Entanglement Growth via Splitting of a Few Thermal Quanta

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Quanta splitting is an essential generator of Gaussian entanglement, exemplified by Einstein-Podolsky-Rosen states and apparently the most commonly occurring form of entanglement. In general, it results from the strong pumping of a nonlinear process with a highly coherent and low-noise external drive. In contrast, recent experiments involving efficient trilinear processes in trapped ions and superconducting circuits have opened the complementary possibility to test the splitting of a few thermal quanta. Stimulated by such small thermal energy, a strong degenerate trilinear coupling generates large amounts of nonclassicality, detectable by more than 3 dB of distillable quadrature squeezing. Substantial entanglement can be generated via frequent passive linear coupling to a third mode present in parallel with the trilinear coupling. This new form of entanglement, outside any Gaussian approximation, surprisingly grows with the mean number of split thermal quanta; a quality absent from Gaussian entanglement. Using distillable squeezing we shed light on this new entanglement mechanism for nonlinear bosonic systems.

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Introduction.—Entangled systems form the backbone of many fields of quantum physics and applications in quantum technology. Since the introduction of the fundamental Einstein-Podolsky-Rosen (EPR) states [1], the most frequently occurring entanglement seems to be Gaussian entanglement. In such cases, a coherent and low-noise external drive is used to linearize nonlinear systems and, therefore, the covariance matrix provides a full description [2]. However, it is widely anticipated that strongly nonlinear quantum interactions exhibit a much richer and exciting variety of phenomena which can potentially produce entanglement from low thermal energy, although typically at the cost of ease of detection, simple explanations, and analytical descriptions. Such novel entanglement might be much more common than anticipated and even appear more autonomously via small local thermal excitation without the need for a strong, coherent, and low-noise external drive.

In this Letter, we analyze the deterministic and thermal creation of just such a new form of entanglement, arising from the splitting of a few thermal quanta in trilinear interactions, with the logarithmic negativity (LN) increasing with *increasing* thermal occupation. We use distillable squeezing instead of conventional squeezing as a new and proper diagnostic tool to analyze such hidden entanglement beyond the Gaussian approximation. We thus begin our discussion from the lowest order nonlinear interactions between linear oscillators, usually the most feasible in experiments. Nondegenerate trilinear couplings, while producing the EPR type of Gaussian entanglement under a classical pump [3], do not produce bipartite entanglement from a thermal pump among any combination of the modes [4,5] [see also Supplemental Material (SM) [6]]. Hence, we focus on a degenerate trilinear interaction generating significant nonclassicality from splitting just a few thermal quanta. When the degenerate interaction is continuously combined with a linear coupling to an auxiliary mode, the resulting enhancement of logarithmic negativity can unconditionally reach ~0.77, saturating the entanglement potential (EP) [22]. Thermally induced entanglement (TIE) thus materializes straightforwardly via the splitting of a few thermal quanta in a regime very different from the Gaussian regime involving a strong coherent pump. To investigate the difference in these regimes we examine in detail the distribution and concentration of distillable squeezing [23] in such entangled states and compare with conventional squeezing from Gaussian entangled states.

Trilinear interactions in bosonic platforms.—The mechanical motion of trapped atoms is an excellent candidate for a proof-of-principle investigation of TIE due to several properties: (i) the strong nonlinear coupling of highly linear mechanical oscillators, (ii) low damping and substantial shielding from ambient background noises, and (iii) the potential scalability of the trapped atoms [24,25]. For the mechanical motion of electrically trapped ions, the intrinsic nonlinearity of the Coulomb coupling can be leveraged into several multimode versions of a trilinear Hamiltonian [26] of the type historically studied to investigate generalizations of squeezing to higher order moments [27]. The trapped ion versions of such Hamiltonians, both partially degenerate and nondegenerate, comprising two-mode and three-mode nonlinearities, respectively, have been experimentally achieved [28] and shown to be effective for tasks such as quantum simulation [29] and quantum refrigeration [30]. For these reasons, we prefer to frame our discussion around systems of trapped ions. In the microwave domain, however, superconducting circuits may take advantage of the nonlinearity of Josephson junctions to generate parametric amplification [31], three-photon down-conversion [32], and to induce trilinear Hamiltonians (SNAILs) [33–35] outside the rotating wave approximation. Furthermore, the TIE we describe may stimulate new nonlinear optics experiments using second- and high-harmonic generation [36,37], three and four wave mixing [4,38,39], or multiphoton Kerr processes [40,41].

