Ground State Nature and Nonlinear Squeezing of Gottesman-Kitaev-Preskill States

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The main bottleneck for universal quantum computation with traveling light is the preparation of Gottesman-Kitaev-Preskill states of sufficient quality. This is an extremely challenging task, experimental as well as theoretical, also because there is currently no single easily computable measure of quality for these states. We introduce such a measure, Gottesman-Kitaev-Preskill squeezing, and show how it is related to the current ways of characterizing the states. The measure is easy to compute and can be easily employed in state preparation as well as verification of experimental results.

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Universal quantum computation is a goal pursued on many experimental platforms [1–3], each one with a set of advantages and disadvantages. Bosonic harmonic oscillators encoded into modes of room temperature traveling light offer unprecedented scalability, arising from their compatibility with modern communication technologies [4–6]. Traveling light has already been used for demonstrating quantum advantage [2,3], but the universal computation requires passing a significant bottleneck—preparation of suitable non-Gaussian states.

Gottesman-Kitaev-Preskill (GKP) states can provide the necessary non-Gaussianity [5,7–10]. They can be effectively multiplexed and error corrected by feasible Gaussian operations and measurements [6,11–16]. However, preparation of high quality GKP states is a task that has so far been achieved only in systems with strong coupling to an auxiliary qubit [17–19]. Unfortunately, traveling light requires a more challenging approach [20]. The currently available optical preparation is based on the boson sampling [21], in which some modes of a multimode entangled Gaussian state are measured by photon number resolving detectors and some outcomes herald preparation of the desired state [20,22–25]. A key component of this approach is the numerical simulation optimization of the state preparation setup with the goal of preparing a suitable state with a feasible success rate [26,27]. However, quantifying the quality of the prepared GKP approximation is still an open problem.

The ideal GKP state is a non-normalizable abstraction with unit value of the stabilizers and infinitely squeezed states [7]. All of these are separate properties, so, rather than using a multiparameter optimization, the current customary approach relies on devising an approximative state with limited quality and aiming to prepare it with high fidelity [20,23–25,28,29]. However, for the sake of both setup optimization and state evaluation, it would be practical to identify a single measure of non-Gaussianity inherent to GKP states, similarly to the cubic squeezing of resource states for the cubic phase gate [7,30–33].

In this Letter we introduce a class of Hermitian positive semidefinite operators, whose ground states are the GKP qubit states in various topologies. We show that their mean values can be interpreted as nonlinear squeezing of the GKP states and that this squeezing neatly ties together the existing methods for evaluation. Furthermore, it can be advantageously used for efficiently evaluating experimentally prepared states, optimizing the state preparation procedures, and even arriving at fundamental relation between the quality of GKP states and their stellar rank.

The ideal GKP qubit state $|0_L\rangle$ is defined as the infinite superposition of quadrature eigenstates,

$$|0_L\rangle \propto \sum_{s\in Z}|x = 2s\sqrt{\pi}\rangle. \tag{1}$$

Its main properties arise from the state being the simultaneous eigenstate of stabilizer displacement operators $\hat{S}_x = e^{-i2\sqrt{\pi}\hat{p}}$ and $\hat{S}_p = e^{-i2\sqrt{\pi}\hat{x}}$, where $\hat{x}$ and $\hat{p}$ are quadrature operators with $[\hat{x},\hat{p}] = i$. Furthermore, the state is preserved under $\sqrt{\hat{S}_p}$ and changes to the orthogonal state $|1_L\rangle$ under $\sqrt{\hat{S}_x}$. Any approximation $|\tilde{0}_L\rangle$ of the state (1) should follow

$$\hat{S}_x|\tilde{0}_L\rangle \approx |\tilde{0}_L\rangle, \qquad \sqrt{\hat{S}_p}|\tilde{0}_L\rangle \approx |\tilde{0}_L\rangle, \tag{2}$$

where "≈" represents approximate equality and would be replaced by the equality for the ideal state $|0_L\rangle$. The main result of this Letter is the realization that the GKP qubit state (1) can be also found as the zero eigenvalue eigenstate of operator

$$\hat{Q}_0 = 2\sin^2\left(\frac{\hat{x}\sqrt{\pi}}{2}\right) + 2\sin^2(\hat{p}\sqrt{\pi}) = \frac{1}{2}\left[4 - \sqrt{\hat{S}_p} - \sqrt{\hat{S}_p}^{\dagger} - \hat{S}_x - \hat{S}_x^{\dagger}\right]. \tag{3}$$

