Topological Quantum Synchronization of Fractionalized Spins

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The gapped symmetric phase of the Affleck-Kennedy-Lieb-Tasaki model exhibits fractionalized spins at the ends of an open chain. We show that breaking SU(2) symmetry and applying a global spin-lowering dissipator achieves synchronization of these fractionalized spins. Additional local dissipators ensure convergence to the ground state manifold. In order to understand which aspects of this synchronization are robust within the entire Haldane-gap phase, we reduce the biquadratic term, which eliminates the need for an external field but destabilizes synchronization. Within the ground state subspace, stability is regained using only the global lowering dissipator. These results demonstrate that fractionalized degrees of freedom can be synchronized in extended systems with a significant degree of robustness arising from topological protection. A direct consequence is that permutation symmetries are not required for the dynamics to be synchronized, representing a clear advantage of topological synchronization compared to synchronization induced by permutation symmetries.

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Introduction.—From neuroscience to chemical reactions, synchronization emerges in an impressively vast variety of seemingly unrelated systems [1–5] and despite its long history continues to be crucial for the development of modern technology [6–13]. In the past decade, the concept of synchronization has been generalized to the quantum regime with studies ranging from classically inspired systems like nonlinear oscillators [14–31] to systems without any classical counterpart like spins [32–36]. Mutual synchronization and forced synchronization have been examined with surprising effects that are absent in the classical regime, such as, for example, the phenomenon of synchronization blockade of two identical systems [37,38], which has recently also been verified experimentally [39]. Promising applications of quantum synchronization range from quantum information [40–43] to quantum thermodynamics [44–46].

One approach to synchronization—followed in particular in the study of quantum many-body systems—is in terms of persistently oscillating eigenmodes of time-independent quantum master equations [35,47,48]. The existence of such eigenmodes is intimately related to dynamical symmetries [47–49], which together with permutation symmetries allow for synchronized dynamics of local observables. An illustrative example investigated in Ref. [47] is a three-site Hamiltonian which nontrivially couples three spin-$1/2$ particles, is reflection symmetric about the central site, and conserves total magnetization. Dissipation acting locally on the central spin forces it to be in the spin-down state. As a consequence, there are two steady states of the master equation, where the two remaining spins are both spin-down or form a singlet. These two pure states form a decoherence-free subspace [50] such that coherent oscillations between these two states are possible even in this dissipative setup. Starting in an initial state that has nonvanishing overlap with both the singlet and the both-down state, results (after a short transient time) in perfectly antisynchronized oscillations of the local transverse spin of the noncentral sites 2 and 3, i.e., $\langle \sigma_2^x(t) \rangle = -\langle \sigma_3^x(t) \rangle$. They are antisynchronized because the singlet state is antisymmetric upon reflection while the spin-down state is symmetric. In the corresponding Bloch sphere representation, the central spin rapidly decays to the south pole, while the other two spins reach the same limit cycle within the Bloch sphere (parallel to the x-y plane) which they orbit perfectly out of phase.

In this Letter, we investigate whether a similar strategy can be exploited to synchronize the fractionalized spin-$1/2$ degrees of freedom localized at the open ends of a spin-1 Affleck-Kennedy-Lieb-Tasaki (AKLT) chain [51–53]. By applying dissipation that acts globally on all sites, we show that lifting the ground state degeneracy through a small external magnetic field leads to stable synchronization of the fractionalized spins. In that case, local spin-1 observables at the ends are perfectly antisynchronized with amplitudes that reflect the topological edge states, i.e., they are exponentially localized at the boundaries. In addition we show that quasilocal dissipators acting on two neighboring sites that dissipate the energetically lowest state of the total spin $S = 2$ subspace are sufficient to depopulate the whole excited subspace and remove unwanted additional oscillations. The observed synchronization is of a topological nature as the underlying mechanism relies on the fractionalization of the spin degrees of freedom and is thus topologically protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the AKLT states. Consequently, the observed synchronization is robust to perturbations that break the inversion (or permutation) symmetry, which can induce additional dissipation and eliminate long-lasting synchronized dynamics. Last, we show that if the fractionalized spins are allowed to interact by decreasing the biquadratic term of the AKLT Hamiltonian, stable synchronization within the ground state manifold of the Haldane-gap phase is still possible even without external magnetic field if one only considers a global spin lowering dissipator. This demonstrates that the dynamic response depends on the microscopic details of systems even though they belong to the same symmetry protected topological phase.