Thermally induced nonclassicality and entanglement.—A pair of ions with equal mass m and charge e, contained in a harmonic trap, are coupled via the Coulomb interaction. If the radial trapping frequencies ω_x and ω_y are assumed to be much greater than the axial trapping frequency ω_z, then the ions distribute themselves along the z axis. The Coulomb interaction can be expanded to second order, inducing a natural transformation to the normal modes of the motion, wherein the collective vibrational modes are decoupled from each other. The first step beyond this harmonic approximation, involving third order terms in the expansion, removes the decoupling of the spatial motion leading to interactions involving both the radial and axial modes. Components of these nonlinear interactions are resonant properties of the collective motion of the ions. From here, we will assume that the most significant components of the potential are those contributing to the interaction between the x and z spatial components. In the rotating wave approximation, the interaction Hamiltonian takes the trilinear form,

$$H = \Omega_T\big(ab^{\dagger 2} + a^\dagger b^2\big), \tag{1}$$

where $\Omega_T = (3e^2/4|Q_0|^4)$. For a more detailed Hamiltonian analysis see, for instance, Refs. [42,43] or the SM [6]. This is a partially degenerate trilinear interaction in which the creation (annihilation) of a phonon in the axial direction results in the annihilation (creation) of a pair of phonons in the radial direction.

The quanta splitting and ensuing nonclassicality (necessary for entanglement) from the nonlinear dynamics may proceed from a small amount of classical thermal noise present in one of the subsystems. By initializing the axial (a) or radial (b) components of the motion in thermal states while the remainder is initialized in the ground state, the degree to which the nonlinearity converts the initial resource of thermal noise into nonclassical behavior can be examined (Fig. 1). In general, if the radial mode is used as a thermal pump of the axial mode, then the nonclassicality and corresponding entanglement potential is small. In contrast, our starting point is that a, using the small incoherent energy of a thermal pump, can drive nonclassicality in b.

In the ideal case, the nonclassicality of mode b can be immediately seen from the phonon distribution $P_k = \langle k|\rho|k\rangle$;

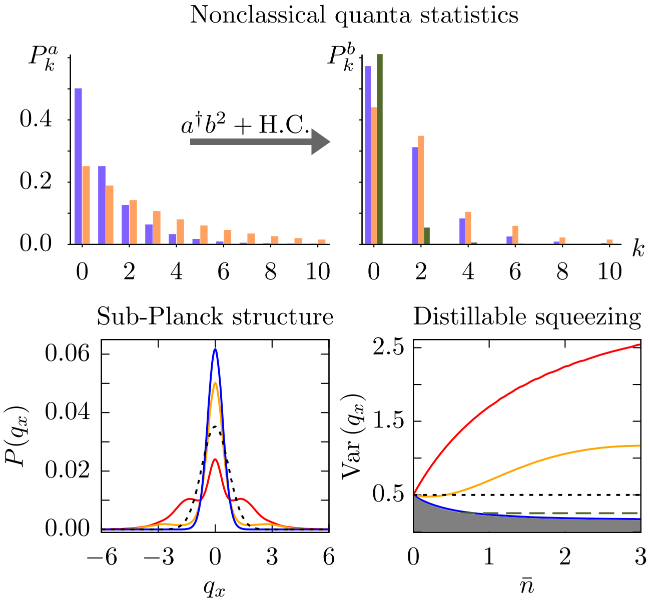

FIG. 1. Nonclassical quanta statistics: The coupling of mode a to mode b through the trilinear coupling [Eq. (1)] dynamically splits thermal quanta in a into pairs in b, generating phase-insensitive nonclassical phonon distributions, verified by Klyshko's criteria. Here, b is in the ground state (for thermal occupation of b, see SM [6]). Increasing $\bar{n}$ makes the nonclassical effects more pronounced. Purple and orange correspond to $\bar{n} = 1$ and 3, respectively, while green corresponds to a squeezed state with 3 dB of squeezing. Sub-Planck structure: The probability densities $P(q_x)$, of the dimensionless quadrature $\hat{q}_x = [(b + b^\dagger)/\sqrt{2}]$, are compared for the ground state (dashed black line) and with thermally induced nonclassicality, $\bar{n} = 3$ (red line), demonstrating the sub-Planck structure around the origin generated by the nonlinear interaction, with the colors corresponding to the resulting distilled distributions on the right. Distillable squeezing: The nonclassicality of mode b can also be quantified in a phase-sensitive manner via distillable squeezing. Universal distillation of squeezing from multiple copies of the nonclassical state in mode b is shown on the lower right, with the asymptotic limit in gray [23] and the 3 dB squeezing limit in dashed green line. Consuming progressively more copies produces a variance moving toward and then below the shot-noise level (dashed black line). The distillation procedure saturates (blue line), limited by the nonclassicality present in mode b after the trilinear interaction.

