Emergent Noninvertible Symmetries in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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One of the simplest examples of noninvertible symmetries in higher dimensions appears in 4D Maxwell theory, where its $SL(2, \mathbb{Z})$ duality group can be combined with gauging subgroups of its electric and magnetic 1-form symmetries to yield such defects at many different values of the coupling. Even though $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory also has an $SL(2, \mathbb{Z})$ duality group, it only seems to share two types of such noninvertible defects with Maxwell theory (known as duality and triality defects). Motivated by this apparent difference, we begin our investigation of the fate of these symmetries by studying the case of 4D $\mathcal{N} = 4$ U(1) gauge theory, which contains Maxwell theory in its content. Surprisingly, we find that the noninvertible defects of Maxwell theory give rise, when combined with the standard U(1) symmetry acting on the free fermions, to defects that act on local operators as elements of the U(1) outer automorphism of the $\mathcal{N} = 4$ superconformal algebra, an operation that was referred to in the past as the "bonus symmetry." Turning to the non-Abelian case of $\mathcal{N} = 4$ SYM theory, the bonus symmetry is not an exact symmetry of the theory, but is known to emerge at the supergravity limit. Based on this observation, we study this limit and show that, if it is taken in a certain way, noninvertible defects that realize different elements of the bonus symmetry emerge as approximate symmetries, in analogy to the Abelian case.

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Introduction.—A recent exciting development in quantum field theory is the understanding that symmetries are represented by topological defects with properties that might go beyond the traditional notion of a symmetry. Noninvertible symmetries, in particular, correspond to defects with exotic non-group-like fusion rules and by now have been identified and investigated in diverse setups in different areas of physics (see [1,2] for reviews).

One of the simplest instances of such noninvertible symmetries in higher dimensions appears in 4D Maxwell theory. As shown in [3,4], when its $SL(2, \mathbb{Z})$ duality is combined with gauging a $\mathbb{Z}_N^{(1)}$ subgroup of its electric 1-form symmetry (with a Dirichlet boundary condition for the $\mathbb{Z}_N^{(1)}$ 2-form gauge field, and possibly with stacking an SPT phase, or symmetry-protected topological phase, for it), one is able to construct noninvertible defects, known as duality and triality defects, at special values of the complexified coupling τ. Later [5] (see also [6]), employing the observation that gauging a subgroup of the electric or magnetic 1-form symmetry results again in Maxwell theory, but with a different coupling, the $SL(2, \mathbb{Z})$ duality was combined with gauging such subgroups (in a nonanomalous way) to yield an $SL(2, \mathbb{Q})$ operation on the theory that acts in an analogous way to $SL(2, \mathbb{Z})$ (this construction will be reviewed in the next section). This, in turn, enabled use to find a new noninvertible defect at any value of τ that is fixed by an element of $SL(2, \mathbb{Q})$ (which acts on τ in the standard way by a fractional transformation), thereby generalizing the results of [3,4].

It is natural at this point to ask what part of this construction has a counterpart in non-Abelian theories. Because of the central role played by $SL(2, \mathbb{Z})$ duality, the natural candidate to examine, which will also be the main focus of this Letter, is $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. Considering for concreteness the theory with $SU(N)$ gauge group, even though it is part of an $SL(2, \mathbb{Z})$ duality orbit, it only has an electric 1-form symmetry which is $\mathbb{Z}_N^{(1)}$ and whose gauging changes the global structure of the theory and cannot be associated with an action on the coupling τ. As a result, we do not seem to have an $SL(2, \mathbb{Q})$ operation as in Maxwell theory, and a noninvertible defect of the type we discuss only has the chance of being found at values of τ that are fixed by $SL(2, \mathbb{Z})$ [7]. Indeed, previous works [4,10] only found the duality and triality defects in $\mathcal{N} = 4$ SYM theory, with the additional defects discussed in [5] being absent.