Synchronization model.—We consider the open spin-1 AKLT chain of size N with an additional external magnetic field B yielding the Hamiltonian

$$H = \sum_{j=1}^{N-1}\left[\frac{1}{2}\vec{S}_j\cdot\vec{S}_{j+1} + \frac{1}{6}(\vec{S}_j\cdot\vec{S}_{j+1})^2 + \frac{1}{3}\right] + \frac{B}{N}S^z, \quad (1)$$

where $S^z = \sum_{j=1}^N S_j^z$ is the total magnetization. For sufficiently small values of B, the Hamiltonian remains gapped even if the chain size is increased, yet breaks SU(2) symmetry (which will become important for synchronization as we explain later). For $B = 0$ the ground state is fourfold degenerate as a consequence of effective spin-1/2 degrees of freedom that are localized at both ends of the chain. The ground states of (1) can be constructed explicitly, e.g., in terms of Schwinger bosons [54] or matrix product states [55–57]. As the spin-1/2 degrees of freedom at the ends are exactly decoupled, there are three ground states with total spin $S = 1$, where the two dangling spin-1/2's form a triplet state with $S_z = 1, 0, -1$ and one with $S = 0$, where the dangling spin-1/2's form a singlet with $S_z = 0$. Thus, we may label the ground states accordingly as $|G_{1,1}\rangle$, $|G_{1,0}\rangle$, $|G_{1,-1}\rangle$, and $|G_{0,0}\rangle$. Note, that while a finite value of B partially lifts the ground state degeneracy, the corresponding manifold is still spanned by $\{|G_{S,S_z}\rangle\}$ as the total magnetization is preserved; $[H, S^z] = 0$.

Synchronization is inherently connected to open system dynamics because it requires dissipation in order to reduce all potential dynamics to only the desired, synchronized ones. To this end, we describe the system by a time dependent density operator $\varrho(t)$ acting on the Hilbert space of the system $\mathcal{H}$. We consider Markovian dynamics such that the evolution may be described via a Lindblad master equation [58,59],

$$\dot{\varrho} = -\mathrm{i}[H,\varrho] + \sum_\mu (2L_\mu \varrho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \varrho\}) = \mathcal{L}[\varrho], \quad (2)$$

where L_μ denotes (for now unspecified) Lindblad operators. The Liouvillian superoperator $\mathcal{L}$ is the generator of a smooth, time-homogeneous, completely positive, and trace-preserving (CPTP) map (or quantum channel), which obeys the semigroup property. The system dynamics described by Eq. (2) is guaranteed to have at least one steady state ϱ_{ss} such that $\mathcal{L}[\varrho_{ss}] = 0$ [58].

A sufficient and necessary condition for the existence of an eigenstate $\varrho = A\varrho_{ss}$ of $\mathcal{L}$ with purely imaginary eigenvalue $\lambda = -\mathrm{i}\omega$, i.e., $\mathcal{L}[\varrho] = -\mathrm{i}\omega\varrho$ with $\omega \in \mathbb{R}$, is given by [47]

$$[L_\mu, A]\varrho_{ss} = 0, \quad (3)$$

$$\left(-\mathrm{i}[H,A] - \sum_\mu [L_\mu^\dagger, A]L_\mu\right)\varrho_{ss} = -\mathrm{i}\omega A\varrho_{ss}. \quad (4)$$

While Eqs. (3) and (4) guarantee the existence of persistent oscillations in the long time limit, one usually demands another condition for (anti)synchronization [47]. Let P_{jk} be an operator that exchanges subsystem j with k and let $\mathcal{P}_{jk}[x] = P_{jk}xP_{jk}$. Then, if P_{jk} is a weak symmetry of the Liouvillian, i.e., $[\mathcal{L}, \mathcal{P}_{jk}] = 0$, and (anti)commutes with the operator A, $P_{jk}AP_{jk} = \pm A$, then we find *stable* synchronization (+) or antisynchronization (−) of the two local operators O_j and O_k if $\mathrm{Tr}[O_j A\varrho_{ss}] \neq 0$ and conditions (3) and (4) are fulfilled, that is after some transient time τ

