Quantum-Enhanced Metrology with Network States

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Armed with quantum correlations, quantum sensors in a network have shown the potential to outclass their classical counterparts in distributed sensing tasks such as clock synchronization and reference frame alignment. On the other hand, this analysis was done for simple and idealized networks, whereas the correlation shared within a practical quantum network, captured by the notion of network states, is much more complex. Here, we prove a general bound that limits the performance of using quantum network states to estimate a global parameter, establishing the necessity of genuine multipartite entanglement for achieving a quantum advantage. The bound can also serve as an entanglement witness in networks and can be generalized to states generated by shallow circuits. Moreover, while our bound prohibits local network states from achieving the Heisenberg limit, we design a probabilistic protocol that, once successful, attains this ultimate limit of quantum metrology and preserves the privacy of involved parties. Our work establishes both the limitation and the possibility of quantum metrology within quantum networks.

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Introduction.—Distributed sensing in a network is a general task of fundamental significance. Remarkably, with Greenberger-Horne-Zeilinger (GHZ) states over a network of M parties, it is possible to estimate a global parameter θ with mean squared error $\Delta^2(\theta) \sim 1/M^2$, achieving the Heisenberg limit of quantum metrology—with a $\Theta(M)$ reduction of error over protocols without entanglement [1]. This fact lies behind the recent interest in distributed quantum sensing [2–10] including several experimental demonstrations [11–14] in both finite dimensional systems and continuous-variable systems.

In practice, however, the distribution of global entangled states, e.g., multipartite GHZ states, is a daunting task [15,16]. The decoherence time of multipartite entanglement leads to experimental limits in transmission and storage [17], especially for remote parties. A feasible solution is to use quantum repeaters [18] and consider distributed settings, i.e., quantum networks [19,20]. Most often the generic resource states are *network states* [21–26], prepared by distributing few-partite entangled sources to different vertices and applying local operations according to a predefined protocol. On the other hand, typical resources for quantum metrology like (multipartite) GHZ states and graph states [27,28] may not be accessible in generic networks [26,29,30]. Consequently, it is natural to ask how to characterize the potential of network states in sensing global parameters, and whether the Heisenberg limit can still be achieved. These critical questions, however, have been largely unexplored due to the much more general and complex nature of network states compared to GHZ states.

In this work, we derive a versatile general upper bound on the precision of any deterministic protocol for estimating a (global) parameter using network states, which leads to sufficient conditions under which the precision is bounded by the standard quantum limit (SQL) $\Delta^2(\theta) \sim 1/M$ (for M parties) and cannot achieve the Heisenberg limit (HL) $\Delta^2(\theta) \sim 1/M^2$. We then design a probabilistic sensing protocol [31–38] using local postselection to achieve the HL, which also features the preservation of local parameters' privacy.

Distributed sensing with network states.—A quantum network state ρ is a multipartite quantum state, whose structure can be efficiently represented by a hypergraph $G(\mathcal{V}, \mathcal{E})$ of $K(= |\mathcal{V}|)$ vertices and $|\mathcal{E}|$ hyperedges (i.e., subsets of $\mathcal{V}$). Each vertex represents a local site, and each hyperedge represents an entanglement source. The network state ρ is generated via a two-step procedure, where each entanglement source, represented by a hyperedge e, distributes an entangled state to every local site in e and then each local site v applies an arbitrary local operation. Moreover, all sites can be classically correlated via a preshared global random variable λ. In other words, a network state is of the generic form

$$\rho = \sum_\lambda p_\lambda \rho_\lambda, \qquad \rho_\lambda = \left(\bigotimes_{v\in\mathcal{V}} \Phi_v^{(\lambda)}\right)\left(\bigotimes_{e\in\mathcal{E}} \sigma_e\right), \quad (1)$$

where $\Phi_v^{(\lambda)}$ is a channel acting on sensor v, and σ_e is an entangled state shared between sensors $v \in e$.