The asymmetry between $\hat{x}$ and $\hat{p}$ follows the asymmetry of the GKP state (1) and the minimal operations (2) that preserve it. For any quantum state, the mean value $\langle\hat{Q}_0\rangle$ directly depends on the value of stabilizers and it can be zero only when eigenvalues of both $\hat{S}_x$ and $\sqrt{\hat{S}_p}$ are equal to 1. At the same time, the asymmetry between $\hat{x}$ and $\hat{p}$ ensures that the eigenstate of $\hat{Q}_0$ is indeed the GKP qubit state $|0_L\rangle$, rather than any other state preserved under the action of the stabilizers. Looking at it from a different perspective, operator (3) can produce the value of zero only for those quantum states with nonzero wave function only for discrete periodic values in both x and p representation. This is a property satisfied only by the GKP qubit state $|0_L\rangle$.

The main advantage of the specific combination of displacement operators is that the resulting operator $\hat{Q}_0$ is Hermitian and positive semidefinite. The operator can be therefore taken as a Hamiltonian and the GKP state as its ground state. Furthermore, the mean value $\langle\hat{Q}_0\rangle$ in some cases equals to the variance of $\sqrt{\hat{Q}_0}$, which is a nonlinear functional of quadrature operators, and can be therefore interpreted as nonlinear squeezing. The concept of squeezing is closely tied to continuous variable quantum optics and it serves as one of its main resources [34–36]. As a figure of merit for preparation of squeezed states, the variance of $\hat{x}$ neatly transitions between the classical limit given by the vacuum state with variance $\langle\mathrm{vac}|\hat{x}^2|\mathrm{vac}\rangle = 1/2$, and the ultimately unachievable limit of the unphysical quadrature eigenstate $\langle x=0|\hat{x}^2|x=0\rangle \to 0$. The degree of squeezing for any quantum state $\hat{\rho}$ with $\langle\hat{x}\rangle = 0$ can be then given by the normalized second moment $2\mathrm{Tr}[\hat{\rho}\hat{x}^2]$, which can be expressed either as a direct value or in decibels. The concept can be expanded to moments of other operators, which has been demonstrated on resource states for the cubic phase gate [15,30,32,33,37].

In analogy to quadrature squeezing, we can define the nonlinear *GKP squeezing* for any quantum state ρ as

$$\xi_0 = \langle\hat{Q}_0\rangle = \mathrm{Tr}[\hat{\rho}\hat{Q}_0]. \tag{4}$$

It can be quickly shown that, for any Gaussian state, the GKP squeezing cannot be lower than one, which is achieved for a quadrature eigenstate [38]. Quantity (4) is also bounded for classical states that can be expressed as mixtures of coherent states and always result in $\xi_0 \geq 2 - e^{-\pi/2} - e^{-\pi}$. Note that definition (4) does employ the second moment of $\sqrt{\hat{Q}_0}$ and not the variance, which is customarily considered in the Gaussian squeezing scenario. The omission of the square of the first moment was done deliberately to keep the GKP squeezing as a linear functional of the density matrix. The choice is also justified as the optimal states have the mean values equal to zero.

