Geometry and Unitarity of Scalar Fields Coupled to Gravity

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We formulate scalar field theories coupled nonconformally to gravity in a manifestly frame-independent fashion. Physical quantities such as the S matrix should be invariant under field redefinitions, and hence can be represented by the geometry of the target space. This elegant geometric formulation, however, is obscured when considering the coupling to gravity because of the redundancy associated with the Weyl transformation. The well-known example is the Higgs inflation, where the target space of the Higgs fields is flat in the Jordan frame but is curved in the Einstein frame. Furthermore, one can even show that any geometry of $\mathrm{O}(N)$ nonlinear σ models can be flattened by an appropriate Weyl transformation. In this Letter, we extend the notion of the target space by including the conformal mode of the metric, and show that the extended geometry provides a compact formulation that is manifestly Weyl-transformation or field-redefinition invariant. We identify the cutoff scale with the inverse of square root of the extended target-space curvature and confirm that it coincides with that obtained from two-to-two scattering amplitudes based on our formalism.

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Introduction.—Physical quantities should remain invariant under transformation between different descriptions of the same system. Such a redundancy has advantages in simplifying calculation. However, it also unavoidably induces opacity of the consistency among different descriptions because the calculation can be drastically different.

Any scalar field theory in general is given the concept of geometry in its field-space manifold, or the target space, which is known as a nonlinear σ model (NLSM) [1–8]. NLSMs frequently arise as low-energy effective field theories in various fields of theoretical physics. They are particularly useful when the system undergoes some symmetry, whose information is encoded in the target space, e.g., the coset space for the effective theory of Nambu-Goldstone (NG) modes [9,10]. Since physical quantities such as the S matrix should be invariant under field redefinitions, we may take whatever field basis of the target space so that the calculations become simple, at a cost of obscuring the field-redefinition invariance. An elegant approach that makes this invariance manifest is formulating the physical observables geometrically since the geometry of target space is also invariant under field redefinitions [11–18]. However, such a geometric formulation is spoiled once we turn on gravity. The fundamental building block of gravitational theory is the metric of spacetime $g_{\mu\nu}$ through which all the fields couple to gravity. Now we can perform the following redefinition of the metric $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with Ω an arbitrary function of matter fields, which is known as the Weyl or frame transformation. It not only modifies the coupling to the Ricci scalar, but changes the geometry of the target space of an NLSM. A famous example in cosmology is the Higgs inflation (HI) [19–21], which is originally defined in the Jordan frame where a large nonminimal coupling between the standard model (SM) Higgs fields and Ricci scalar is introduced to fit the observation of cosmic microwave background [22] while the target space of the Higgs fields is flat. One may perform a Weyl transformation to the Einstein frame [23] where the Higgs fields are minimally coupled but the target space is curved. This redundancy makes the determination of the target-space geometry ambiguous.

As an NLSM is merely a low-energy effective field theory (EFT), its validity is restricted up to a certain energy scale corresponding to the target-space curvature. This scale Λ can be extracted as the unitarity-violation scale of tree-level scattering amplitudes, and again the geometric formulation provides a powerful toolkit [12,14,15,18]. Along the same way, to verify the validity of HI as an EFT for inflation, the inflation scale needs to stay below Λ (the momenta of the gauge bosons produced during preheating can exceed the low cutoff scale, so the system becomes strongly coupled [24]). However, owing to the presence of gravity, the geometric formulation is not applicable, and we had to calculate all relevant processes in different frames and compare the results to guarantee consistency [25–32], which has caught intensive attention and debate due to the ambiguity (see also [33,34]).

In this Letter, we extend the notion of target-space geometry to preserve the advantages of the geometric approach even with gravity. Specifically, we include the conformal mode of the spacetime metric $\det(g_{\mu\nu})$ as a coordinate of the target space [35,36]. The extended target-space geometry is manifestly invariant even under the Weyl transformation because it corresponds to redefining the conformal mode. We provide the geometrical meaning of the cutoff scale, which is manifestly independent of the frame and state.