$$\langle O_j(t)\rangle = \pm\langle O_k(t)\rangle \quad \forall\ t \geq \tau \quad (5)$$

up to exponentially small corrections. In the example referred to in the introduction the local transverse spin of the noncentral sites 2 and 3 will be perfectly antisynchronized, i.e., $\langle\sigma_2^x(t)\rangle = -\langle\sigma_3^x(t)\rangle \propto \cos(\omega t)$, where the oscillation frequency ω depends on the specific choice of Hamiltonian [47]. As we discuss later, the necessity of permutation symmetry is omitted for topological synchronization in the AKLT chain as long as the $\mathbb{Z}_2 \times \mathbb{Z}_2$ is preserved.

In the following, we first focus on the ground state manifold and show how a single, globally acting dissipator L_G leads to the fulfilment of conditions (3)–(5) within the ground state manifold and thus to stable synchronization. In a second step we will then show that additional, locally acting dissipators force the dynamics into the ground state manifold.

In order to find adequate dissipators such that Eqs. (3) and (4) are fulfilled, we utilize the fractionalized spins of the AKLT ground states: since the triplet and singlet states have different respective total spin $S = 1$ and $S = 0$, a global lowering operator $S^- = \sum_{j=1}^N S_j^-$ leaves the singlet state $|G_{0,0}\rangle$ invariant while lowering the magnetization S_z of the triplet states. Repeated application of S^- will then force the population into the state with the lowest weight, i.e., $|G_{1,-1}\rangle$, which is also invariant upon acting with S^-. Hence, a globally acting Lindblad dissipator $L_G = \sqrt{\gamma}S^-$ with dissipation rate γ, establishes two steady states of the master Eq. (2) given by the pure states $\varrho_0 = |G_{0,0}\rangle\langle G_{0,0}|$ and $\varrho_1 = |G_{1,-1}\rangle\langle G_{1,-1}|$. Together with the operator $A = |G_{1,-1}\rangle\langle G_{0,0}|$ conditions (3) and (4) are fulfilled; in particular, it holds that

$$\mathcal{L}[\varrho_{10} = A\varrho_0] = \mathrm{i}\frac{B}{N}\varrho_{10}, \quad \mathcal{L}[\varrho_{01} = \varrho_0 A^\dagger] = -\mathrm{i}\frac{B}{N}\varrho_{01}. \tag{6}$$

Note, that $\varrho_1 = A\varrho_0 A^\dagger$. We now also recognize that lifting the ground state degeneracy is necessary to observe synchronization, i.e., without the external magnetic field in Eq. (1) the oscillation frequency would be zero.

Depopulating the excited states.—So far, we have discussed how synchronization may arise within the ground state manifold with the help of a dissipative channel in terms of L_{G}. However, in addition the excited states need to be depopulated. This can be done either in a two-step process where one prepares the ground state subspace first using established approaches [60–69] or via depopulation during the dissipative evolution. As a proof of principle, we here opt for the latter and construct the simplest possible operators by exploiting that the Hamiltonian (1) preserves total angular momentum. In particular, for $B = 0$, each term in H can be written as $P^{(2)}_{j,j+1}$, where $P^{(2)}_{j,j+1}$ denotes the projector of two spin-1's on sites j and $j+1$ onto total spin-2. Hence, the ground states are reached by driving two adjacent spin-1 particles out of the $S = 2$ subspace. The previously introduced dissipative channel ($L_{\mathrm{G}} = \sqrt{\gamma}S^-$) forces all population within the $S = 2$ subspace to eventually reach the $S_z = -2$ state. Thus, we only need to depopulate these states to dissipatively reach the ground state manifold. An exemplary choice is the Lindblad dissipators $L_{j,j+1} = \sqrt{\kappa}|00\rangle\langle --|_{j,j+1}$ written in the S_z basis $\{|+\rangle, |0\rangle, |-\rangle\}$.