The GKP squeezing (4) is defined for the GKP state $|\tilde{0}\rangle$. However, since other GKP states and other encodings are only a Gaussian operation away, we can easily define operators for the other GKP states by replacing the arguments of the functions. We can thus define

$$\begin{aligned}
\hat{Q}_1 &= 2\cos^2\left(\frac{\hat{x}\sqrt{\pi}}{2}\right) + 2\sin^2\left(\hat{p}\sqrt{\pi}\right),\\
\hat{Q}_{s0} &= 2\sin^2\left(\hat{x}\sqrt{\frac{\pi}{2}}\right) + 2\sin^2\left(\hat{p}\sqrt{\frac{\pi}{2}}\right),\\
\hat{Q}_{s1} &= 2\cos^2\left(\hat{x}\sqrt{\frac{\pi}{2}}\right) + 2\sin^2\left(\hat{p}\sqrt{\frac{\pi}{2}}\right),\\
\hat{Q}_h &= 2\sin^2(\kappa_+\hat{x} - \kappa_-\hat{p}) + 2\sin^2(\kappa_+\hat{p} - \kappa_-\hat{x}),\\
\hat{Q}_G &= \hat{U}_G^\dagger\hat{Q}_0\hat{U}_G,\\
\hat{Q}_j &= 2\sin^2(a\hat{x}) + 2\sin^2(b\hat{p}),
\end{aligned} \tag{5}$$

which are the operators whose ground states are, respectively, GKP qubit state $|1_L\rangle$, symmetrical GKP states $|0_S\rangle$ and $|1_S\rangle$ on a square grid, hexagonal GKP on a triangular grid with $\kappa_\pm = \sqrt{(\pi/8)}(3^{1/4} \pm 3^{-1/4})$ [39], a GKP state transformed by an arbitrary unitary Gaussian operation $\hat{U}_G$ to fit into any desired encoding, and, finally, a general GKP type state on a grid with arbitrary spacing. Operators for qubit states 0 and 1 generally differ by displacement in $\hat{x}$, symmetrical grid is then obtained by squeezing [38]. The last operator stands aside because it cannot be always obtained from the others by a Gaussian unitary transformation and therefore does not always have a zero eigenvalue, but it is relevant for scenarios in which the grid is affected by nonunitary transformations, such as in the case of losses. Among these new operators, ξ_{s0} plays a prominent role—since the operator is symmetric, it has identical properties in both quadratures and therefore makes it easier to compare it to the existing figures of merit, which is what we are going to do now.

The GKP squeezing is directly tied to the *grid squeezing* proposed by [23,28], which is also expressed as the function of stabilizers. For any quadrature operator $\hat{q}$ and the corresponding displacement operator $e^{-iu\hat{q}}$ that translates the conjugate quadrature by a grid constant u, the grid squeezing is defined as

$$\Delta_{q,u}^2 = -\frac{4}{u^2}\ln\left|\langle e^{-iu\hat{q}}\rangle\right|. \tag{6}$$

Please note that we have deliberately changed the notation from [23,28] to allow for easier discussion. The GKP squeezing can be directly expressed as a function of grid squeezing values for quadratures $\hat{x}$ and $\hat{p}$:

$$\xi_{s0} = 2 - e^{-\frac{\pi}{2}\Delta^2_{x,\sqrt{2\pi}}} - e^{-\frac{\pi}{2}\Delta^2_{p,\sqrt{2\pi}}} \approx \frac{\pi}{2}\Delta^2_{x,\sqrt{2\pi}} + \frac{\pi}{2}\Delta^2_{p,\sqrt{2\pi}}, \tag{7}$$

where the approximation holds for states with high squeezing. This correspondence can be used to gain some insight about the suitability of the respective states for error correction [5]. The exact equivalence will need to be derived by further research, but we can expect −8.69 dB (−5.06 dB) of GKP squeezing to be sufficient (necessary) for fault tolerance [38]. These thresholds are invariant under Gaussian operations if they are accompanied by suitable transformation of the grid. A sufficient amount of GKP squeezing therefore indicates that the state can be transformed into an error correctable form by Gaussian transformations.