the reduced state of mode b is nonclassical due to its population of only even Fock states irrespective of the strength of the thermal pump. This is verified directly by Klyshko's criteria [44,45]. The nonclassicality is phase insensitive, reflecting the incoherent phase of the thermal pump (see SM for comparison with a coherent pump [6]). Simultaneously, the position probability density, despite having a variance greater than the ground state, displays sub-Planck structure [46] around the global maximum verified by the presence of universally distillable squeezing [23], with the degree of squeezing available increasing with the thermal energy. Notably, despite the presence of such sub-Planck structures, the Wigner function is positive at the level of the nonclassical state of b. The impurity of such non-Gaussian states precludes detailed categorization of the quasiprobability distribution [47,48] and is another motivation to use distillable squeezing. The lower right-hand panel of Fig. 1 shows that the degree of distillable squeezing stimulated by quanta splitting is *enhanced* by increases in $\bar{n}$ beyond 3 dB. The nonclassicality is remarkably robust, remaining detectable by both Klyshko's criteria and distillable squeezing despite an initial thermal occupation greater than one phonon in the quanta splitting mode (see SM [6]).

The amount of nonclassicality capable of being converted to entanglement in a particular mode can be further characterized through the EP [22], denoted $\mathcal{E}$. The EP of mode a or b is evaluated through its capacity to produce LN [49] via passive interactions with the ground state of an oscillator; fully classical states will not produce entanglement this way. The LN is calculated by determining $\mathcal{L}(\rho) = \log_2\|\rho^{T_A}\|$, where T_A denotes the partial transpose with respect to a subsystem A. For the EP, the LN should be calculated for the following state:

$$\rho_{\mathcal{E}} = U_{\mathrm{BS}}^{\dagger}(\rho \otimes |0\rangle\langle 0|)U_{\mathrm{BS}}, \tag{2}$$

where $U_{\mathrm{BS}} = e^{(\pi/4)(c^{\dagger}A - cA^{\dagger})}$, c is an auxiliary mode prepared in the ground state $|0\rangle$, and ρ is the reduced state corresponding to the mode $A = a, b$. EP links the nonclassicality produced by the trilinear interaction to the upper bound [50] on the entanglement generated between the nonclassical mode b and an auxiliary classical mode c and will allow us to directly compare the EP with the dynamically achievable TIE [51].

With this in mind, the nonclassicality converted from the thermal noise resource can be leveraged to create entanglement with another mode c which interacts simultaneously with b via the passive linear interaction:

$$H_{\mathrm{aux}} = H + g(bc^{\dagger} + b^{\dagger}c). \tag{3}$$

In our analysis, we have selected $g = \Omega_T$, which appears to give the best performance. Deviation from equality simply results in reduced LN, although the qualitative effects remain the same. The rise in LN is not a short-term effect, in the sense that it arises after an initial quiescent period in the dynamics, and optimization of the interaction time necessarily occurs over anharmonically oscillating dynamics. To compromise we select the first peak of LN in order to balance finding significant entanglement while remaining closer to the short times available to experimental settings, where decoherence will be less relevant.

The EP generated in b by a thermal pump in a is shown in Fig. 2(a) as the blue line. If a Gaussian squeezed state is prepared with the asymptotic value of the distillable squeezing (Fig. 1), then this squeezed state generates similar EP (black line) to the non-Gaussian state from the trilinear interaction. The auxiliary mode c simultaneously coupled to b, as in Eq. (3), results in LN, $\mathcal{L}_{b:c}$, that dynamically fulfills the EP of b. This can be seen as the red line in Fig. 2(a) achieves at least the value of $\mathcal{E}_b$. This persists even though residual LN is generated with the thermal pump mode ($\mathcal{L}_{a:c}$ and $\mathcal{L}_{a:b}$) and is subsequently traced out. The Gaussian entanglement, extracted from the covariance matrix of the state, is always zero during the evolution due to the thermal pump, indicating that the TIE involves correlations beyond the covariance matrix and is properly referred to as non-Gaussian entanglement.