Metric Higgs inflation.—We begin our discussion by considering HI in a metric formulation of gravity, which is based on the following action:

$$S = \int \mathrm{d}^4x\sqrt{-g_\mathrm{J}}\left(\frac{M_\mathrm{Pl}^2 + \xi\phi^2}{2}R_\mathrm{J} - \frac{g_\mathrm{J}^{\mu\nu}}{2}\delta_{ij}\partial_\mu\phi^i\partial_\nu\phi^j - V\right). \tag{1}$$

Here, $g_{\mathrm{J}\mu\nu}$ is the metric of spacetime in the Jordan frame with its determinant being g_J, R_J is the Ricci curvature determined by $g_{\mathrm{J}\mu\nu}$, M_Pl is the reduced Planck mass, ϕ^i is a multicomponent scalar field whose index i runs through $i = 1, \ldots, N$ with $N \geqslant 2$, $\phi^2 \equiv \delta_{ij}\phi^i\phi^j$, and V is a potential of the scalar field invariant under $\mathrm{O}(N)$ rotation. The target space spanned by ϕ^i is trivial, i.e., δ_{ij}. By identifying ϕ^i as the SM Higgs doublet for $N = 4$ and V as the SM Higgs potential [19–21], one can show that the large expectation value of the SM Higgs fields exhibits the cosmic inflation perfectly consistent with observations [22]. In this case, the SM Higgs fields couples to, e.g., gauge bosons which acquire mass terms for a finite vacuum expectation value (VEV) of Higgs fields. In the discussion of unitarity, we are interested in the behavior at a higher energy than the VEV. For this reason, only the longitudinal modes are important, and thereby the Goldstone equivalence theorem guarantees that our action (1) is sufficient [26,27,32,37,38].

We can write down the same model in a seemingly different form by performing the following Weyl transformation:

$$g_{\mathrm{E}\mu\nu} = \Omega^2 g_{\mathrm{J}\mu\nu}, \qquad \Omega^2 \equiv 1 + \frac{\xi\phi^2}{M_\mathrm{Pl}^2}, \tag{2}$$

which leads to the action in the Einstein frame

$$S = \int \mathrm{d}^4x\sqrt{-g_\mathrm{E}}\left(\frac{M_\mathrm{Pl}^2}{2}R_\mathrm{E} - \frac{g_\mathrm{E}^{\mu\nu}}{2}G^\mathrm{E}_{ij}\partial_\mu\phi^i\partial_\nu\phi^j - \frac{V}{\Omega^4}\right), \tag{3}$$

where the target-space metric is given by

$$G^\mathrm{E}_{ij} \equiv \frac{1}{\Omega^2}\left(\delta_{ij} + \frac{6\xi^2}{\Omega^2}\frac{\phi_i\phi_j}{M_\mathrm{Pl}^2}\right). \tag{4}$$

One may readily confirm that the target space of ϕ^i is curved in this frame although it is flat in the Jordan frame (1). Nevertheless, the physical quantities should be unchanged as the Weyl transformation (2) is merely a field redefinition.

This observation motivates us to extend the notion of the target space so that the extended geometry is also invariant under the Weyl transformation. For this purpose, we extract the conformal mode of the metric as [35,36,39]

$$g_{\bullet\mu\nu} = \frac{\Phi_\bullet^2}{6M_\mathrm{Pl}^2}\tilde{g}_{\mu\nu}, \qquad \det(\tilde{g}_{\mu\nu}) = -1, \tag{5}$$

where the black dot implies a subscript associated with the frame, e.g., $\bullet = \mathrm{J}, \mathrm{E}$. The Weyl transformation of Eq. (2) now turns into the field redefinition of

$$\Phi_\mathrm{E}^2 = \Omega^2\Phi_\mathrm{J}^2. \tag{6}$$

Once the target space is extended to involve $\Phi_\bullet$, its geometry is manifestly invariant under not only the field redefinition of ϕ_i but the Weyl transformation.

We rewrite the action (1) in a field basis of $(\varphi^a_\mathrm{J}) = (\Phi_\mathrm{J}, \phi^i)$. A straightforward calculation leads to the following action:

$$S = \int \mathrm{d}^4x\left(\frac{\Phi_\mathrm{J}^2}{12}\Omega^2\tilde{R} - \frac{\tilde{g}^{\mu\nu}}{2}G^\mathrm{J}_{ab}\partial_\mu\varphi^a_\mathrm{J}\partial_\nu\varphi^b_\mathrm{J} - \frac{\Phi_\mathrm{J}^4 V}{36M_\mathrm{Pl}^4}\right), \tag{7}$$

where the extended target-space metric reads

$$(G^\mathrm{J}_{ab}) \equiv \begin{pmatrix} -\Omega^2 & -\xi\Phi_\mathrm{J}\phi_j/M_\mathrm{Pl}^2 \\ -\xi\Phi_\mathrm{J}\phi_i/M_\mathrm{Pl}^2 & \frac{\Phi_\mathrm{J}^2}{6M_\mathrm{Pl}^2}\delta_{ij} \end{pmatrix}. \tag{8}$$

Here the Ricci curvature $\tilde{R}$ is given by $\tilde{g}_{\mu\nu}$. Physical quantities should be represented by the geometry specified by G^J_{ab}.

