\mathrm{Gauge}} = (x^2 + x^3 + 3x^4 + 3x^5 + 3x^6 + 4x^7 + 4x^8 + 3x^9 + 3x^{10} + 3x^{11} + x^{12} + x^{13})/D_5(x). \tag{17}$$

However, in this case the number of distinct resulting gauge parameters (at 1 order higher) is somewhat smaller and given by

$$(x^3 + x^4 + 2x^5 + 3x^6 + 2x^7 + 3x^8 + 4x^9 + 3x^{10} + 2x^{11} + 2x^{12} + x^{13})/D_5(x). \tag{18}$$

This difference comes about as some BCJ parameters can be split into Mandelstam times [2,2,1] in multiple ways. The resulting number of physical BCJ parameters at every order is given by the difference of (16) and (18) and can be written as a sum of fractions with positive coefficients,

$$H_5^{\mathrm{Phys}}(x) = (x^4 + 2x^5 + 2x^6 + 4x^7 + 3x^8)/D_5(x) + \frac{(x^9 + x^{10})}{(1-x^2)(1-x^4)(1-x^5)(1-x^6)}, \tag{19}$$

but this decomposition is not unique. Modulo the primary invariants of the denominators, this series consists of 14 different hook structures. However, one of these can be written in terms of a secondary invariant, implying that

TABLE III. The number of six-point BCJ-compatible scalar numerators (split into physical and gauge parameters) and invariants at $\mathcal{O}(s^p)$. The last column lists the number of scalar numerators that are compatible with the BCJ bootstrap.

p	BCJ	Phys	Gauge	Inv	Bootstrap
1	1	1	0	0	0
2	3	3	0	1	1
3	9	8	1	2	2
4	23	18	5	4	3
5	54	38	16	6	8
6	121	79	42	13	24
7	246	151	95	19	53

there are 13 independent hook structures that can be used as five-point seed interactions.

BCJ bootstrap.—From *six-point* on, a systematic classification of scalar numerators becomes more complicated. The six-point Molien series for invariants is

$$H_6^{\mathrm{Inv}}(x) = (1 + 2x^5 + 5x^6 + 7x^7 + 9x^8 + 11x^9 + 13x^{10} + 14x^{11} + 21x^{12} + 24x^{13} + 28x^{14} + 32x^{15} + 26x^{16} + 22x^{17} + 13x^{18} + 7x^{19} + 3x^{20} + x^{21} + x^{22})/D_6(x) \tag{20}$$

in terms of the following denominator containing the primary invariants

$$D_6(x) = (1-x^2)(1-x^3)^2(1-x^4)^3(1-x^5)^2(1-x^6). \tag{21}$$

Thus, there are nine primary invariants and 239 secondary ones. The Molien series of the BCJ irreps and gauge parameters can be established in a similar manner. However, deriving the final number of independent structures requires the explicit forms of the primary and secondary invariants and is problematic due to complicated relations (with syzygies of syzygies [34,42]). We will not attempt such a general classification to all orders, and only list the number of physical and gauge parameters at lowest orders in Table III.

Moreover, we will focus on the subset of six-point interactions that follow from four-point seeds; in other words, we will require that they are BCJ "bootstrappable" from four-point seed interactions, similar to [14]. This implies in particular that at singular channels such as $s_{abc} \to 0$, the amplitude should factorize into two four-point amplitudes. In turn, this implies that a scalar numerator of $\mathcal{O}(s^p)$ factorizes as

$$\lim_{s_{abc}\to 0} N_6^{(p)} = \sum_q c_q N_4^{(p-q)}(abcx) N_4^{(p+q)}(xdef), \tag{22}$$

where x denotes the internal leg. As the lowest four-point scalar numerator is linear in Mandelstam, the first N_6 that can be bootstrapped is quadratic. Beyond that, we find up to $\mathcal{O}(s^4)$