The GKP squeezing also directly relates to the fidelity with which it is possible to prepare the approximate GKP state

$$|0_{s,g}\rangle \propto \sum_{s\in Z} e^{-\frac{g}{2}(s\sqrt{2\pi})^2} \frac{1}{(\pi g)^{1/4}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s\sqrt{2\pi})^2}{2g}} |x\rangle dx, \quad (8)$$

with grid squeezing $\Delta^2_{x,\sqrt{2\pi}} = \Delta^2_{p,\sqrt{2\pi}} = g$, which is equal to the normalized quadrature variance of the individual squeezed states forming the superposition. For such state, the GKP squeezing can be found analytically as

$$\xi_{s0,g,F=1} = \langle 0_{s,g}|\hat{Q}_{s0}|0_{s,g}\rangle = 2 - 2e^{-\frac{\pi g}{2}}. \quad (9)$$

The linearity of the operator and the bounded range of its eigenvalues guarantee that, for any state that has fidelity $F = f$ to approximate state (8) with parameter g, the GKP squeezing is bounded by

$$f\xi_{s0,g,F=1} \le \xi_{s0,g,F=f} \le f\xi_{s0,g,F=1} + 4(1-f). \quad (10)$$

See [38] for details. The upper and lower bounds are shown in Fig. 1. The spread suggests that squeezing might be a more suitable figure of merit for evaluation of quantum states than the fidelity.

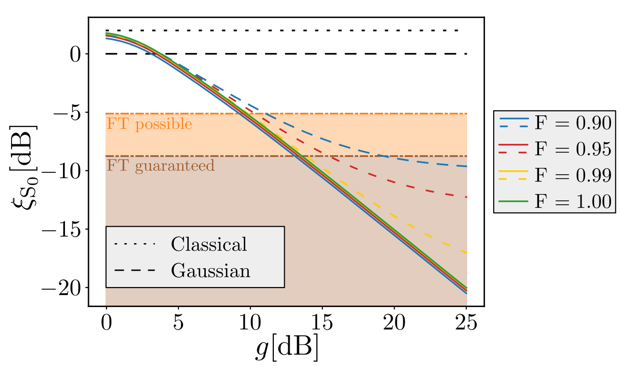

FIG. 1. The range of GKP squeezing ξ_{s0} for quantum states that have fidelity F with an approximate state (8) relative to quadrature squeezing g (see the legend). Solid lines show the lower bound, dashed lines the upper bound. The black lines represent the classical and Gaussian thresholds. Background colors mark areas in which fault tolerance is guaranteed (brown) and possible (orange).

The operators (3) and (5) are Hermitian and can be, in principle, directly measured. Interestingly, for the sake of evaluation, the measurement can be replaced by direct measurements of two quadratures. From their statistics it is now possible to obtain the values of sine and cosine functions, together with their respective error bars, and use them to construct the value of GKP squeezing. Any value smaller than one witnesses the non-Gaussian nature of the state and low enough value ensures fault tolerance. The measurement is also much more feasible than quantum tomography required for evaluating the fidelity.

Furthermore, when trying to assess the non-Gaussian nature of experimental data, one can calculate the GKP squeezing pertinent to any operator from (3) and (5) and look for the lowest value:

$$\xi_{\mathrm{opt}} = 2\ \min\langle \sin^2(c_{11}\hat{x} + c_{12}\hat{p} + d_1) + \sin^2(c_{21}\hat{x} + c_{22}\hat{p} + d_2)\rangle, \quad (11)$$

where the minimization is taken over the vector of parameters $(c_{11}, c_{12}, c_{21}, c_{22}, d_1, d_2)$. While the coefficients c_{ij} can be, in principle, arbitrary, they should form a symplectic matrix multiplied by $\sqrt{\pi/2}$ for GKP states. In all cases, the mean value (11) can be converted to a linear combination of mean values of displacement operators, which can be evaluated efficiently [37]. The minimum (11) then represents the best achievable GKP squeezing and the optimal vector determines the relevant grid. Taking $M_{\mathrm{GKP}} = -\ln \xi_{\mathrm{opt}}$ now gives us an operationally defined convex monotone of non-Gaussianity specific to GKP states [40–42].

On a theoretical side, any of operators (3) and (5) can be used as a straightforward cost function for numerical optimization of state preparation protocols. The optimization numerically simulates the experimental circuits planned for the state preparation for various parameters of the setups with the goal of finding parameters for which the produced quantum state has the required properties. Such properties can be efficiently evaluated by GKP squeezing. For example, it can be quickly seen that the breeding protocol [23], which is often cited as a deterministic way to prepare GKP states, actually requires the postselection to work [38].