Synchronized dynamics.—Combining all Lindblad operators, the dissipative evolution of the density matrix which eventually leads to the synchronization of the fractionalized spins is given by

$$\dot{\varrho} = -\mathrm{i}[H, \varrho] + \mathcal{D}[L_{\mathrm{G}}]\varrho + \sum_{j=1}^{N-1}\mathcal{D}[L_{j,j+1}]\varrho = \mathcal{L}[\varrho], \tag{7}$$

where $\mathcal{D}[L]\varrho = 2L\varrho L^\dagger - \{L^\dagger L, \varrho\}$. Its solution, given that the system is initialized in the state $\varrho(0)$, may be expressed using the spectral decomposition of the Liouvillian superoperator as

$$\varrho(t) = \sum_k C_k \exp(\lambda_k t)\varrho_k, \tag{8}$$

where ϱ_k is the right eigenstate of $\mathcal{L}$ with corresponding eigenvalue λ_k, i.e., $\mathcal{L}[\varrho_k] = \lambda_k\varrho_k$. As $\mathcal{L}$ is non-Hermitian, the left eigenstates defined by $\mathcal{L}^\dagger[\sigma_k] = \lambda_k^*\sigma_k$ may differ from the right ones. However, it holds that $\mathrm{Tr}(\sigma_k^\dagger\varrho_{k'}) = \delta_{kk'}$. The constant C_k in Eq. (8) denotes the overlap of the eigenstates with the initial state $\varrho(0)$, i.e., $C_k = \mathrm{Tr}[\sigma_k^\dagger\varrho(0)]$. Note that because $\mathcal{L}$ generates a CPTP map, the eigenvalues λ_k can lie only in the left half of the complex plane with $\mathrm{Re}[\lambda_k] \leq 0$, and they always come in pairs, i.e., if λ_k is an eigenvalue, so is λ_k^*. All eigenstates of $\mathcal{L}$ with negative real part of the corresponding eigenvalues will experience selective decay, and only the ones which lie on the imaginary axis contribute to the dynamics in the long time limit.

As discussed previously, the dynamics given by Eq. (7) will eventually terminate in the decoherence-free subspace [50] spanned by $\{\varrho_0, \varrho_1, \varrho_{10}, \varrho_{01}\}$. Thus, the expectation value of some observable O is given by

$$\lim_{t\to\infty}\langle O\rangle(t) = C_0\mathrm{Tr}(O\varrho_0) + C_1\mathrm{Tr}(O\varrho_1) + [\mathrm{e}^{\mathrm{i}Bt/N}C_{01}\langle G_{0,0}|O|G_{1,-1}\rangle + \mathrm{c.c.}]. \tag{9}$$

Because the subspace is decoherence free, $C_i = \mathrm{Tr}[\sigma_i^\dagger\varrho(0)] = \mathrm{Tr}[\varrho_i^\dagger\varrho(0)]$. In order to observe stable synchronization, not only does the initial state need to have nonvanishing overlap with the eigenstate ϱ_{01}, but also the observable is nonzero in that state, i.e., $\mathrm{Tr}[OA\varrho_0] = \langle G_{0,0}|O|G_{1,-1}\rangle \neq 0$. A suitable choice of local operators that may be used as witnesses of the fractionalized spin synchronization are given by the transverse spin S_j^x, for which the first two terms in Eq. (9) are identical to zero, and only $C_{01} = \langle G_{0,0}|\varrho(0)|G_{1,-1}\rangle = C_{10}^*$ and $\langle G_{0,0}|S_j^x|G_{1,-1}\rangle$ contribute to the long-time dynamics.

As the dynamical symmetry operator $A = |G_{1,-1}\rangle\langle G_{0,0}|$ is antisymmetric upon inversion of the chain, an operator acting locally on site j will be antisynchronized with the corresponding site at the other end of the chain located at $(N+1) - j$. Figures 1(a) and 1(b) show the time evolution of the transverse spin $\langle S_j^x\rangle$ for a chain of length $N = 6$ with a random pure state as initial condition (solid lines correspond to the left half of the chain $j = 1, 2, 3$, dashed lines to the right half $j = 4, 5, 6$). The oscillations are perfectly antisynchronized upon inversion of the chain. As a consequence of the fractionalized spin, the amplitudes decay exponentially into the bulk. As seen in Fig. 1(b) for short times there is no synchronization. However, the transient time is short compared to the oscillation frequency $\omega = B/N$. For random initial conditions the oscillation amplitudes even at the boundaries are small. The reason is that the overwhelming majority of states has no overlap with the ground state coherences such that $C_{01} = \langle G_{0,0}|\varrho(0)|G_{1,-1}\rangle \ll 1$. However, one may maximize this overlap by choosing $\varrho(0) = |\psi\rangle\langle\psi|$ as initial state, where $|\psi\rangle$ represents an equal superposition of $|G_{0,0}\rangle$ and $|G_{1,S_z}\rangle$. This suggests that the previously mentioned two-step process may be better suited for actual experimental implementations. Figure 1(c) shows the dynamics for $|\psi\rangle = (|G_{0,0}\rangle + |G_{1,-1}\rangle)/\sqrt{2}$ as initial state. As this