$$\begin{aligned} \lim_{s_{abc}\to 0} N_6^{(2)} &= (s_{ac} - s_{bc})(s_{de} - s_{df}), \\ \lim_{s_{abc}\to 0} N_6^{(3)} &= (s_{ac} - s_{bc}) s_{ab} (s_{de} - s_{df}) \\ &\quad + (s_{ac} - s_{bc})(s_{de} - s_{df}) s_{ef}, \\ \lim_{s_{abc}\to 0} N_6^{(4)} &= (s_{ac} - s_{bc}) s_{ab} (s_{de} - s_{df}) s_{ef}. \end{aligned} \tag{23}$$

Imposing this BCJ bootstrap fixes $N_6^{(2)}$ uniquely to

$$N_6^{(2)} = (s_{ac} - s_{bc})(s_{de} - s_{df}) + \frac{1}{2} s_{abc}(s_{ae} - s_{af} - s_{be} + s_{bf}), \tag{24}$$

while in other cases it still leaves some free parameters that can be seen as contact interactions that are separately BCJ compatible. We provide an overview of these numbers in Table III.

Single copy: The gauged NLSM.—The systematic classification of seed scalar numerators at four-point and the corresponding bootstrapped ones at six and higher point allows for the construction of novel theories that feature interactions with different soft limits. As a first illustration, we will propose a single copy theory for an adjoint Goldstone scalar field, with amplitudes generated by the product of a color factor with a linear combination of the different elementary solutions involving Mandelstam. Moreover, we will use the requirement of $\sigma_{\min} = 0$ as a guiding principle. This will result in a gauged version of the chiral NLSM with symmetry breaking $G \times G \to G$, with additional interactions due to gluon exchange.

At four-point this theory is generated by $C_4 \times (N_4^{(1)} + N_4^{(2)})$, and therefore has two different contributions to the amplitudes. The corresponding Lagrangian is

$$\mathcal{L}_4 = -\frac{1}{2}(D\phi)^2 + \frac{1}{6} f^2 \phi^2 (D\phi)^2 - \frac{1}{4} F^2 \tag{25}$$

up to this order.

Moving to six-point, we consider the schematic form $C_6 \times (N_6^{(2)} + N_6^{(3)} + N_6^{(4)})$. While the quadratic scalar numerator is unique, that is not the case for the cubic and quartic ones. To unambiguously determine the theory, we impose the soft limit $\sigma_{\min} = 0$ at cubic and $\sigma_{\max} = 1$ at quartic order; as promised in the introduction, this theory has different soft degrees with $\sigma_{\max} = \sigma_{\min} + 1$. We have to extend the Lagrangian of the gauged NLSM with terms of the following form [43]

$$\mathcal{L}_6 = \mathcal{L}_4 + \frac{1}{45} f^4 \phi^4 (D\phi)^2 - 2 f^2 F^3 + \frac{1}{6} f^2 \phi^2 F^2 \tag{26}$$

to generate these amplitudes correctly.

Moving to higher multiplicities, we conjecture that this pattern continues. For instance, at eight-point, one can take the amplitude generated by BCJ numerators of the form $C_8 \times (N_8^{(3)} + .. + N_8^{(6)})$. The corresponding Lagrangian will include all terms above plus $\phi^6(D\phi)^2$ and possibly F^4 and $F^2\phi^4$. Note that all terms are gauge covariant, and will thus result in $\sigma_{\min}=0$ amplitudes. Moreover, the purely scalar two-derivative part reduces to the NLSM with $\sigma_{\max}=1$. Thus one should think of this theory as the NLSM with subleading terms included. These are dictated by a combination of nonlinear symmetries (e.g., the structure of two-derivative terms), gauge invariance (i.e., the covariant derivatives), and BCJ consistency (e.g., the F^3 and $F^2\phi^2$) [46].

Double copy: DBI-Lovelock.—As a second example we propose a double copy theory that involves a single scalar field and that is fully determined by two different nonlinear symmetries, with BCJ compatibility arising as a result. This theory turns out to be the double copy of a gauged and an ungauged NLSM.