The operators (3) and (5) can also give us a valuable insight on how the GKP squeezing of quantum states depends on their stellar rank [43]. We can see that by taking any one of the operators in Fock representation and turning it into a square Hermitian matrix by projecting it on a finite dimensional Hilbert space. The most straightforward way of doing that relies on decomposing the operator into a sum of displacement operators (2), constructing their matrices on a larger Hilbert space, and then truncating them [37,38]. The minimal eigenvalue of this matrix now determines the maximal achievable GKP squeezing and the corresponding

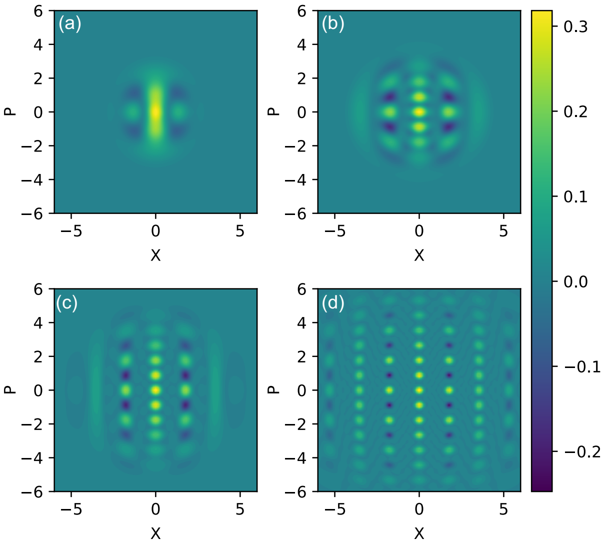

FIG. 2. Wigner functions of the lowest eigenvalue eigenstates of $\hat{Q}_0$ restricted to dimension (a) $\mathcal{N} = 5$, (b) $\mathcal{N} = 10$, (c) $\mathcal{N} = 20$, and (d) $\mathcal{N} = 50$.

eigenstate is the optimal quantum state. Figure 2 shows few examples of quantum states found in this way for operator (3) limited to several chosen dimensions. We can also quickly compare the maximal achievable GKP squeezing for different operators (5) to see which one of them is best suitable for state preparation, see Fig. 3. It can be seen that the GKP state $|1\rangle$ is the first to show a non-Gaussian behavior, already for $\mathcal{N} = 3$ corresponding to maximal photon number 2. The symmetrical GKP states $s0$ and $s1$ can be prepared slightly more easily than the other forms, but the difference is not large and the Hilbert space dimension related to the stellar rank is the determining factor.

The GKP squeezing also offers a straightforward way to evaluate effects of the dominant forms of decoherence suffered by GKP states—Gaussian loss and additive noise. Both of these can be efficiently described with the help of the Heisenberg picture; the quadrature operators of the mode after the decoherence evolution are given in terms of the initial quadrature operators as $\hat{x}_{\mathrm{out}} = \sqrt{\eta}\hat{x}_{\mathrm{in}} + \hat{x}_V$ and $\hat{p}_{\mathrm{out}} = \sqrt{\eta}\hat{p}_{\mathrm{in}} + \hat{p}_V$, where η represents the intensity loss coefficient and $\hat{x}_V$ and $\hat{p}_V$ represent the quadrature operators of the effective environmental mode with variance $V = \langle\hat{n}\rangle + [(1-\eta)/2]$, where $\langle\hat{n}\rangle$ is the number of added thermal photons. The GKP squeezing after such a channel can be expressed as

$$\xi_{s0,\mathrm{out}} = 2\gamma\left\langle \sin^2\left(\sqrt{\frac{\pi\eta}{2}}\hat{x}_{\mathrm{in}}\right)\right\rangle + 2\gamma\left\langle \sin^2\left(\sqrt{\frac{\pi\eta}{2}}\hat{p}_{\mathrm{in}}\right)\right\rangle + 2 - 2\gamma, \tag{12}$$