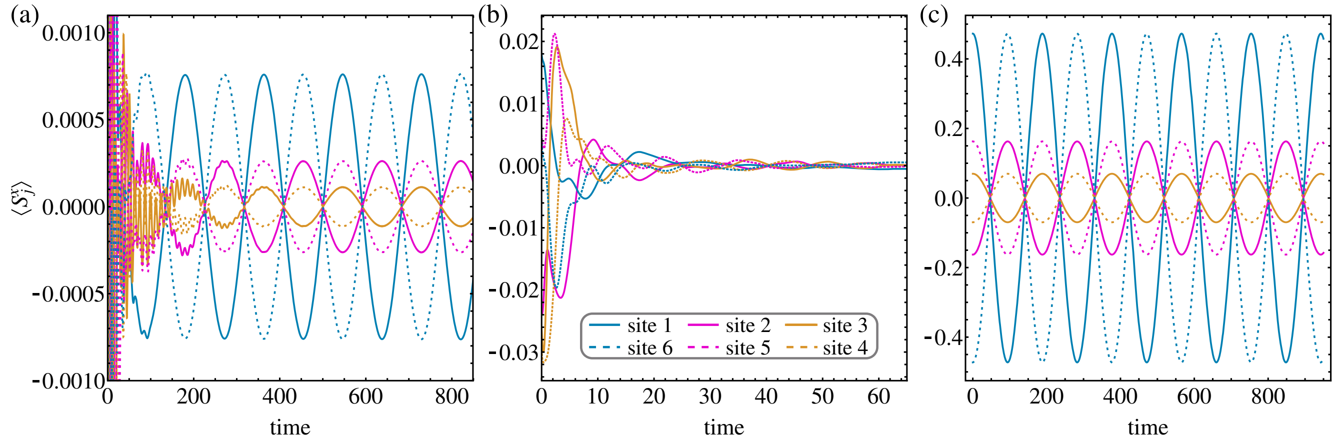

FIG. 1. Evolution of the local transverse spin $\langle S_j^x \rangle$ of the synchronized AKLT model for an open chain of length $N = 6$ (sites $j = 1$, 2, 3 in solid lines, sites $j = 4$, 5, 6 in dashed lines). (a) Starting from a random pure state, i.e., a vector of dimension 3^6 with random complex amplitudes (we also investigated random mixed states with equivalent results), the two halves of the chain are perfectly antisynchronized with each other after a transient time because the dynamical symmetry operator $A = |G_{1,-1}\rangle\langle G_{0,0}|$ is antisymmetric upon inversion of the chain. The (anti)synchronized amplitudes after the transient time decay exponentially into the bulk. (b) Same plot as in (a) but focusing on the early time dynamics: The random initial conditions result in transient random spin dynamics. (c) The balanced superposition of $|G_{0,0}\rangle$ and $|G_{1,-1}\rangle$ as initial state is immune to dissipation and maximizes the observed (anti)synchronization amplitudes. The oscillation frequency is $\omega = B/N$. Parameters: $B = 0.2$, $\gamma = \kappa = 0.2$.

state is decoherence free, the amplitudes are unaffected by the dissipation and antisynchronization is stable. Note, that the transient relaxation time, related to the Liouvillian gap Δ, scales in general exponentially with system size. However, within the ground state subspace $\Delta = 2\gamma$. Thus, in combination with the previously mentioned two-step process, where the ground state manifold can be prepared dissipatively in polynomial time [61] or via a constant-depth quantum circuit [68], the synchronized dynamics can be achieved efficiently in general.