Here, in order to investigate this apparent difference between Maxwell and $\mathcal{N} = 4$ SYM theories in a more systematic way, and to try to extract a clue that will guide us toward the fate of these additional symmetries in $\mathcal{N} = 4$

SYM theory, we begin by examining more closely a theory that is, in a sense, the intersection of these two: 4D $\mathcal{N}=4$ Abelian U(1) gauge theory. This theory has an $\mathcal{N}=4$ superconformal algebra and contains free Maxwell theory in its content. Surprisingly, we will find that the additional symmetries of [5], when combined with the standard U(1) symmetry acting on the free fermions in the theory, realize at different values of the coupling different elements of a noninvertible U(1) R symmetry that acts on the local operators as the U(1) outer automorphism of the $\mathcal{N}=4$ superconformal algebra. Moreover, since correlators of local operators do not depend on the coupling in a nontrivial way in this theory (i.e., one can normalize the operators such that the correlators are coupling independent), they will satisfy the selection rules of the entire U(1) at any value of the coupling.

A priori, the U(1) outer automorphism of the $\mathcal{N}=4$ superconformal algebra may or may not be a symmetry of the theory. This question was discussed in detail in the past in [11,12] (see also [13–16]), where this potential symmetry was referred to as the bonus symmetry. It was demonstrated that in the Abelian theory we consider, the various local operators have definite charges under this U(1) and that it is respected by the equations of motion and supersymmetry transformations. However, the nature of this symmetry seemed to be mysterious, as it was clearly not a symmetry of the Lagrangian and the field strength appeared to be charged under it. One of the observations made in this Letter is therefore the identification in modern terms of the bonus symmetry of the U(1) gauge theory as a noninvertible symmetry, with different elements realized at different values of the coupling. These elements, in turn, mainly correspond to the defects discovered in [5], and while their action on local operators is the one identified in the past in [11,12], their action on line operators is highly nontrivial [5].

Once we have identified the noninvertible defects of Maxwell theory discussed above as the key ingredient giving rise to the bonus symmetry of the $\mathcal{N}=4$ theory with U(1) gauge group, it is time to turn to the non-Abelian theory and use the bonus symmetry of its algebra as our guide in searching for new noninvertible symmetries analogous to those of Maxwell theory. Unlike the Abelian case, in $\mathcal{N}=4$ SYM theory the bonus symmetry is not an exact symmetry of the theory [11,12]. This is indeed consistent with the fact that only the duality and triality defects have been identified as exactly topological defects in the past. However, as discussed in detail in [11], based on holographic duality the bonus symmetry is expected to emerge as an approximate symmetry in the limit where the gravity dual of $\mathcal{N}=4$ SYM theory is approximated by type IIB supergravity (i.e., when both N and $g_{\mathrm{YM}}^2 N$ are very large). This emergence, in turn, follows from the enhancement of the $\mathrm{SL}(2,\mathbb{Z})$ duality symmetry of type IIB string theory to $\mathrm{SL}(2,\mathbb{R})$ in this supergravity limit (at least when the fields are not treated as quantized) [17], of which the U(1) bonus symmetry is the maximal compact subgroup. This observation then suggests that the place in $\mathcal{N}=4$ SYM theory in which we should look for new defects analogous to those of Maxwell theory is exactly at this limit. Indeed, as we will show below, when this limit is taken in an appropriate way, such noninvertible defects emerge as approximate symmetries and the picture we obtain is analogous to the Abelian case.

$\mathcal{N}=4$ Abelian U(1) gauge theory.—In this section, we would like to investigate the way in which the noninvertible defects of Maxwell theory that include $\mathrm{SL}(2,\mathbb{Z})$ duality transformations appear in the 4D $\mathcal{N}=4$ theory with U(1) gauge group. As discussed in the previous section, we are mainly interested in the relation between these defects and the bonus (or outer-automorphism) symmetry of the $\mathcal{N}=4$ theory.

Before beginning with reviewing the noninvertible defects of Maxwell theory, let us first identify the following hint for them in the classical theory. Defining the self-dual and anti-self-dual field strengths $F^{\pm}_{mn}=\frac{1}{2}(F_{mn}\pm \frac{1}{2}\epsilon_{mnpq}F^{pq})$, we observe that the classical operation $F^{\pm}\to e^{\pm i\varphi}F^{\pm}$ leaves the stress tensor invariant (see Supplemental Material [19] for more details). Since our modern understanding of symmetries in quantum field theory is in terms of topological defects, which act trivially on the stress tensor (alternatively, the displacement operator on the defect vanishes), this suggests that such an operation might correspond to an exact symmetry in the quantum theory. Notice that this operation is also a symmetry of the equations of motion, but a bit strange at first sight since it is not a symmetry of the Lagrangian (see Supplemental Material [19]) and the field strength is charged under it.