The goal of distributed sensing is for a group of far-apart sensors, each having access to an unknown local signal, to estimate one (usually global) parameter, e.g., the average of all local parameters [39]. To accomplish this goal, the local sensors have access to a joint network state in each round of the experiment. In general, a *deterministic protocol* consists of three phases: (i) *Network state distribution:* Each entanglement source e distributes an entangled state (e.g., Bell pairs or GHZ states) among its associated local sites (sensors) $\{v \in \mathcal{V} | v \in e\}$. Each sensor performs a local operation. Eventually, the sensors share a network state ρ. (ii) *Signal acquisition:* The sensors obtain local signals. Explicitly, the state goes through a unitary evolution $U(\vec{\theta}) = \exp\{-i \sum_{s\in\mathcal{S}} H_s \theta_s\}$, where $\{\theta_s\}$ are unknown parameters with generators $\{H_s\}$ and $\mathcal{S}$ consists of subsets of $\mathcal{V}$. We denote by $M \coloneqq |\mathcal{S}|$ the cardinality of $\mathcal{S}$. A generator H_s acts trivially on a sensor v if $v \notin s$. The global state becomes $\rho(\{\theta_s\})$ after signal acquisition. (iii) *Parameter estimation:* The parameter of interest is a function $f(\{\theta_s\})$ of $\{\theta_s\}$. The form of f is known to all sensors. Depending on this function, a measurement is performed on the global state $\rho(\{\theta_s\})$ and an unbiased estimate $\hat{\theta}$ of θ is extracted from the measurement outcome statistics. Note that, in practice, the type of measurements that could be performed is often restricted (e.g., to be local). Here we show a stronger result (Theorem 1) that holds without any constraint on what kind of measurements may be performed.

For example, Fig. 1(a) shows a network of $K = 3$ sensors (v_1, v_2, v_3) with a shared bipartite entangled state ($\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$) between each pair of sensors. There are $M = 3$ parameters, each collected by an individual sensor ($\mathcal{S} = \{\{v_1\}, \{v_2\}, \{v_3\}\}$).

A general lower bound on the estimation error.—Consider the task of estimating a global parameter that is an arbitrary linear combination of the local parameters $\theta(\vec{\alpha}) \coloneqq \vec{\alpha}^T\vec{\theta} = \sum_s \alpha_s\theta_s$, where $\vec{\alpha}$ is a vector of dimension M [40]. The mean squared error can be expressed as

$$\Delta^2(\theta(\vec{\alpha})) = \vec{\alpha}^T \mathrm{Cov}\Big(\{\hat{\theta}_s\}\Big)\vec{\alpha}, \tag{2}$$

where $\mathrm{Cov}(\{\hat{\theta}_s\})$ is the covariance matrix of the estimators $\{\hat{\theta}_s\}$ for $\{\theta_s\}$: $(\mathrm{Cov})_{ij} \coloneqq \mathbb{E}[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j)]$ for $1 \le i, j \le M$.

Our first main result is a general lower bound on the error of any deterministic protocol for an arbitrary network state. Compared with works in distributed sensing [2–9], our bound captures not only the impact of the sensing task, but also the architecture of the network. More explicitly, we identify a key quantity in determining the error scaling, named the *influence of the local signal s*: $k_s \coloneqq \max_{e\in\mathcal{E}, e\cap s \ne \emptyset} |\{t \in \mathcal{S} | t \cap e \ne \emptyset\}|$. Intuitively, k_s is the maximum number of local signals influenced by s via an entanglement source within the network. Note that k_s depends on both the sensing task and the network.

Theorem 1.—When estimating a global parameter $\theta(\vec{\alpha}) = \vec{\alpha}^T\vec{\theta}$ given a network state ρ with structure $G(\mathcal{E}, \mathcal{V})$, the mean squared error of any deterministic protocol is lower bounded as

$$\Delta^2(\theta(\vec{\alpha})) \ge \sum_s \frac{\alpha_s^2}{4\nu k_s \mathrm{Var}(\rho, H_s)}. \tag{3}$$

Here H_s is the generator of θ_s, $\mathrm{Var}(\rho, H_s)$ is the variance of H_s with respect to ρ, and ν is the number of rounds that the experiment is repeated.

In principle, the bound (3) can be extended to the more general multiparameter case where the cost function is of the form $\mathrm{Tr}[W\mathrm{Cov}(\{\hat{\theta}_s\})]$ for some weight matrix $W \ge 0$. In fact, we prove the bound by combining the (matrix) Cramér-Rao bound [42,43] $\mathrm{Cov}(\{\hat{\theta}_s\}) \ge (1/\nu)\mathcal{F}_Q^{-1}$ with the following bound on the quantum Fisher information (QFI) matrix (see [44] for its explicit definition) for the parameters $\{\theta_s\}$:

$$\mathcal{F}_Q(\rho, \{H_s\}) \le \mathrm{diag}\{4k_s \mathrm{Var}(\rho, H_s)\}_s, \tag{4}$$

where ρ denotes the network state, and $\mathcal{F}_Q(\rho, \{H_s\})$ denotes the quantum Fisher information matrix of the state $\rho_{\vec{\theta}} \coloneqq U(\vec{\theta})\rho U(\vec{\theta})^\dagger$ with $U(\vec{\theta}) \coloneqq e^{-i\sum_s H_s\theta_s}$. Its proof can be found in the Supplemental Material [44].