Haldane chain.—The AKLT Hamiltonian (1) exhibits spin-1/2 degrees of freedom that are perfectly localized at boundaries and do not interact. In the following we investigate the impact of interactions by decreasing the value of the biquadratic term in Eq. (1), i.e., we consider the Hamiltonian

$$H_\varepsilon = H - \varepsilon \sum_{j=1}^{N-1} (\vec{S}_j \cdot \vec{S}_{j+1})^2. \tag{10}$$

For finite values of ε, the Hamiltonian cannot simply be expressed via projection operators and the Lindblad operators $L_{j,j+1}$ induce additional dissipation. Figure 2(a) shows the complex eigenvalues of $\mathcal{L}$ close to the imaginary axis for different values of ε in the range of $[0, 1/6]$, where $\varepsilon = 1/6$ removes the biquadratic term completely, and corresponds to the spin-1 Heisenberg chain (with additional magnetic field). Upon increasing ε the initially purely imaginary eigenvalues move away from both the real and the imaginary axis, i.e., the oscillation frequency increases, yet the synchronization is damped. However, the real part remains small and in particular the eigenvalues with second smallest real part also move away from the imaginary axis. Thus, there exists a time range for which all eigenstates but the synchronized ones are damped. Such damped synchronized dynamics has also been termed metastable synchronization [47].

Stable synchronization may, however, be restored under certain conditions even for the Heisenberg chain ($\varepsilon = 1/6$). To this end we consider the case of $B = 0$, such that for $\varepsilon = 0$ the ground state is fourfold degenerate. Perturbations to the biquadratic term of the AKLT Hamiltonian partially lift the ground degeneracy such that the $S = 0$ state is energetically distinct from the states within the $S = 1$ subspace. In the following, we will refer to both the fourfold degenerate ground state for $\varepsilon = 0$ as well as the threefold degenerate subspace together with the ground state for $\varepsilon \neq 0$ as the ground state subspace. As H_ε and the dissipator L_G preserve the total angular momentum, the dynamics is confined to their respective total angular momentum subspace for $\kappa = 0$. Then, there exists again a dynamical symmetry operator connecting the threefold degenerate subspace ($S = 1$) with the $S = 0$ state. Similar to the previous discussion, this results in perfect antisynchronization if the initial state is chosen to be within the ground state subspace. Figures 2(b) and 2(c) show the time evolution of the transverse spin $\langle S_j^x \rangle(t)$ for a chain of length $N = 6$ for $\varepsilon = 0.1$ and $\varepsilon = 1/6$, respectively. The dynamics show perfect antisynchronization for the infinite temperature state within the ground state subspace as initial state. The oscillation frequency in Fig. 2(c) is larger compared to Fig. 2(b) as the energy gap between the $S = 1$ and $S = 0$ subspaces opens.