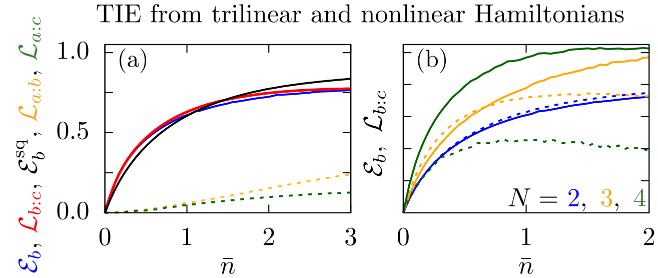

FIG. 2. TIE from a compound trilinear and linear process. (a) For the trilinear coupling given by Eq. (1), the entanglement potential $\mathcal{E}_b$ (blue line) of mode b increases as the thermal noise is increased. After passively interacting [Eq. (3)] with an auxiliary mode c, entanglement measured by logarithmic negativity $\mathcal{L}$ is shown between b and c (red line), a and b (dashed yellow line), and a and c (dashed green line). Both b and c are initially in their respective ground states. The values plotted here correspond to the first peak in the temporal evolution of their respective quantities $\mathcal{E}$ and $\mathcal{L}$ which otherwise undergo rather complex evolution. $\mathcal{L}_{bc}$ closely mimics the behavior of $\mathcal{E}_b$, indicating that the entanglement potential of b can be realized dynamically, despite residual entanglement among the remaining modes. $\mathcal{E}_b$ is further compared against the EP of a squeezed vacuum state (black line), whose degree of squeezing is selected in accord with the asymptotic distillation of squeezing available for the corresponding trilinear state. (b) Entanglement potential $\mathcal{E}_b$ (solid curves) and $\mathcal{L}_{b:c}$ (dashed curves) generated by incoherent thermal noise in a for Hamiltonians H_{NL}. The entanglement extracted by the auxiliary linear coupling ceases to closely follow the EP for $N > 2$.

For Gaussian entanglement either quadrature squeezing or a generalized squeezing emerging from concentration of squeezing via linear interference can always be used to detect the nonclassicality of the correlations. To analyze non-Gaussian entanglement beyond the covariance matrix in an operational and systematic way, backward compatible with Gaussian entanglement and quadrature squeezing, we have relied on distillable squeezing. We have dedicated a detailed discussion in the SM [6] to support the use of this new and extendable tool while this Letter focuses on the significant result that substantial non-Gaussian entanglement can arise from feasible and thermally driven nonlinear interactions.

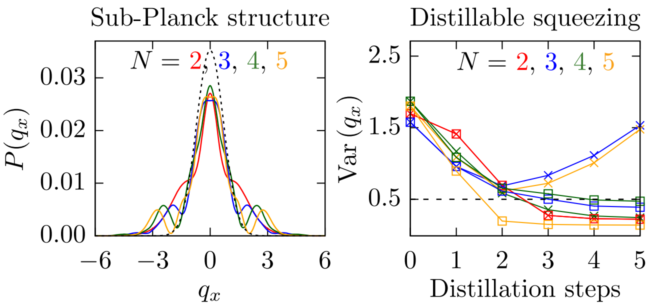

FIG. 3. (a) The probability densities $P(q_x)$ from $H_{\mathrm{NL}} = \Omega_T(a^\dagger b^N + a b^{\dagger N})$ with $N = 2$ (red line), 3 (blue line), 4 (green line), and 5 (yellow line) are compared at the optimal time (see text), compared with the ground state distribution (dashed black line). (b) Universal (squares) and nonuniversal (crosses) squeezing distillation as a function of number of copies of the state. For odd N the universal method does not find distillable squeezing around the global maximum of the distribution, unlike the case of even N. Instead, the nonuniversal method searches the whole distribution for nonclassical structures and distills them to find distillable squeezing. In both panels mode a is initially in a thermal state with $\bar{n} = 1$ and mode b is in the ground state.