We now check how the error scales with respect to M, the number of local parameters. In Eq. (3), the variance $\mathrm{Var}(\rho, H_s)$ can be bounded as $\mathrm{Var}(\rho, H_s) \le \|H_s\|^2 \le h_{\max}^2$ (with $\|\cdot\|$ being the operator norm), where $h_{\max} \coloneqq \max_s \|H_s\|$ is independent of M. For estimating the mean of $\{\theta_s\}$, we have $\vec{\alpha} = (1/M, \ldots, 1/M)^T$, and thus $\vec{\alpha}^T\vec{\alpha} \sim 1/M$. Further, denoting by $k_{\max}$ the maximum of the influence k_s, Eq. (3) implies

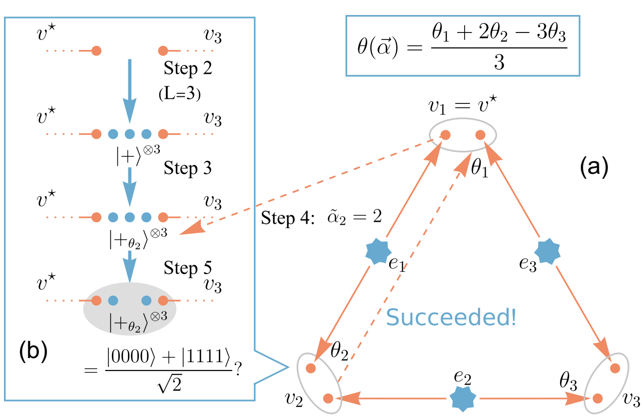

FIG. 1. (a) A cyclic network consisting of $M = 3$ sensors. Here each pair of sensors share a Bell state $|B_0\rangle = [(|00\rangle + |11\rangle)/\sqrt{2}]$ via a source. (b) Steps 2–5 of Protocol 1 are illustrated for one of the noncenter sensors, v_2.

$$\Delta^2(\theta(\vec{\alpha})) = \Omega\left(\frac{1}{M \cdot k_{\max} \cdot h_{\max}^2}\right). \tag{5}$$

A key implication of our result is that genuine M-partite *network* entanglement is necessary for achieving the HL in a network. When $\Delta^2 \sim 1/M^2$, Eq. (5) requires $k_{\max}$ to scale with M. As long as $s \cap s' = \emptyset$ for any pair of local signals, $k_{\max}$ is upper bounded by $\max_{e \in \mathcal{E}} |e|$, which captures the range of genuine entanglement in the network [21]. Our result thus establishes a crucial connection between this core property of a generic network and quantum metrology, requiring it to scale with M to achieve the HL. Since network entanglement is a stronger and more natural resource for network scenarios than multipartite entanglement [21], our result extends the main result of Ref. [48], where the necessity of M-partite entanglement was established. Our bound also shows that local pre-processing cannot be used to gain an advantage in metrology with shared entangled states between limited numbers of parties.

Furthermore, it is revealed by our bound that not only the amount of entanglement but also the *architecture of the network* is important, when analyzing each influence k_s instead of $k_{\max}$. As an example, consider the task shown in Fig. 2, where the state processes an amount of genuine network entanglement that scales with M but fails to achieve the HL. A sufficient condition for the bound (5) to attain the HL is to have all vertices covered by M hyperedges. Constructing a sensing protocol for such network states is an interesting direction for future work.

Error bound as an entanglement witness.—The bound (4) on $\mathcal{F}_Q$ (and, equivalently, Theorem 1) can also be used as a witness of genuine multipartite network entanglement [21]. Previous works have shown that QFI with respect to a noninteracting Hamiltonian can detect genuine k-body entanglement, which cannot be produced by probabilistic mixing of pure states in which no more than $(k-1)$ parties are entangled [49,50], and that an interacting Hamiltonian can be used to rule out fully separable states [51]. Our result extends both types of results to cover genuine network k entanglement [21], defined as entanglement that cannot be generated in a network in which sources are distributed to at most $k-1$ parties at a time. Because of the entanglement distribution and the inclusion of the parties' local channels, witnessing genuine k-network entanglement is stronger than witnessing genuine k-body entanglement.