where $\gamma = e^{-\pi V}$ [38]. The two main effects of decoherence are the scaling of the grid and addition of a noise floor. The scaling of the grid can be ignored in single quadrature measurements, because it can be compensated by Gaussian phase sensitive amplification or by transforming the data. In this way, the purely lossy channel with η can be converted to purely noisy channel with $\eta = 1$ and $V = (1-\eta)/(2\eta)$. The effect is illustrated in Fig. 4, where we can see that roughly 10% losses can be expected to prevent fault tolerance. For the symmetrical GKP squeezing, the two quadratures are affected in the similar manner. This is not true in general; different topologies behave differently under decoherence. It could be therefore practical to take this into consideration when designing experiments [44,45].

In summary, from their very introduction it was realized that GKP states are simultaneous eigenstates of the commuting stabilizer operators. However, these operators, individually representing displacements in two quadratures, were always treated independently. Consequently, the figures of merit, apart from the fidelity to a finite energy

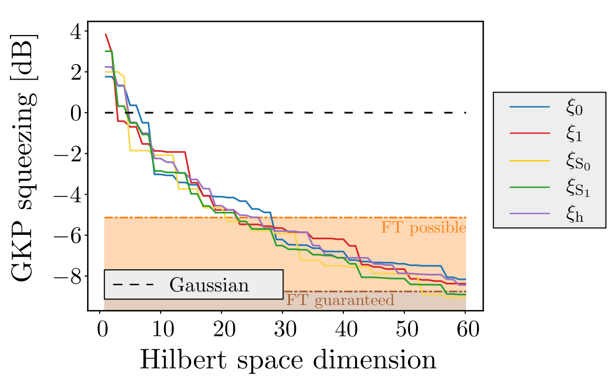

FIG. 3. Maximal GKP squeezing (4) in various topologies (see the legend) achievable for quantum states from a Hilbert space with dimension $\mathcal{N}$. The black lines represent the classical and Gaussian thresholds. Background colors mark areas in which fault tolerance is guaranteed (brown) and possible (orange).

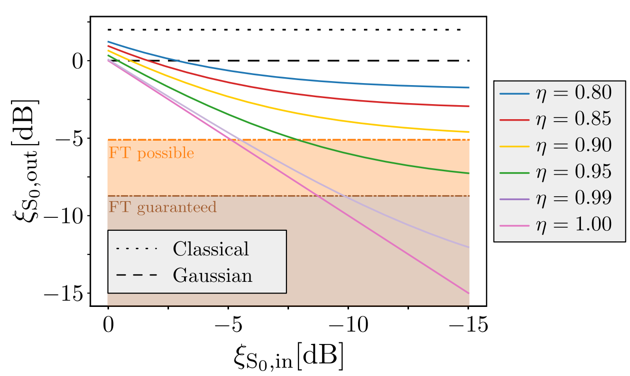

FIG. 4. The GKP squeezing $\xi_{s0,\mathrm{out}}$ after decoherence relative to the GKP squeezing before decoherence $\xi_{s0,\mathrm{in}}$ for various lossy channels (see the legend). The black lines represent the classical and Gaussian thresholds. Background colors mark areas in which fault tolerance is guaranteed (brown) and possible (orange).

approximate states, were also mostly focused on their independent properties, which led to difficult trade-offs, as each stabilizer could be, on its own, saturated by a Gaussian state. We have shown that it is possible to merge the stabilizers into a single positive semidefinite Hermitian operator. In any quantum state, the mean value of this operator can be interpreted as the amount of GKP squeezing that is a monotone for the particular non-Gaussianity relevant to GKP states. At the same time, the mean value directly depends on the values of stabilizers, which allows us to keep the methods of characterization developed so far. The new operator can be feasibly evaluated for experimentally prepared states as it requires only measurement in two orthogonal bases, offers an easy-to-compute cost function for the numerical methods employed in optimization of state preparation circuits, and its eigenstates in any given dimension are the best possible approximations of GKP states. The operator can be also treated as the Hamiltonian of a system for which the GKP states arise naturally as its ground states.

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