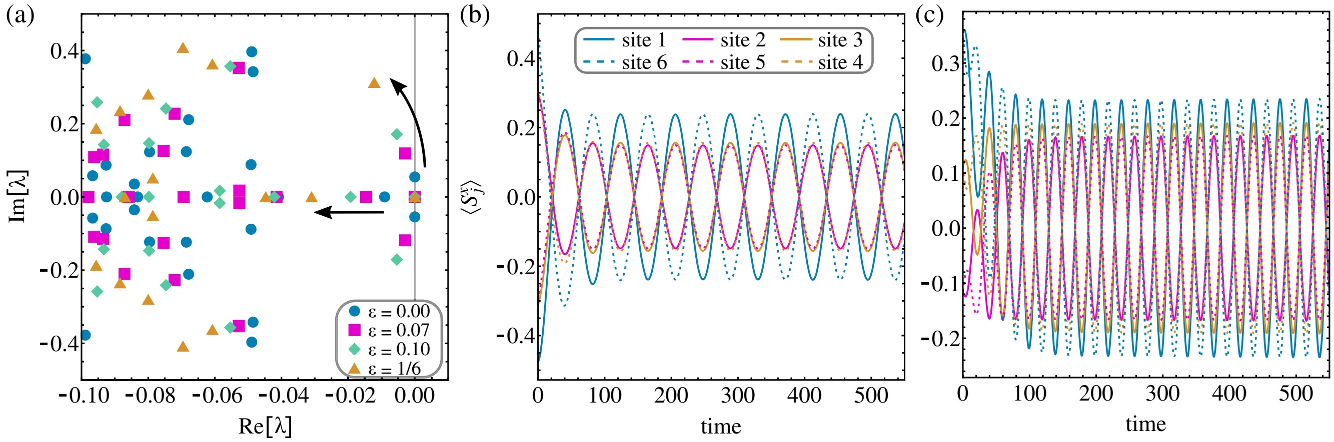

FIG. 2. (a) Eigenvalues of the Liouvillian superoperator $\mathcal{L}$ close to the imaginary axis for an open chain of length $N = 6$ and different values of ε, considering both global and local dissipators. The purely imaginary eigenvalues for $\varepsilon = 0$ move away from the imaginary axis as the biquadratic term is decreased. Simultaneously, the oscillation frequency increases resulting in fast but damped (anti) synchronization. Parameters: $B = 0.2$, $\kappa = \gamma = 0.2$. (b),(c) Stable synchronization may be recovered for finite values of ε within the ground state subspace for $B = 0$ if one considers only the global dissipator L_G (i.e., $\kappa = 0$). Sites $j = 1, 2, 3$ are in solid lines, sites $j = 4, 5, 6$ in dashed lines. The oscillation frequency in panel (c) with $\varepsilon = 1/6$ is larger compared to panel (b) with $\varepsilon = 0.1$ because of the increased gap above the threefold degenerate ground state. The initial state is the infinite temperature state within the ground state manifold.

A few remarks are in order. First, the synchronization observed in Figs. 2(b) and 2(c) is distinct from regular coherent dynamics. While without dissipation, coherent (antiphase) oscillations are still present, there also exists an additional constant shift depending on the initial conditions. Different locally acting observables may not exhibit the same shift, so in the strict definition of Eq. (5) they are not synchronized. Open system dynamics are thus necessary for perfect (anti)synchronization even within the ground state subspace of the Haldane chain. Second, perfect antisynchronization in the Haldane chain ($\varepsilon \neq 0$) is possible without an additional magnetic field ($B = 0$), which demonstrates the significance of microscopic details for synchronization even for systems belonging to the same (Haldane) phase. Third, the Haldane phase is protected as long as any *one* of three symmetries is preserved [53,70]. That is time-reversal symmetry, link inversion symmetry (lattice inversion about the center of a bond), and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (π rotations about two orthogonal axes). As a consequence, synchronization within the ground state subspace is robust even if the inversion symmetry is broken, for example by perturbing the interactions between neighboring spins or via an inhomogenous magnetic field [71]. This is in clear contrast to previous spin chain models [35,47,48] where permutation symmetry of the Liouvillian is necessary for synchronized dynamics of local observables as defined in Eq. (5). In particular, synchronization induced by permutation symmetries becomes, in general, unstable upon symmetry breaking perturbations (i.e., the purely imaginary eigenvalues acquire a negative real part), whereas it remains stable in the presence of symmetry-protected topological order. However, additional single site dissipation or decoherence is not protected by the topology and will render synchronization metastable, similar to other spin chains with dynamical symmetries.

Conclusions.—We have shown that it is possible to synchronize the *fractionalized* spin degrees of freedom in the spin-1 AKLT chain via engineered dissipation and an external magnetic field. The observed synchronization is stable and topologically protected. While perturbations to the biquadratic term result in an additional dissipation channel, stable synchronization is restored within the ground state subspace of the chain even without magnetic field via a single global spin lowering operator. Given recent experimental advancements in the preparation of the ground state of the AKLT model on a digital quantum computer [68] and the Heisenberg chain in cold atoms [72], coupled with the experimental capability to implement collective decay processes on these platforms (either through a combination of ancillary systems and measurements [73] or via a collective cavity mode in the "bad cavity" limit [74,75]) the prospect of realizing topological synchronization seems attainable in the near future [71]. Our results illuminate on the possibility to utilize dissipation in order to control the dynamics of fractionalized degrees of freedom, not only to prepare them, and provide a pathway to topologically induced quantum synchronization.

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