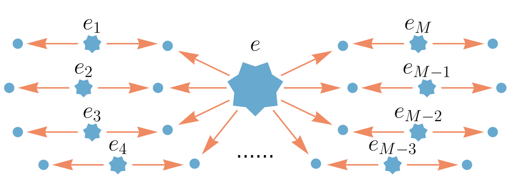

FIG. 2. Genuine network entanglement of order M is not sufficient for Heisenberg scaling. Consider a task of estimating the average of $2M$ local parameters in a network, which consists of a hyperedge e with cardinality M and M edges $e_1, \ldots, e_M$. The network state has genuine M-partite network entanglement, but the squared error scales as $1/M$ by Eq. (5) since there are M vertices with $k_s = 2$.

For example, consider a one-dimensional spin model with nearest-neighbor coupling: $H = \sum_{i=1}^{M} H_i$ such that each H_i acts nontrivially only on sites i and $i+1$. In terms of a network, each vertex is a site i and each s is a pair $(i, i+1)$. If the state can be prepared from r-partite sources, then it is easily seen that $k_s \leq 2r$, where the upper bound is achieved when the particles from the same source do not interact with each other. For such a state ρ, Theorem 1 yields

$$\Delta^2(\theta(\vec{\alpha})) \geq \sum_{i=1}^{M} \frac{\alpha_i^2}{8\nu r \mathrm{Var}(\rho, H_i)}. \tag{6}$$

If the model is translationally invariant and $\alpha_i = 1/M$ for every i (i.e., to estimate the average $\bar{\theta}$), the above bound is reduced to $\Delta^2(\bar{\theta}) \geq 1/[8\nu r M \mathrm{Var}(\rho, H_1)]$. More generally, for a model with bipartite interactions and τ nearest neighbors per site, we see that $\Delta^2(\bar{\theta}) \geq 1/[4\nu\tau r M \mathrm{Var}(\rho, H_1)]$; for instance, a d-dimensional cubic lattice has $\tau = 2^d$. The case of an Ising model was studied in Ref. [51] for $r = 1$, taking $H_i = \frac{1}{2} Z_i + (\epsilon/4) Z_i Z_{i+1}$. By optimizing over pure fully separable states, they find the upper bound on the QFI to be $M[1 + (5\epsilon^2/4)]$ for $\epsilon \ll 1$ and $M[\frac{1}{2} + \epsilon + (\epsilon^2/2)]$ for $\epsilon > \epsilon_c \approx 0.7302$. Our bound gives $M[2 + 2\epsilon + (\epsilon^2/2)]$, which is less tight, but has the advantage of being easily extended to $r > 1$ without the need to search for optimal states.

Precision bounds for shallow circuits.—Our technique can also be applied to scenarios where an entangled probe state is prepared from a circuit with local gates of a finite depth. We first consider an unentangled (i.e., fully separable) state ρ input into a circuit composed of l-local gates. This has depth D, meaning that there are D layers such that the gates are nonoverlapping within each layer. The gates have corresponding unitaries $U_{j,\alpha}$, where j labels the layer and α the index within the layer. The unitary for the full circuit is $U = (\otimes_{\alpha_D} U_{d,\alpha_D}) \ldots (\otimes_{\alpha_i} U_{1,\alpha_1})$. The output from the circuit $\sigma = U \rho U^\dagger$ is used as a probe for sensing rotations generated by a p-local Hamiltonian $H = \sum_{i=1}^{M} H_i$, where each H_i acts locally. Then the QFI is $\mathcal{F}_Q(\sigma, H) = \mathcal{F}_Q(\rho, U^\dagger H U)$. The locality q of the transformed Hamiltonian $U^\dagger H U$ can be bounded using the depth D. We can give different bounds depending on the structure of the circuit. This follows from a light cone argument as described in Ref. [52].

With no particular geometry, the weight of each Hamiltonian term (i.e., number of subsystems acted upon) increases by a factor of at most l under the application of each layer. Therefore, we have $q \leq l^D p$. A more useful bound cannot be obtained unless we know the circuit structure. If, for example, the circuit forms a one-dimensional chain with

$l = 2$ and H being 1-local, then we see that $U^\dagger H U$ has terms interacting $q \leq 2D+1$ neighbors. It follows that $\mathcal{F}_Q(\sigma, H) \leq 4q \sum_{i=1}^M \mathrm{Var}(\rho, U^\dagger H_i U) \leq 4(2D+1) \sum_{i=1}^M \mathrm{Var}(\rho, U^\dagger H_i U)$. The HL can therefore only be approached if $D \sim M$. A two-dimensional square lattice works similarly, replacing $q \leq D^2 + (D+1)^2$ by counting the number of lattice points reachable from a given starting point in D unit steps. Hence, we now only need a shallow circuit with $D \sim \sqrt{M}$ to get the HL. In general, with local geometry, we would require D to be similar to the size of the system. Using this idea, we can also bound the QFI when the parameter of interest is embedded in the circuit U (see [44] for details).