To understand if and how this U(1) symmetry can appear in the quantum theory, we notice that its action on $F^{\pm}$ implies that it rotates between the electric and magnetic fields. This suggests that if this symmetry (or a part of it) is realized at the quantum level, it involves the $\mathrm{SL}(2,\mathbb{Z})$ duality of Maxwell theory. This was indeed shown to be the case in [5], by observing that combining (nonanomalous) gauging of discrete subgroups of the electric and magnetic 1-form symmetries with the usual $\mathrm{SL}(2,\mathbb{Z})$ duality extends it to the following $\mathrm{SL}(2,\mathbb{Q})$ operation (see Supplemental Material [19]):

$$\begin{pmatrix} e \\ m \end{pmatrix} \to \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}\begin{pmatrix} e \\ m \end{pmatrix}, \qquad \tau \to \frac{q_1\tau+q_2}{q_3\tau+q_4} \tag{1}$$

with the corresponding matrix being an element of $\mathrm{SL}(2,\mathbb{Q})$. Then, every coupling of the form

$$\tau=\frac{q_1-q_4+i\sqrt{4-(q_1+q_4)^2}}{2q_3} \tag{2}$$

with q_1, q_3, and q_4 any rationals satisfying $2 > q_1 + q_4$ and $q_3 \neq 0$ (such that the coupling g is kept real) is invariant under the corresponding $\mathrm{SL}(2,\mathbb{Q})$ element, with $q_2 = (q_1 q_4 - 1)/q_3$. We therefore see that topological defects corresponding to different elements of $\mathrm{SL}(2,\mathbb{Q})$ are realized at different values of the coupling. Moreover, such defects rotate between the electric and magnetic fields with different rotation angles, and since correlators of local operators do not depend on τ in a nontrivial way in this theory [and since rational rotations are dense inside U(1)], such correlators will respect the selection rules of the entire U(1) operation discussed above. Notice, however, that this U(1) is not really a symmetry of the theory and is not associated with a topological operator.

Turning now to the $\mathcal{N} = 4$ U(1) gauge theory, the status of its U(1) bonus symmetry [11] is very similar to that of the U(1) "symmetry" of Maxwell theory we discussed above. To see it, let us begin with the basic description of the theory. We consider a free theory consisting of a U(1) gauge field with field strength $F_{(\alpha\beta)}$, fermions $\psi_{I\alpha}$, $\bar{\psi}^I_{\dot{\alpha}}$, and real scalars $\phi_{[IJ]}$, where I is the index of the fundamental representation of $\mathrm{SU}(4)_R$ and α, $\dot{\alpha}$ are the usual indices of $\mathrm{SU}(2)_{L,R}$. The Lagrangian is simply given by the sum of the kinetic terms of the fields, with the exactly marginal coupling τ (and $\bar{\tau}$) an overall factor. Since this theory contains Maxwell theory, it also contains the defects we discussed above, realizing when acting on local operators different elements of a U(1) symmetry, which we will denote by $\mathrm{U}(1)_F$. As its name suggests, among the basic fields of the theory only the field strength is charged under $\mathrm{U}(1)_F$ (with charge 1 for $F_{(\alpha\beta)}$ and -1 for $\bar{F}_{(\dot{\alpha}\dot{\beta})}$). In addition, there is a $\mathrm{U}(1)_\psi$ symmetry under which the free fermions $\psi_{I\alpha}$ are charged with charge 1, and a particularly natural combination of these two symmetries is $\mathrm{U}(1)_Y = -\mathrm{U}(1)_\psi - 2\mathrm{U}(1)_F$ under which the supercurrent has a well-defined charge of -1 (indeed, this will be the charge of terms like $F_{(\alpha\beta)}\bar{\psi}^I_{\dot{\alpha}}$ and $\partial_{\alpha\dot{\alpha}}\phi^{[IJ]}\psi_{J\beta}$ in it). Since the supercharges are charged under $\mathrm{U}(1)_Y$, it is an R symmetry, and in fact this is exactly the bonus symmetry as defined in [11]. We therefore see that, in this $\mathcal{N} = 4$ U(1) gauge theory, the bonus symmetry appears as an ordinary 0-form U(1) symmetry at the level of local correlators, but is in fact given by (in general) noninvertible defects realizing at different values of the coupling different elements of it.