Let us estimate the cutoff scale of this theory in the geometric language with the extended target space. Hereafter, we assume the contribution from the potential

TABLE I. Target-space Riemann tensor at $(\bar{\varphi}^a_{\rm J})=(\sqrt{6}\Lambda_{\rm G}/\bar{\Omega},v,0,\ldots,0)$ in several examples. The field vector is expanded as $(\varphi^a_{\rm J})=(\bar{\varphi}^a_{\rm J})+(\delta\Phi_{\rm J},h,\pi^1,\ldots,\pi^{N-1})$. Note that the results are independent of frames, so we drop the frame index. $\bar{R}_{\rm others}$ means that at least one of the indices is $\delta\Phi_{\rm J}$. We parametrize the Riemann tensor as $\bar{R}_{ikjl}=\bar{R}_{ik}(\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk})$.

	Metric HI	Einstein-Cartan HI	NLSM
$\bar{R}_{h\pi^k}$	$\{(1+6\xi)^2\Lambda_{\rm G}^2/6(M_{\rm Pl}^2+\xi v^2)[M_{\rm Pl}^2+(1+6\xi)\xi v^2]\}$	$\Lambda_{\rm G}^2(\{[\xi(1+6r^2\xi)v^2+(1+6\xi)M_{\rm Pl}^2]^2-36(1-r^2)\xi^2M_{\rm Pl}^4\}/6(M_{\rm Pl}^2+\xi v^2)^3[M_{\rm Pl}^2+\xi(1+6r^2\xi)v^2])$	$\Lambda_{\rm G}^2[(\bar{G}^2+12M_{\rm Pl}^2\bar{G}(\overline{f'}+v^2\overline{f''})+6M_{\rm Pl}^2\{\bar{f}\,\overline{G'}+\overline{f'}[6M_{\rm Pl}^2(\overline{f'}+2v^2\overline{f''})-v^2\overline{G'}]\})/6M_{\rm Pl}^4\bar{f}^2(\bar{G}+6M_{\rm Pl}^2v^2\overline{f'}^2\bar{f}^{-1})]$
$\bar{R}_{\pi^i\pi^k}$	$\{(1+6\xi)^2\Lambda_{\rm G}^2/6(M_{\rm Pl}^2+\xi v^2)[M_{\rm Pl}^2+(1+6\xi)\xi v^2]\}$	$\Lambda_{\rm G}^2(\{\xi(1+6\xi)(1+6r^2\xi)v^2+[1+12\xi(1+3r^2\xi)]M_{\rm Pl}^2\}/6(M_{\rm Pl}^2+\xi v^2)^2[M_{\rm Pl}^2+\xi(1+6r^2\xi)v^2])$	$\Lambda_{\rm G}^2\{[(6M_{\rm Pl}^2\bar{f}+v^2)(\bar{G}-1)+v^2(1+6M_{\rm Pl}^2\overline{f'})^2]/6M_{\rm Pl}^4v^2\bar{f}^2(\bar{G}+6M_{\rm Pl}^2v^2\overline{f'}^2\bar{f}^{-1})\}$
$\bar{R}_{\rm others}$	0	0	0

is subdominant so we can drop it. This is actually true for HI whose potential is up to quartic order with sufficiently small coupling. One convenient way of extracting the cutoff scale is to consider the two-to-two scattering of ϕ^i as it grows in proportion to $E_{\rm c.m.}^2/\Lambda_{\rm G}^2$ with $E_{\rm c.m.}$ being the center-of-mass energy. $\Lambda_{\rm G}$ is the unavoidable UV cutoff scale above which the graviton becomes strongly coupled, and to which the ratio of dimensionful quantities acquires physical meaning, usually chosen as $M_{\rm Pl}$. Thus, one can extract the cutoff scale by requiring each amplitude to be smaller than unity. Consider scatterings around the background of $(\bar{\varphi}^a_{\rm J})=(\sqrt{6}\Lambda_{\rm G}/\bar{\Omega},v,0,\ldots,0)$ [we choose $\bar{\Omega}(v)\bar{\Phi}_{\rm J}=\sqrt{6}\Lambda_{\rm G}$ as background such that the kinetic term of the graviton is expressed as $\Lambda_{\rm G}^2(\partial_\rho h_{\mu\nu})^2/8$ with $\tilde{g}_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$] and $\bar{\tilde{g}}_{\mu\nu}=\eta_{\mu\nu}$, which is relevant for the unitarity violation during HI. Geometric language provides the following elegant expression of general four-point amplitudes [14–16,18]:

$$\mathcal{M}_{IJ\leftrightarrow KL}=\frac{2}{3}\left[s_{IJ}\bar{R}_{I(KL)J}+s_{IK}\bar{R}_{I(JL)K}+s_{IL}\bar{R}_{I(JK)L}\right],\qquad(9)$$

with $s_{IJ}\equiv(p_I+p_J)^2$. The subscripts I specify the states, where the capital letter indicating that the states should be canonically normalized, e.g., $I=H$ for the Higgs mode. The relation between a field basis φ^a and the canonically normalized states at $\bar{\varphi}^a$ are provided by the vielbein, e.g., $\bar{G}^{\rm J}_{ab}=\bar{e}^A_a\bar{e}^B_b\eta_{AB}$. The parentheses in the subscripts denote the symmetrization. The Riemann tensor is

$$\bar{R}_{ABCD}=\bar{e}^a_A\bar{e}^b_B\bar{e}^c_C\bar{e}^d_D\bar{R}_{abcd},\qquad(10)$$

with the vielbein and Riemann tensor being evaluated at $\bar{\varphi}^a$. To estimate the scattering amplitudes among Higgs H and NG bosons Π^i at $\bar{\varphi}^a_{\rm J}$, all we need is the following vielbein:

$$(\bar{e}^a_A)=\frac{M_{\rm Pl}\bar{\Omega}}{\Lambda_{\rm G}}\left(\begin{array}{cc|c}\frac{\Lambda_{\rm G}}{M_{\rm Pl}\bar{\Omega}^2} & -\frac{\sqrt{6}\xi\Lambda_{\rm G}v/M_{\rm Pl}^2}{\bar{\Omega}^3\sqrt{1+6\xi^2v^2/(M_{\rm Pl}^2\bar{\Omega}^2)}} & \mathbf{0}\\ 0 & \frac{1}{\sqrt{1+6\xi^2v^2/(M_{\rm Pl}^2\bar{\Omega}^2)}} & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbb{1}\end{array}\right).\qquad(11)$$

In the literature, e.g., [28,29,31,32], the scattering amplitudes among the Higgs and NG bosons are computed explicitly to confirm the invariance between the Einstein and Jordan frames. Owing to the extended geometry of the target space, the invariance under the frame transformation with fixed incoming and outgoing states now becomes manifest. This is because the Weyl transformation is a particular coordinate transformation with respect to the field indices a in the extended target space, which are already contracted as given in Eq. (10).

Furthermore, the physical quantities such as the cutoff scale should not even depend on the choice of incoming and outgoing states. This motivates us to consider the Ricci scalar of the extended target space, which is frame, coordinate, and states independent,

$$\bar{R}=\bar{G}^{ab}\bar{G}^{cd}\bar{R}_{acbd}=2(N-1)\left(\bar{R}_{H\Pi^k}+\frac{N-2}{2}\bar{R}_{\Pi^i\Pi^k}\right).\qquad(12)$$

In the second equality, we have used $\bar{R}_{ABCD}\neq0$ only if all the indices are $I=H,\Pi^k$, and $\bar{R}_{IKJL}=\bar{R}_{IK}(\delta_{IJ}\delta_{KL}-\delta_{IL}\delta_{JK})$, which follows from Eq. (11) and Table I. The cutoff scale of the theory is then given by $\Lambda_{\rm metric}/\Lambda_{\rm G}\sim\sqrt{N^2/\bar{R}}/\Lambda_{\rm G}$.