Achieving the Heisenberg limit via postselection.—Next we introduce a concrete protocol. Consider a network state with a hypergraph topology $G(\mathcal{V}, \mathcal{E})$, where each $e \in \mathcal{E}$ represents a GHZ state shared by all sensors $v \in e$. Each sensor v_j can locally access a signal θ_j, which is a phase gate $e^{-i\theta_j Z_j/2}$ with Z_j being the Pauli-Z operator located at the jth sensor's place. That is, $\mathcal{S}$ is the collection of all singletons and $|\mathcal{S}| = |V|$. The task is for a center, which could be any one of the sensors, to estimate an arbitrary combination of parameters $\theta(\vec{\alpha}) \coloneqq (1/M) \sum_{j=1}^M \tilde{\alpha}_j \theta_j$ with the assistance of all sensors, where with every $\tilde{\alpha}_j \in \mathbb{Z} \backslash \{0\}$ and $|\tilde{\alpha}_j|$ is upper bounded by L for some known constant $L \in \mathbb{N}^*$.

Besides the precision of estimation, we require the parameter estimation to respect the privacy of involved parties. Concretely, the center requires no other party to obtain the full knowledge of $\vec{\alpha}$. On the other hand, each sensor wishes to keep its local value θ_j private while still assisting the center to estimate $\theta(\vec{\alpha})$.

The standard deterministic protocol of measuring locally and communicating the outcomes to the center fails the second requirement unless the network state is of specific global topology (e.g., being a large GHZ state spanning all vertices) [53]. We present a probabilistic protocol that achieves both privacy requirements and, as a bonus, attains HL when successful, with almost no restriction on the network topology. Probabilistic protocols are prevalent in quantum sensing [54–56] and quantum information processing [31–38]. For metrology, it has been shown to boost the precision to the HL [31] and beyond [34]. Whether such an appealing feature persists under the constraint of quantum networks, however, remains largely unexplored. The protocol runs as described by Protocol 1. Note that, if $\vec{\alpha}$ is known *a priori*, we need assume only that the sensors have universal local control and the classical communication of measurement outcomes to the center can be delayed until the end. Therefore the sensors also do not need a quantum memory. However, if the "task allocation" step is included, the choice of $\vec{\alpha}$ is communicated to the sensors before the measurement step.

For Protocol 1, it is obvious that the first privacy requirement is fulfilled, as each sensor learns only a single entry $\tilde{\alpha}_j$. Note that in the (most interesting) scenario of a large network, it can be extremely difficult for the sensors to conspire and communicate the entire $\vec{\alpha}$. In addition, the probability that an arbitrary clique of sensors succeeds (and other sensors fail) is independent of the values of $\{\theta_i\}$. The conditional state at the center also depends only on $\theta(\vec{\alpha})$ (see [44] for the proof). As a consequence, no additional information on the local parameters $\{\theta_i\}$ is leaked to the center whether or not the protocol is successful. The second requirement of privacy is thus fulfilled.

Protocol 1. Probabilistic metrology of a global parameter over a network state.

1: (State preparation.) Each source e prepares a $|e|$-qubit GHZ state and distributes it among $v \in e$.
2: (Center election and local pre-processing.) An arbitrary sensor v^* is selected as the center. Each of the other sensors locally prepares L copies of the plus state $|+\rangle \coloneqq (1/\sqrt{2})(|0\rangle + |1\rangle)$.
3: (Signal acquisition.) Each $v_j \neq v^*$ passes each plus states once through the signal θ_j. The resultant state is (up to an irrelevant global phase) $|+_{\theta_j}\rangle^{\otimes L}$ with $|+_{\theta_j}\rangle \coloneqq (1/\sqrt{2})(|0\rangle + e^{i\theta_j}|1\rangle)$.
4: (Task allocation.) The center sends the weight $\tilde{\alpha}_j$ to the jth vertex.
5: (Local measurement and postselection.) For each sensor v_j (including v^*), it first discards $L - |\tilde{\alpha}_j|$ plus states and keeps the state $|+_{\theta_j}\rangle^{\otimes \tilde{\alpha}_j}$. Next, if $\tilde{\alpha}_j < 0$, the sensor performs X on each qubits of the plus states. If $v_j = v^*$ in this case, it performs X on the single qubit that acquired the signal. Finally, the sensor performs a binary measurement $\{|\mathrm{GHZ}_j\rangle\langle \mathrm{GHZ}_j|, I - |\mathrm{GHZ}_j\rangle\langle \mathrm{GHZ}_j|\}$, where $|\mathrm{GHZ}_j\rangle$ denotes the GHZ state of all its local qubits (including the plus states with signals and the qubits from the sources). For every $v_j \neq v^*$, it declares success if the first outcome (i.e., $|\mathrm{GHZ}_j\rangle\langle \mathrm{GHZ}_j|$) is obtained.
6: (Estimation.) Conditioning on all other sensors declaring success, the probability that v^* yields $|\mathrm{GHZ}^*\rangle\langle \mathrm{GHZ}^*|$ is a function of θ, from which an unbiased estimate $\hat{\theta}$ on $\theta(\vec{\alpha})$ can be obtained.