We again start from the full classification at four-point. The scalar numerators $N_4^{(1)}\times N_4^{(2)}$ yields DBI with the quartic operator $(\partial\phi)^4$. In order to retain the $\sigma_{\min}=2$ generalized Adler zero, one must add $(\partial\phi)^{2n}$ with specific coefficients at every order [8]. The full DBI theory is then given by the measure of

$$g_{\mu\nu}=\eta_{\mu\nu}+\partial_\mu\phi\partial_\nu\phi, \tag{27}$$

which can be seen as a brane-induced metric and is covariant under [48]

$$\delta\phi=c_\mu x^\mu+c_\mu\phi\partial^\mu\phi, \tag{28}$$

that generate a nonlinear realisation of 5D Poincare symmetries.

The BCJ product $N_4^{(2)}\times N_4^{(2)}$, instead, yields the special Galileon (SG) theory, with operator $(\partial\phi)^2([\Pi]^2-[\Pi^2])$, $\Pi_{\mu\nu}=\partial_\mu\partial_\nu\phi$, $[\ldots]=\mathrm{Tr}[\ldots]$, and the nonlinear symmetry [49]

$$\delta\phi=s_{\mu\nu}x^\mu x^\nu+s_{\mu\nu}\partial^\mu\phi\partial^\nu\phi, \tag{29}$$

resulting in the soft degree $\sigma_{\max}=3$.

Can these be combined into a single, extended Goldstone theory that is BCJ-compatible based on the product $N_4^{(2)}\times(N_4^{(1)}+N_4^{(2)})$? We will provide evidence that the answer to this question is affirmative. The defining property, similar to DBI and SG, will be the soft limit: all interactions are required to have at least $\sigma_{\min}=2$. This is naturally satisfied when taking curvature invariants of the metric (27); a general effective field theory would therefore be

$$\mathcal{L}=\sqrt{-g}[c_0+c_1R+c_2R^2+\ldots]. \tag{30}$$

However, this does not display the $\sigma_{\max}=3$ scaling in any limit. In order to ensure that the highest-derivative terms have the SG scaling, one needs to restrict to the specific Lovelock invariants at every order; these have the special property of being degenerate and hence do not generate any corrections with more than two derivatives on a given field [50]. When evaluated at the induced metric (27), the Lovelock invariants become total derivatives:

$$R^n\equiv\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\prod_{i=1}^{n}R^{\alpha_r\beta_r}_{\mu_r\nu_r}=\partial_\mu j^{(n)\mu}. \tag{31}$$

For illustration, the currents for $n=1$, 2 are given by

$$\begin{aligned}
j^{(1)\mu}&=\frac{2}{1+(\partial\phi)^2}(\phi^\mu[\Pi]-\phi_\nu\Pi^{\nu\mu}),\\
j^{(2)\mu}&=\frac{4}{(1+(\partial\phi)^2)^2}(2\phi^\mu[\Pi^3]-3\phi^\mu[\Pi^2][\Pi]\\
&\quad+\phi^\mu[\Pi]^3+3\phi_\nu\Pi^{\nu\mu}[\Pi^2]-6\phi_\kappa\Pi^\kappa_\lambda\Pi^\lambda_\nu\Pi^{\nu\mu}\\
&\quad+6\phi_\kappa\Pi^\kappa_\nu\Pi^{\nu\mu}[\Pi]-3\phi_\nu\Pi^{\nu\mu}[\Pi]^2),
\end{aligned} \tag{32}$$

with $\phi_\mu=\partial_\mu\phi$. At lowest order in ϕ, these operators become

$$\sqrt{-g}R^n\simeq(\partial\phi)^2\delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\prod_{r=1}^{n}(\partial^{\mu_r}\partial_{\nu_r}\phi), \tag{33}$$

up to an overall constant. Note that these are exactly the SG invariants; indeed, both the Lovelock and the SG invariants trivialize in sufficiently low dimensions D.