Motivated by this link between the noninvertible defects in Maxwell theory associated with its $\mathrm{SL}(2,\mathbb{Q})$ operation and the bonus symmetry of the $\mathcal{N} = 4$ U(1) gauge theory, we continue in the next section to non-Abelian $\mathcal{N} = 4$ theories with the aim of using their bonus symmetry as a guide for finding new analogous defects.

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory.—As discussed in the Introduction, the bonus symmetry is not an exact symmetry of $\mathcal{N} = 4$ SYM theory, in the sense that the corresponding selection rules are clearly violated, in general, by local correlators. However, in the supergravity limit of the theory (when both N and $g^2_{\mathrm{YM}}N$ are very large), we obtain a description in terms of type IIB supergravity on five-dimensional Anti-de Sitter space (or $\mathrm{AdS}_5 \times S^5$), and when the fields are not treated as quantized there is a well-known $\mathrm{SL}(2,\mathbb{R})$ symmetry acting on them [enhancing the standard $\mathrm{SL}(2,\mathbb{Z})$ duality of IIB string theory]. The bonus symmetry then emerges as the U(1) subgroup of this $\mathrm{SL}(2,\mathbb{R})$ that fixes a given value of the coupling τ, and the corresponding selection rules are expected to be satisfied by correlators of local operators that can be computed using the supergravity approximation [11].

This suggests that defects analogous to the ones discussed in the previous section might emerge as approximate symmetries in the supergravity limit. For this to be the case, the possible global structures of the theory at this limit should allow for transformations analogous to the $\mathrm{SL}(2,\mathbb{Q})$ of Maxwell theory. As we will show, such a description can indeed be obtained if the large-N limit is taken in a certain way.

Let us begin with recalling the case of finite N, considering for concreteness the gauge group $\mathrm{SU}(N)$. Here there is an electric $\mathbb{Z}_N^{(1)}$ 1-form symmetry associated with the $\mathbb{Z}_N$ center of $\mathrm{SU}(N)$, acting on Wilson lines according to the N-ality of their representation. Gauging this 1-form symmetry or a subgroup of it, possibly with stacking an SPT phase, changes the spectrum of line operators in the theory and correspondingly also the 1-form symmetry. The way these different global forms are encoded holographically is through the topological theory [21] (also known as the SymTFT [22–24] of the theory),

$$S = \frac{2\pi}{N}\int_{\mathrm{AdS}_5} \mathsf{B}_N \cup \delta\mathsf{C}_N \tag{3}$$

obtained near the boundary of AdS_5, where $\cup$ is the cup product and B_N and C_N are both $\mathbb{Z}_N$ 2-cochains. The different global forms then correspond to different topological boundary conditions for the theory (3), which in turn are classified by the Lagrangian subgroups of its surface operators

$$S_{(e,m)}(\sigma) = e^{(2\pi i e/N)\int_\sigma \mathsf{B}_N} e^{(2\pi i m/2\pi i m)\int_\sigma \mathsf{C}_N}. \tag{4}$$

Surfaces that can end on the boundary (that is, the ones in the chosen Lagrangian subgroup) then correspond to the line operators of the boundary theory that are charged under the 1-form symmetry, while the rest of the surfaces give rise to the symmetry generators when pushed to the boundary.

Let us now examine the large-N limit of this story. Instead of considering a specific value of N that is finite but large (in which case the discussion remains the same as above), we will take the limit in a way that formalizes more precisely the intuition that combinations such as $(e\int_\sigma \mathsf{B}_N \mathrm{mod} N)/N$ [which appears in the expression for the surface $S_{(e,m)}$, see (4)] are approximately valued in all of $\mathbb{Q}/\mathbb{Z}$ as N is taken to be very large. In order to do it, let us focus on the $\mathbb{Z}_N$ center of $\mathrm{SU}(N)$ and begin by discussing two different natural ways of taking such a large-N limit of it (limits of these types have been recently discussed in a physical context in [25]).