As for the precision, Protocol 1 makes ML queries in total to the local signal sources for each run (when estimating the average, in particular, we have $L = 1$). When $\{\tilde{\alpha}_j\}$ are (noninteger) rational numbers, one may still apply the protocol with $\tilde{\alpha}_j$ replaced by $\tilde{L}\tilde{\alpha}_j$, where $\tilde{L}$ is the smallest positive integer such that $\{\tilde{L}\tilde{\alpha}_j\}$ are all integers. The queries per run becomes $M\tilde{L} \max_j \tilde{\alpha}_j$. For a generic irrational $\tilde{\alpha}_j$, one may round it to a close rational number prior to applying the protocol. The protocol achieves the HL (i.e., QFI $\sim M^2$) as long as G remains connected after removing the center (i.e., when v^* is not a cut vertex). The proof can be found in [44].

With this, we see that combining local resources and postselection allows us to achieve the optimal sensing precision up to a constant. In contrast, deterministic protocols are bound by the standard quantum limit (i.e., QFI $\sim M$) by Theorem 1. Note that the few-partite entanglement in network states plays an essential role in achieving the (probabilistic) HL, as a completely local state remains local after postselection. As a concrete example, one may consider G being a cyclic network of M vertices, where each vertex (sensor) v_j holds a local signal with parameter θ_j [see Fig. 1(a)], and each edge represents one Bell state $|B_0\rangle \coloneqq (1/\sqrt{2})(|00\rangle + |11\rangle)$. This state is obviously local, and by Theorem 1 the performance of any deterministic protocol is bounded by the standard quantum limit. On the other hand, Protocol 1, when applied to this cyclic network [see Fig. 1(b) for an illustration], achieves the HL.

The overall success probability of Protocol 1 can be lower bounded as (see [44] for details)

$$\log_2 p_{\text{succ}} \geq -\left[\sum_{v\neq v^*}|\tilde{\alpha}_v| + \sum_{e\in\mathcal{E}}|e| - |\mathcal{E}(v^*)|\right], \qquad (7)$$

with $\mathcal{E}(v^*)$ being the set of edges containing v^* and $|e|$ being the number of vertices contained in e. The probability p_{succ} vanishes with increasing network size. The advantage thus vanishes if one takes into account the failed cases, which is a general limitation of postselected metrology [33] rather than a defect of Protocol 1.

Protocol 1 also features other interesting advantages. One is that the sensors could decide which $\theta(\vec{\alpha})$ to estimate after each local signal is applied. Via postselection, they could "steer" the state to maximize the precision of $\theta(\vec{\alpha})$ for one particular configuration of $\vec{\alpha}$. Whenever needed, the election of the center can even be delayed to after the acquisition of the signal by a minor modification of the protocol (i.e., to let every sensor prepare $[(|0\rangle + |1\rangle)/\sqrt{2}]$ states to host the signal). We emphasize that no extra classical communication (at the "task allocation" step) is needed if $\vec{\alpha}$ is known *a priori*. Therefore, the extra classical communication is not the reason for the precision enhancement.

Existing works of cryptographic quantum metrology required global entanglement (e.g., GHZ states spanning the network) [57–61], while our work shows the possibility of achieving desired privacy when global entanglement is not accessible. It should also be noted that our discussion does not cover the verification of probe states, which can be discussed independently.