We can therefore tune the Lovelock coefficients to have a theory that propagates a single scalar field, has $\sigma_{\min}=2$ for all amplitudes (ensuring that it is DBI plus higher-derivative corrections), and moreover the higher-derivative amplitude has $\sigma_{\max}=3$ (such that it asymptotes to the SG). This *DBI-Lovelock* theory is thus defined to all orders (and in all dimensions) uniquely by its nonlinear symmetries and associated soft degrees.

We have checked that the amplitudes resulting from the above theory can be written in terms of a linear combination of scalar numerators, at least up to this order. Moreover, the specific linear combination of quadratic, cubic, and quartic six-point terms that are needed for the gauged NLSM and the DBI-Lovelock theory are identical.

Conclusion.—This Letter opens up the tantalizing possibility that various Goldstone theories with hybrid soft degrees σ adhere to the double copy paradigm. Moreover, these can be systematically classified using representation theory of S_n: at four and five-point, respectively, there are two and 13 scalar numerators in terms of Mandelstam variables.

Focusing on the former, we have identified a new sector of scalar field theories that can be BCJ-bootstrapped, over and beyond the analysis of [14]: the inclusion of the linear four-point seed interaction introduces lower (instead of higher) derivative corrections to the NLSM. These allow for the construction of a gauged NLSM. BCJ compatibility then requires specific F^3 and ϕ^2F^2 terms. Moreover, the two four-point seeds can be double copied into an extension of the special Galileon theory with lower-derivative corrections, which can be phrased in terms of Lovelock invariants of the DBI metric.

We have demonstrated the bootstrap construction of our two example theories explicitly at six-point [51]. In both cases, the theories are strongly constrained by two soft degrees with $\sigma_{\max} = \sigma_{\min} + 1$: this plays a crucial role in uniquely determining the scalar numerators. Our theories therefore appear to be closely related to the so-called extended DBI theory [18,52], which involves the NLSM and DBI in specific limits. Indeed this theory can also be phrased in terms of BCJ numerators [19], and only has a finite number of operators contributing to n-point amplitudes. Moreover, the on-shell constructibility of this kind of theory follows from the graded soft theorem proposed in [20].

Based on the current results, it thus appears there is a fundamental difference between the higher-derivative corrections of [12,14,15] on the one hand, and hybrid Goldstone theories such as extended DBI, the gauged NLSM, and DBI-Lovelock on the other hand: the latter category does not require an infinite set of higher-derivative corrections at given multiplicity. We expect that this difference arises due to the special nature of the highest-derivative terms in these theories: these have a softer degree $\sigma_{\max}$, are related by the nonlinear symmetry and are separately BCJ-compatible. In contrast, the leading terms of the higher-derivative corrections F^3 and F^4 in [12] [at vanishing gauge coupling, i.e., of the form $(\partial A)^n$] do not have a nonlinear symmetry (beyond Abelian gauge symmetry) and are not separately BCJ-compatible. A similar discussion applies to the scalar BCJ bootstrap [14] and the Kawai-Lewellen-Tye kernel bootstrap [15], whose four-point seed structures do not include the linear exchange interaction (6).

Finally, the general analysis of this Letter suggests a number of novel theories beyond the two examples that we outlined. At four-point, one can instead take the product $N_4^{(1)} \times (N_4^{(1)} + N_4^{(2)})$ leading to a combination of DBI with gravitational interactions; moreover, this can be extended to have an $SO(N)$ flavor along the lines of [22]. More generally, one can consider the case where both BCJ numerators have multiple terms of different order in Mandelstam. At first sight one might expect this to lead to a theory with three soft degree sectors; it would be interesting to investigate how this relates to the graded soft theorem of [20] that only allows for $\sigma_{\min}$ and $\sigma_{\max}$ to differ by one. We leave this question to future research.

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