The first, known as the "direct limit" and denoted by $\varinjlim$, results in the limit group of the family [26] $\{(1/N)\mathbb{Z}_N\}_{N\in\mathbb{N}}$ being $\mathbb{Q}/\mathbb{Z}$, corresponding to the fact that it can be written as the union of all cyclic groups. The second, called the "inverse limit" and denoted by $\varprojlim$, is the Pontryagin dual of the direct limit and results in the group

$$\varprojlim \mathbb{Z}_N = \left\{\vec{a}\in\prod_{N\in\mathbb{N}}\mathbb{Z}_N | a_M = a_N \mathrm{mod} M \ \ \forall \ \ M|N\right\}. \tag{5}$$

This simply means that any element in the resulting group is specified by the set of its residues modulo N for all N, in a way that is consistent with the mod map relating different residues. This group is called the "profinite integers" and is denoted by $\hat{\mathbb{Z}}$. It is a certain completion of the integers $\mathbb{Z}$ and contains them as a subgroup. In particular, there are profinite integers that are not ordinary integers.

The natural question at this point is what type of limit should be used for the large-N limit of the $\mathbb{Z}_N$ center of the $\mathrm{SU}(N)$ gauge group. In taking this limit, we would like to keep the fields B_N and C_N on equal footing such that the $\mathrm{SL}(2,\mathbb{Z}_N)$ symmetry of the action in (3) is maintained. Leaving an analysis of the general case for future work, we will here focus on gauge groups of the form $\mathrm{SU}(N^2)$. Then in order to take such a limit of the $\mathbb{Z}_{N^2}$ center, we view it as the following extension:

$$0\to\mathbb{Z}_N\to\mathbb{Z}_{N^2}\to\mathbb{Z}_N\to 0 \tag{6}$$

where the second arrow denotes multiplication by N. Defining the bilinear pairing $(a,b)_{N^2} = ab/N^2 \mathrm{mod} 1$ in $\mathbb{Z}_{N^2}$ [which is a map $(,)_{N^2}\colon \mathbb{Z}_{N^2}\times\mathbb{Z}_{N^2}\to\mathbb{Q}/\mathbb{Z}$], we observe that the first $\mathbb{Z}_N$ group in the sequence (6) is a Lagrangian subgroup with respect to it, and that the two $\mathbb{Z}_N$ groups can be regarded as Pontryagin dual to each other using it since an element $a\in\mathbb{Z}_N$ can be understood as $a\in \mathrm{Hom}(\mathbb{Z}_{N^2}/\mathbb{Z}_N,\mathbb{Q}/\mathbb{Z})$ with $a(b)=(a,b)_{N^2}$ for $b\in\mathbb{Z}_{N^2}/\mathbb{Z}_N$ (which is well defined since $\mathbb{Z}_N$ is Lagrangian). We can then take the large-N limit of the sequence (6) in such a way that the inverse limit is taken for the first $\mathbb{Z}_N$ group, while the direct one is taken for the other $\mathbb{Z}_N$ (i.e., for $\mathbb{Z}_{N^2}/\mathbb{Z}_N$). This, in turn, is possible due to the fact that the two $\mathbb{Z}_N$ groups are Pontryagin dual to each other and by using the property that Pontryagin duality exchanges direct and inverse limits, $\mathrm{Hom}(\lim_{\rightarrow}A_n,B)=\lim_{\leftarrow}\mathrm{Hom}(A_n,B)$. We then end up at this large-N limit with the sequence

$$0\to\hat{\mathbb{Z}}\to\hat{\mathbb{Q}}\to\mathbb{Q}/\mathbb{Z}\to 0, \tag{7}$$

where the extension $\mathbb{Z}_{N^2}$ turns at the limit to the extension $\hat{\mathbb{Q}}$ of $\mathbb{Q}/\mathbb{Z}$ by $\hat{\mathbb{Z}}$. The group $\hat{\mathbb{Q}}$ is called the group of profinite rationals and is defined in an analogous way to the profinite integers in (5),

$$\hat{\mathbb{Q}} = \left\{\vec{q}\in\prod_{N\in\mathbb{N}}(\mathbb{Q}/N\mathbb{Z})|q_M = q_N \mathrm{mod} M \ \forall \ M|N\right\}. \tag{8}$$

It is self-dual under Pontryagin duality and has a natural ring structure extending that of $\hat{\mathbb{Z}}$. Moreover, it includes $\mathbb{Q}$ and $\hat{\mathbb{Z}}$ as subrings and can be written as $(\hat{\mathbb{Z}}\oplus\mathbb{Q})/\mathbb{Z}$.