Now we confirm that the cutoff scale extracted from the scattering amplitudes coincides with the Ricci scalar by explicit computations. From Table I and Eq. (11), the nonvanishing scattering amplitudes read

$$\mathcal{M}_{\Pi^i\Pi^i\leftrightarrow\Pi^j\Pi^j}=-\frac{s_{12}}{6\Lambda_{\rm G}^2}\frac{(1+6\xi)^2(M_{\rm Pl}^2+\xi v^2)}{M_{\rm Pl}^2+(1+6\xi)\xi v^2}\quad\text{for } i\neq j,\qquad(13)$$

$$\mathcal{M}_{HH\leftrightarrow\Pi^i\Pi^i} = -\frac{s_{12}}{6\Lambda_G^2}\frac{(1+6\xi)^2(M_{\rm Pl}^2+\xi v^2)^2}{[M_{\rm Pl}^2+(1+6\xi)\xi v^2]^2}. \tag{14}$$

Consequently, we obtain the cutoff scale of the metric HI

$$\frac{\Lambda_{\rm metric}}{\Lambda_G}\sim\frac{\bar{R}^{-1/2}_{\Pi^i\Pi^k}}{\Lambda_G}\sim\begin{cases}1/\xi & \text{for } v\lesssim M_{\rm Pl}/\xi,\\ v/M_{\rm Pl} & \text{for } M_{\rm Pl}/\xi\lesssim v\lesssim M_{\rm Pl}/\sqrt{\xi},\\ 1/\sqrt{\xi} & \text{for } M_{\rm Pl}/\sqrt{\xi}\lesssim v,\end{cases} \tag{15}$$

for $N>2$, and

$$\frac{\Lambda_{\rm metric}}{\Lambda_G}\sim\frac{\bar{R}^{-1/2}_{H\Pi^k}}{\Lambda_G}\sim\begin{cases}1/\xi & \text{for } v\lesssim M_{\rm Pl}/\xi,\\ \xi v^2/M_{\rm Pl}^2 & \text{for } M_{\rm Pl}/\xi\lesssim v\lesssim M_{\rm Pl}/\sqrt{\xi},\\ 1 & \text{for } M_{\rm Pl}/\sqrt{\xi}\lesssim v,\end{cases} \tag{16}$$

for $N=2$ where we do not have $\Pi^i\Pi^i\leftrightarrow\Pi^j\Pi^j$ scattering. Our results are consistent with the literature, such as Refs. [26–28,31,32,41]. It is clear that the obtained cutoff scale indeed coincides with the Ricci scalar given in Eq. (12), which is frame and state independent. We emphasize that the physically relevant quantity is the ratio $\Lambda_{\rm metric}/\Lambda_G$, which is frame independent.

Other Higgs inflation.—Our discussion is also applicable to alternative formalisms of gravity like Palatini HI [42,43]. It is recently realized that HI in the Einstein-Cartan gravity with a nonminimally coupled Nieh-Yan term [44] can serve as a general setup that includes the well-known metric and Palatini cases [45], so here we consider Einstein-Cartan HI for a general discussion. In Einstein-Cartan HI, the affine connection $\Gamma^\rho_{\mu\nu}$ is treated *a priori* independently of $g_{\mu\nu}$, although the action is of the same form as Eq. (1). The nonminimally coupled Nieh-Yan term is [45]

$$-\frac{\xi_\eta}{4}\int d^4x\,\phi_i^2\partial_\mu(\epsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma}), \tag{17}$$

where ξ_η is the coupling constant, $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol such that $\epsilon^{0123}=1$, and $T^\rho_{\mu\nu}\equiv\Gamma^\rho_{\mu\nu}-\Gamma^\rho_{\nu\mu}$ is the torsion tensor. One can solve the constraint equation for Γ to obtain an equivalent action in the same form as Eq. (7) but G^J_{ab} is replaced by [46]

$$(G^{\rm EC}_{ab})\equiv\begin{pmatrix}-\Omega^2 & -\xi\Phi_J\phi_j/M_{\rm Pl}^2\\ -\xi\Phi_J\phi_i/M_{\rm Pl}^2 & \frac{\Phi_J^2}{6M_{\rm Pl}^2}\left(\delta_{ij}-\frac{6(1-r^2)\xi^2}{M_{\rm Pl}^2\Omega^2}\phi_i\phi_j\right)\end{pmatrix}, \tag{18}$$

where we have defined $r\equiv\xi_\eta/\xi$. Thus, $r=1$ recovers the results in the previous section, while $r=0$ reproduces the Palatini HI.