Conclusion.—In this work, we explored both the limitation (by deriving a general bound) and the potential (by designing a probabilistic protocol) of network states in metrology. Our work opens a new line of research and hints at more future possibilities along it. For example, it is an interesting open question whether the protocol can be extended to the case of reference frame alignment. The entanglement of the network state and its usefulness for metrology can be enhanced by LOCC, which depends on the quality of quantum memory. Since the range of entanglement affects the success probability, the relation between the quality of quantum memory and the success probability is also desirable to investigate as the technology develops.

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[1] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).

[2] P. Komar, E. M. Kessler, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Nat. Phys. 10, 582 (2014).

[3] T. J. Proctor, P. A. Knott, and J. A. Dunningham, Phys. Rev. Lett. 120, 080501 (2018).

[4] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss-Feig, Phys. Rev. Lett. 121, 043604 (2018).

[5] Z. Eldredge, M. Foss-Feig, J. A. Gross, S. L. Rolston, and A. V. Gorshkov, Phys. Rev. A 97, 042337 (2018).

[6] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Phys. Rev. A 97, 032329 (2018).

[7] K. Qian, Z. Eldredge, W. Ge, G. Pagano, C. Monroe, J. V. Porto, and A. V. Gorshkov, Phys. Rev. A 100, 042304 (2019).

[8] P. Sekatski, S. Wölk, and W. Dür, Phys. Rev. Res. 2, 023052 (2020).

[9] Z. Zhang and Q. Zhuang, Quantum Sci. Technol. 6, 043001 (2021).

[10] M. Fadel, B. Yadin, Y. Mao, T. Byrnes, and M. Gessner, New J. Phys. 25, 073006 (2023).