We therefore obtain that both B_{N^2} and C_{N^2} turn at this limit into $\hat{\mathbb{Q}}$ 2-cochains, and that the symmetry between them is preserved. In order to identify this symmetry in full and find what $\mathrm{SL}(2,\mathbb{Z}_{N^2})$ turns into at this limit, we should first find the limit of the action (3). We can do it by rewriting this finite-N action using the pairing $(,)_{N^2}$ of $\mathbb{Z}_{N^2}$ we defined above as

$$S = 2\pi\int_{\mathrm{AdS}_5}(\mathsf{B}_{N^2},\delta\mathsf{C}_{N^2})_{N^2} \tag{9}$$

and observing that this pairing turns at the limit we are taking into the pairing $(q_1,q_2)_\infty = q_1q_2 \mathrm{mod}\hat{\mathbb{Z}}\in\mathbb{Q}/\mathbb{Z}$ in $\hat{\mathbb{Q}}$, where $q_1,q_2\in\hat{\mathbb{Q}}$ and q_1q_2 is their product using the standard ring structure of $\hat{\mathbb{Q}}$. We therefore find the large-N action

$$S_\infty = 2\pi\int_{\mathrm{AdS}_5}(\mathsf{B}_{\hat{\mathbb{Q}}},\delta\mathsf{C}_{\hat{\mathbb{Q}}})_\infty \tag{10}$$

and can identify an $\mathrm{SL}(2,\hat{\mathbb{Q}})$ symmetry acting on $\mathsf{B}_{\hat{\mathbb{Q}}}$ and $\mathsf{C}_{\hat{\mathbb{Q}}}$ as a doublet using the standard ring structure of $\hat{\mathbb{Q}}$. Let us also comment that the surfaces of the theory, which for finite N are specified by the pair of charges $(e,m)\in\mathbb{Z}_{N^2}\times\mathbb{Z}_{N^2}$ [see (4) for the case of $\mathrm{SU}(N)$ gauge group], are now specified by a pair $(\mathfrak{e},\mathfrak{m})\in\hat{\mathbb{Q}}\times\hat{\mathbb{Q}}$ and take the form

$$S_{(\mathfrak{e},\mathfrak{m})}(\sigma) = e^{2\pi i\int_\sigma(\mathfrak{e},\mathsf{B}_{\hat{\mathbb{Q}}})_\infty}e^{2\pi i\int_\sigma(\mathfrak{m},\mathsf{C}_{\hat{\mathbb{Q}}})_\infty}. \tag{11}$$

We have found that the group $\mathrm{SL}(2,\mathbb{Z}_{N^2})$, corresponding to gauging the 1-form symmetry and stacking an SPT phase at finite N, turns into $\mathrm{SL}(2,\hat{\mathbb{Q}})$ at the large-N limit we are considering. Notice, however, that the duality group of $\mathcal{N}=4$ SYM theory, which acts also on the rest of the theory (e.g., on the coupling τ), is still $\mathrm{SL}(2,\mathbb{Z})$ for general values of the 't Hooft coupling λ. At this point, we are making use of the supergravity approximation of the string-theory dual of $\mathcal{N}=4$ SYM theory, which is valid when λ is taken to be large, and of its $\mathrm{SL}(2,\mathbb{R})$ enhanced symmetry when the fields are treated as classical, to identify an $\mathrm{SL}(2,\mathbb{Q})\subset\mathrm{SL}(2,\mathbb{R})$ duality group that is consistent at the quantum level with the global structures (or charge quantization) we have found at the large-N limit we described. Therefore, taking the large-N limit as detailed above and the ‘t Hooft coupling to be large, we expect to have an $\mathrm{SL}(2,\mathbb{Q})$ duality group [27] with elements that leave values of λ corresponding to a τ of the form (2) invariant. Performing such a duality operation in half-space and accompanying it with the corresponding element of $\mathrm{SL}(2,\mathbb{Q}) \subset \mathrm{SL}(2,\hat{\mathbb{Q}})$ that brings the global structure to its original form, we obtain a noninvertible topological defect in the original theory [28]. Different such defects, which are realized at different values of λ, implement different elements (corresponding to certain rotation angles) of the U(1) bonus symmetry of $\mathcal{N}=4$ SYM theory, in analogy to the case of the $\mathcal{N}=4$ U(1) gauge theory discussed in the previous section.

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[28] This is similar to the construction of the duality and triality defects. For example, the duality defect at $\tau=i$ is obtained by performing S duality in half-space, which is then accompanied by gauging the 1-form symmetry (or performing an S operation in the modular group of gauging and stacking SPT phases) in the same half-space.