Following previous procedures, the frame-independent cutoff scale is obtained by calculating the Ricci scalar for the extended target space, whose structure is the same as Eq. (12): $\Lambda_{\rm EC}/\Lambda_G\sim\bar{R}^{-1/2}/\Lambda_G$. For $1/\sqrt{\xi}\lesssim r\leqslant 1$,

$$\frac{\Lambda_{\rm EC}}{\Lambda_G}\sim\begin{cases}1/(r\xi) & \text{for } v\lesssim M_{\rm Pl}/(r\xi),\\ v/M_{\rm Pl} & \text{for } M_{\rm Pl}/(r\xi)\lesssim v\lesssim M_{\rm Pl}/\sqrt{\xi},\\ 1/\sqrt{\xi} & \text{for } M_{\rm Pl}/\sqrt{\xi}\lesssim v,\end{cases} \tag{19}$$

for $N>2$, and

$$\frac{\Lambda_{\rm EC}}{\Lambda_G}\sim\begin{cases}1/(r\xi) & \text{for } v\lesssim M_{\rm Pl}/(r\xi),\\ r\xi v^2/M_{\rm Pl}^2 & \text{for } M_{\rm Pl}/(r\xi)\lesssim v\lesssim M_{\rm Pl}/\sqrt{\xi},\\ r\sqrt{\xi}v/M_{\rm Pl} & \text{for } M_{\rm Pl}/\sqrt{\xi}\lesssim v\lesssim M_{\rm Pl}/(r\sqrt{\xi}),\\ 1 & \text{for } M_{\rm Pl}/(r\sqrt{\xi})\lesssim v,\end{cases} \tag{20}$$

for $N=2$. As for $0\leqslant r\lesssim 1/\sqrt{\xi}$, the cutoff is basically $\sim 1/\sqrt{\xi}$ except for $v\gtrsim M_{\rm Pl}/\sqrt{\xi}$ in the $N=2$ case where $\Lambda_{\rm EC}/\Lambda_G\sim v/\sqrt{v^2+12M_{\rm Pl}^2}$. These results are all frame independent.

General nonlinear σ model.—We can easily apply our approach to the general NLSM with gravity. Consider, e.g., multiple scalars with curved target space and nonminimal coupling in metric formalism [47]

$$S=\int d^4x\sqrt{-g_J}\left(\frac{M_{\rm Pl}^2}{2}fR_J-\frac{g_J^{\mu\nu}}{2}\mathcal{G}_{ij}\partial_\mu\phi^i\partial_\nu\phi^j-V\right), \tag{21}$$

where f is a positive-definite scalar function with $f=1$ for $\phi^i=0$, and $\mathcal{G}_{ij}$ is a general nondegenerate target-space metric both of which depend on ϕ^i. We further assume the theory respects $O(N)$ symmetry ($N\geqslant 2$) as a simple example for NLSM. This symmetry allows us to rewrite the metric as $\mathcal{G}_{ij}d\phi^i d\phi^j=G(h^2)dh^2+(h^2/v^2)[d\vec{\pi}^2+(\vec{\pi}\cdot d\vec{\pi})^2/(v^2-\vec{\pi}^2)]$ with $G(0)=1$ in the spherical coordinate of $\phi^i=(h,\vec{\pi})$, and restricts the form of the nonminimal coupling to be $f=f(h^2)$ [48]. Here, h is the radial mode, and $\vec{\pi}$ are the coordinates on S^{N-1}. v is a parameter to give a mass dimension one to $\vec{\pi}$. For $h=v$ and $\vec{\pi}=\vec{0}$, the Riemann tensor of $\mathcal{G}_{ij}$ denoted as $\mathcal{R}_{ikjl}$ is readily obtained as $\bar{\mathcal{R}}_{\pi^i\pi^k}=(\bar{G}-1)/(v^2\bar{G})$ and $\bar{\mathcal{R}}_{h\pi^k}=\overline{G'}/\bar{G}$ with $\bar{G}=G(v^2)$ and $\overline{G'}=G'(v^2)$. We again parametrize the Riemann tensor as $\bar{\mathcal{R}}_{ikjl}=\bar{\mathcal{R}}_{ik}(\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk})$. One may confirm the restoration of $O(N)$ symmetry in the limit of $v\to 0$ as $\bar{\mathcal{R}}_{\pi^i\pi^k}=\bar{\mathcal{R}}_{h\pi^k}=\overline{G'_0}$ with $\overline{G'_0}=G'(0)$.