To illustrate the robustness of these phenomena we examined the deviation of mode b from the ideal ground state. Thermal occupation of mode b, while reducing the absolute value of the EP, maintains the enhancement of nonclassicality due to increasing the thermal energy in a. Remarkably, this occurs even when the thermal occupation of b is greater than 1 and can occur when the thermal occupation of b is greater than that of a. This allows experimental tests of these phenomena without being bound to the ground state of mode b. Additionally, we examined the effect on the EP and LN when the trilinear interaction occurs outside the deep-strong coupling regime, so that the free evolution of the oscillators contributes, something not present in trapped ions [42]. While the EP is robust to these effects, the LN decreases, while still retaining the property that it increases with $\bar{n}$. Finally, we also examined the effect of coupling the modes to a thermal environment. While both EP and LN are reduced by the interaction with the environment, significant values of both quantities remain (see SM [6]).

Higher order nonlinearities.—It is possible that higher nonlinearities will exhibit diverse phenomena, and so we examine highly nonlinear Hamiltonians of the form $H_{\mathrm{NL}} \propto a(b^\dagger)^N + a^\dagger b^N$, where $N > 2$. Continuing the logic, the internal thermal driving of a will drive higher order quanta splitting, resulting in generalized squeezing [27] in b and producing highly nonclassical and non-Gaussian states with large entanglement potential.

In Fig. 3 we show the probability densities in q_x for the nonclassical states produced by these higher order nonlinearities. They are clearly non-Gaussian, with embedded sub-Planck structures. However, the distillable squeezing shows features distinct from the $N = 2$ case considered above. The universal method of distillation [23], which extracts nonclassical features from the curvature around the global maximum of the distribution, does not produce squeezing for odd N. Indeed, this is reflected in the distributions which have low curvature around their maxima. Instead, the nonuniversal method [23,52], which optimizes the distillation around arbitrary features of the distribution, can extract squeezing even from the odd parity Hamiltonians. However, the distillable squeezing found through the nonuniversal method does not increase with $\bar{n}$, in contrast with the universal method analyzed in detail in Fig. 1.

This nonclassicality can again be used to generate entanglement, again using the simultaneous linear coupling as used in Eq. (3). Figure 2(b) shows the entanglement potential and logarithmic negativity via coupling to an auxiliary state for various N. Increasing N tends to increase the nonclassicality generated by the same thermal energy; however, the anticipated generation of entanglement fails to reach the EP for larger values of $\bar{n}$. This suggests that among these the trilinear Hamiltonian of Eq. (1), combined with a linear coupling to another mode, is the most effective in terms of generating non-Gaussian entanglement from small thermal energy.

Discussion.—We have singled out the simplest nonlinearity coupling linear oscillators, the degenerate trilinear interaction, combined with a linear coupling in order to transmute the thermal energy of a few quanta into nonclassicality and non-Gaussian TIE, taking us simultaneously beyond the more common Gaussian EPR-type states and non-Gaussian pure states understood by Hudson's theorem [47,48]. Remarkably, the quantum non-Gaussianity manifests differently and positively as entanglement which grows as the thermal energy is increased. Other possibilities among three mode interactions are surprisingly limited in their capacity to generate bipartite entanglement [4,5] (see SM [6]). That larger thermal fluctuations are converted into larger coherent quantum phenomena, such as entanglement [53–56], functions as a direct proof of genuine nonlinear quantum effects, as it cannot happen for linearized dynamics within the Gaussian approximation provided by the covariance matrix. While such dynamics intrinsically involves higher than quadratic interaction terms, and shows quantum non-Gaussian effects [28], such a direct experimental proof is yet to be demonstrated. In the past, such features were predicted in systems taking advantage of discrete nonlinearities in the level structure of atoms or superconducting circuits with very strong saturation [55–57] while here entanglement materializes without the direct use of saturation in such discrete nonlinearities. It is also conceptually different from TIE using controlled-SWAP gates [58,59]. The thermally induced nonclassicality and entanglement studied here arises purely from the natural Coulomb force between trapped ions. Analogous effects from the direct coupling, without any assisting drive, for fully discrete variable systems remains to be investigated.

This spearhead analysis, inspired by recent experimental achievements regarding trilinear interactions in trapped ion [28–30] and superconducting circuit platforms [32,35], provides a clear example of the hidden power of multimode nonlinear systems to achieve, autonomously and unconditionally, increasing nonclassical quantum behavior leveraging only small thermal energy. Furthermore, the linear coupling required to directly test the parallel generation of entanglement has already been proposed for optical systems [60] and is likely to be easily adapted to phononic systems and superconducting circuits.

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