[11] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S. Neergaard-Nielsen, and U. L. Andersen, Nat. Phys. **16**, 281 (2020).
[12] Y. Xia, W. Li, W. Clark, D. Hart, Q. Zhuang, and Z. Zhang, Phys. Rev. Lett. **124**, 150502 (2020).
[13] L.-Z. Liu, Y.-Z. Zhang, Z.-D. Li, R. Zhang, X.-F. Yin, Y.-Y. Fei, L. Li, N.-L. Liu, F. Xu, Y.-A. Chen *et al.*, Nat. Photonics **15**, 137 (2021).
[14] S.-R. Zhao, Y.-Z. Zhang, W.-Z. Liu, J.-Y. Guan, W. Zhang, C.-L. Li, B. Bai, M.-H. Li, Y. Liu, L. You *et al.*, Phys. Rev. X **11**, 031009 (2021).
[15] Y. Zhong, H.-S. Chang, A. Bienfait, É. Dumur, M.-H. Chou, C. R. Conner, J. Grebel, R. G. Povey, H. Yan, D. I. Schuster *et al.*, Nature (London) **590**, 571 (2021).
[16] S. Zhang, Y.-K. Wu, C. Li, N. Jiang, Y.-F. Pu, and L.-M. Duan, Phys. Rev. Lett. **128**, 080501 (2022).
[17] A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).
[18] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).
[19] H. J. Kimble, Nature (London) **453**, 1023 (2008).
[20] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Nature (London) **464**, 45 (2010).
[21] M. Navascués, E. Wolfe, D. Rosset, and A. Pozas-Kerstjens, Phys. Rev. Lett. **125**, 240505 (2020).
[22] J. Åberg, R. Nery, C. Duarte, and R. Chaves, Phys. Rev. Lett. **125**, 110505 (2020).
[23] M.-X. Luo, Adv. Quantum Technol. **4**, 2000123 (2021).
[24] T. Kraft, S. Designolle, C. Ritz, N. Brunner, O. Gühne, and M. Huber, Phys. Rev. A **103**, L060401 (2021).
[25] T. Kraft, C. Spee, X.-D. Yu, and O. Gühne, Phys. Rev. A **103**, 052405 (2021).
[26] K. Hansenne, Z.-P. Xu, T. Kraft, and O. Gühne, Nat. Commun. **13**, 496 (2022).
[27] K. M. Audenaert and M. B. Plenio, New J. Phys. **7**, 170 (2005).
[28] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Nest, and H.-J. Briegel, in *Proceedings of the International School of Physics "Enrico Fermi"* (2006), Vol. 162, pp. 115–218; arXiv:quant-ph/0602096.
[29] O. Makuta, L. T. Ligthart, and R. Augusiak, npj Quantum Inf. **9**, 117 (2023).
[30] Y.-X. Wang, Z.-P. Xu, and O. Gühne, npj Quantum Inf. **10**, 11 (2024).
[31] B. Gendra, E. Ronco-Bonvehi, J. Calsamiglia, R. Munoz-Tapia, and E. Bagan, Phys. Rev. Lett. **110**, 100501 (2013).
[32] B. Gendra, J. Calsamiglia, R. Munoz-Tapia, E. Bagan, and G. Chiribella, Phys. Rev. Lett. **113**, 260402 (2014).
[33] J. Combes, C. Ferrie, Z. Jiang, and C. M. Caves, Phys. Rev. A **89**, 052117 (2014).
[34] D. R. Arvidsson-Shukur, N. Yunger Halpern, H. V. Lepage, A. A. Lasek, C. H. Barnes, and S. Lloyd, Nat. Commun. **11**, 1 (2020).
[35] N. Lupu-Gladstein, Y. B. Yilmaz, D. R. M. Arvidsson-Shukur, A. Brodutch, A. O. T. Pang, A. M. Steinberg, and N. Y. Halpern, Phys. Rev. Lett. **128**, 220504 (2022).
[36] S.-J. Xiong, P.-F. Wei, H.-Q.-C. Wang, L. Shao, Y.-N. Sun, J. Liu, Z. Sun, and X.-G. Wang, arXiv:2212.05285.
[37] G. Chiribella, Y. Yang, and A. C.-C. Yao, Nat. Commun. **4**, 1 (2013).
[38] J. Calsamiglia, B. Gendra, R. Muñoz-Tapia, and E. Bagan, New J. Phys. **18**, 103049 (2016).
[39] Note that, in general, there could be more than one parameter of interest, while in this work we focus on the fundamental case of estimating only one parameter.
[40] To estimate $f(\{\theta_s\})$ that is nonlinear in the parameters, one can assume w.l.o.g. that the parameters lie in the vicinity of the true values (see, e.g., [41], Sec. 9) and reduce to the linear case via Talyor expansion.
[41] Y. Yang, G. Chiribella, and M. Hayashi, Commun. Math. Phys. **368**, 223 (2019).
[42] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, NY, 1976).
[43] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer Science & Business Media, New York, 2011), Vol. 1.
[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.210801, which contains Refs. [45–47], proofs of Theorems 1 and 2, general case of precision bounds for shallow circuits, derivation of the success probability of Protocol 1, and proof of the privacy statement.
[45] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, J. Phys. A **53**, 023001 (2020).
[46] G. Tóth and D. Petz, Phys. Rev. A **87**, 032324 (2013).
[47] M. Gessner, L. Pezzè, and A. Smerzi, Phys. Rev. Lett. **121**, 130503 (2018).
[48] A. Ehrenberg, J. Bringewatt, and A. V. Gorshkov, Phys. Rev. Res. **5**, 033228 (2023).
[49] G. Tóth, Phys. Rev. A **85**, 022322 (2012).
[50] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Phys. Rev. A **85**, 022321 (2012).
[51] L. Pezzè, Y. Li, W. Li, and A. Smerzi, Proc. Natl. Acad. Sci. U.S.A. **113**, 11459 (2016).
[52] N. Yu and T.-C. Wei, arXiv:2303.08938.
[53] The privacy of the local parameters may be preserved by a modification of the deterministic protocol: The jth sensor first estimates θ_j and then sends a protected phase $\Theta_j = \theta_j + v_j$ to the center with some randomly generated value v_j. Provided $\vec{v}^T\vec{\alpha} = 0$, the center can effectively learn $\theta(\vec{\alpha}) = \vec{\alpha}^T\vec{\Theta}$ without knowing $\vec{\theta}$. However, to distribute to each sensor the desired v_j, it requires a trusted third party who learns $\vec{\alpha}$, violating the first privacy requirement.
[54] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature (London) **429**, 161 (2004).
[55] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature (London) **429**, 158 (2004).
[56] T. Nagata, R. Okamoto, J. L. O'brien, K. Sasaki, and S. Takeuchi, Science **316**, 726 (2007).
[57] P. Yin, Y. Takeuchi, W.-H. Zhang, Z.-Q. Yin, Y. Matsuzaki, X.-X. Peng, X.-Y. Xu, J.-S. Xu, J.-S. Tang, Z.-Q. Zhou *et al.*, Phys. Rev. Appl. **14**, 014065 (2020).
[58] N. Shettell, E. Kashefi, and D. Markham, Phys. Rev. A **105**, L010401 (2022).
[59] N. Shettell and D. Markham, Phys. Rev. A **106**, 052427 (2022).
[60] N. Shettell, M. Hassani, and D. Markham, arXiv:2207.14450.
[61] H. Kasai, Y. Takeuchi, Y. Matsuzaki, and Y. Tokura, arXiv:2305.14119.