The extended target-space metric for $\varphi^a_J=(\Phi_J,h,\vec{\pi})$ is given as

$$
(G_{ab}^{\mathrm{NL}}) \equiv \left(\begin{array}{cc|c} -f & -f'\Phi_{\mathrm{J}}h & \\ -f'\Phi_{\mathrm{J}}h & \frac{\Phi_{\mathrm{J}}^2}{6M_{\mathrm{Pl}}^2}\mathcal{G}_{hh} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \frac{\Phi_{\mathrm{J}}^2}{6M_{\mathrm{Pl}}^2}(\mathcal{G}_{\pi^i\pi^j}) \end{array}\right). \tag{22}
$$

The relevant components of the target-space metric and Riemann tensor are shown in Table I, which are invariant under frame transformation. We can calculate the frame-independent cutoff scale in the same way as previous sections, although the results are now more involved. As an illustration, we compare our results to those without gravity in the limit $v \to 0$, where the Riemann tensor respects the $\mathrm{O}(N)$ symmetry $\bar{R}_{\pi^i\pi^k} = \bar{R}_{h\pi^k}$ as expected. The frame-independent cutoff scale is

$$
\frac{\Lambda_{\mathrm{NLSM}}}{\Lambda_{\mathrm{G}}} \sim \sqrt{\frac{N^2}{\bar{R}\Lambda_{\mathrm{G}}^2}} \sim \sqrt{\frac{N}{N-1}}\left(M_{\mathrm{Pl}}^2\overline{G_0'} + \frac{(1+6M_{\mathrm{Pl}}^2\overline{f_0'})^2}{6}\right)^{-1/2}. \tag{23}
$$

The first term corresponds to the result without gravity, and the second term is the correction from the nonminimal coupling to gravity. The latter vanishes for $M_{\mathrm{Pl}}^2\overline{f_0'} = -1/6$, i.e., the conformal coupling, as expected.

As a final remark, we consider a general frame transformation

$$
\Phi_{\mathrm{F}}^2 = \Omega_{\mathrm{F}}^2\Phi_{\mathrm{J}}^2, \qquad \Omega_{\mathrm{F}}^2 = \frac{f}{f_{\mathrm{F}}}, \tag{24}
$$

where f_{F} being an arbitrary function of h^2. The extended-target space metric for $(\varphi_{\mathrm{F}}^a) = (\Phi_{\mathrm{F}}, h, \vec{\pi})$ becomes

$$
(\hat{G}_{ab}^{\mathrm{NL}}) = \left(\begin{array}{cc|c} -f_{\mathrm{F}} & -f_{\mathrm{F}}'\Phi_{\mathrm{F}}h & \\ -f_{\mathrm{F}}'\Phi_{\mathrm{F}}h & \frac{\Phi_{\mathrm{F}}^2\left[\frac{\mathcal{G}_{hh}}{6M_{\mathrm{Pl}}^2} + h^2 f\left(\frac{f'^2}{f^2} - \frac{f_{\mathrm{F}'}^2}{f_{\mathrm{F}}^2}\right)\right]}{\Omega_{\mathrm{F}}^2} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \frac{\Phi_{\mathrm{F}}^2(\mathcal{G}_{\pi^i\pi^j})}{6M_{\mathrm{Pl}}^2\Omega_{\mathrm{F}}^2} \end{array}\right). \tag{25}
$$

Interestingly, we can obtain an apparently flat target space for ϕ^i by choosing f_{F} such that $\hat{G}_{hh}^{\mathrm{NL}} = (6M_{\mathrm{Pl}}^2\Omega_{\mathrm{F}}^2)^{-1}\Phi_{\mathrm{F}}^2$ for given f and $\mathcal{G}_{hh}$, and redefining $\mathrm{d}\tilde{\phi}^i \equiv \sqrt{f_{\mathrm{F}}/f}\mathrm{d}\phi^i$. In other words, for a given $\mathrm{O}(N)$ NLSM, there always exists a certain frame where the target space of ϕ^i is completely flat (this particular frame is the "Jordan frame" for the metric and Einstein-Cartan HI) [48].

This observation emphasizes the significance of the inclusion of a conformal mode in the discussion of NLSM with gravity. As we have shown, all the different target-space geometries of ϕ^i in $\mathrm{O}(N)$ NLSM are connected by the frame transformation. Hence, the target space spanned only by ϕ^i is clearly unphysical once we introduce the coupling to gravity. To tell the difference, we have to consider the geometry of the extended target space including the conformal mode, which is manifestly invariant under the frame transformation or field redefinition [48]. As we have seen, we only need one straightforward calculation of the curvature of the extended-target space.

Conclusions.—We propose a geometric method for calculation of physical quantities, e.g., the unitarity-violation scale, of theories where scalar fields nonconformally coupled with gravity, from which the results are manifestly Weyl-transformation or field-redefinition independent as they should be. The cutoff scale of multiple-scalar theories is characterized by the geometry of the target space of scalar fields, which is invariant manifestly under the redundancy description of field redefinition. However, coupling to gravity introduces a new redundancy, i.e., Weyl transformation or frame choice, which spoils the advantages of the geometric method. We show that including the conformal mode of the spacetime metric to extend the notion of target space can help regain the merits of geometric method, because the Weyl transformation now becomes simply a field redefinition (of the conformal mode) in the extended geometry which, by definition, does not change the geometric or physical quantities. We show several examples of frame-independent unitarity-violation scales, such as HI in metric and Einstein-Cartan formalisms (Table I). These results are consistent with those calculated in either the Jordan or Einstein frame in the literature. We also discuss general NLSM where one can freely choose a frame in which the original target space is flat or curved, but the extended geometry and physical quantities are invariant under field redefinition of both conformal mode and scalar fields.

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[1] K. Meetz, Realization of chiral symmetry in a curved isospin space, J. Math. Phys. (N.Y.) **10**, 589 (1969).
[2] J. Honerkamp, Chiral multiloops, Nucl. Phys. **B36**, 130 (1972).
[3] J. Honerkamp and K. Meetz, Chiral-invariant perturbation theory, Phys. Rev. D **3**, 1996 (1971).

[4] G. Ecker and J. Honerkamp, Application of invariant renormalization to the nonlinear chiral invariant pion Lagrangian in the one-loop approximation, Nucl. Phys. **B35**, 481 (1971).

[5] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model, Ann. Phys. (N.Y.) **134**, 85 (1981).

[6] L. Alvarez-Gaume and D. Z. Freedman, Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model, Commun. Math. Phys. **80**, 443 (1981).

[7] D. G. Boulware and L. S. Brown, Symmetric space scalar field theory, Ann. Phys. (N.Y.) **138**, 392 (1982).

[8] P. S. Howe, G. Papadopoulos, and K. S. Stelle, The background field method and the nonlinear σ model, Nucl. Phys. **B296**, 26 (1988).

[9] S. R. Coleman, J. Wess, and B. Zumino, Structure of phenomenological Lagrangians. 1., Phys. Rev. **177**, 2239 (1969).

[10] C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, Structure of phenomenological Lagrangians. 2., Phys. Rev. **177**, 2247 (1969).

[11] L. J. Dixon, V. Kaplunovsky, and J. Louis, On effective field theories describing (2,2) vacua of the heterotic string, Nucl. Phys. **B329**, 27 (1990).

[12] R. Alonso, E. E. Jenkins, and A. V. Manohar, A geometric formulation of Higgs effective field theory: Measuring the curvature of scalar field space, Phys. Lett. B **754**, 335 (2016).

[13] R. Alonso, E. E. Jenkins, and A. V. Manohar, Geometry of the scalar sector, J. High Energy Phys. 08 (2016) 101.

[14] R. Nagai, M. Tanabashi, K. Tsumura, and Y. Uchida, Symmetry and geometry in a generalized Higgs effective field theory: Finiteness of oblique corrections versus perturbative unitarity, Phys. Rev. D **100**, 075020 (2019).

[15] T. Cohen, N. Craig, X. Lu, and D. Sutherland, Unitarity violation and the geometry of Higgs EFTs, J. High Energy Phys. 12 (2021) 003.

[16] C. Cheung, A. Helset, and J. Parra-Martinez, Geometric soft theorems, J. High Energy Phys. 04 (2022) 011.

[17] A. Helset, E. E. Jenkins, and A. V. Manohar, Geometry in scattering amplitudes, Phys. Rev. D **106**, 116018 (2022).

[18] R. Alonso, A primer on Higgs effective field theory with geometry, arXiv:2307.14301.

[19] J. L. Cervantes-Cota and H. Dehnen, Induced gravity inflation in the standard model of particle physics, Nucl. Phys. **B442**, 391 (1995).

[20] F. L. Bezrukov and M. Shaposhnikov, The Standard Model Higgs boson as the inflaton, Phys. Lett. B **659**, 703 (2008).

[21] A. O. Barvinsky, A. Y. Kamenshchik, and A. A. Starobinsky, Inflation scenario via the standard model Higgs boson and LHC, J. Cosmol. Astropart. Phys. 11 (2008) 021.

[22] Planck Collaboration, Planck 2018 results. X. Constraints on inflation, Astron. Astrophys. **641**, A10 (2020).

[23] K.-i. Maeda, Towards the Einstein-Hilbert action via conformal transformation, Phys. Rev. D **39**, 3159 (1989).

[24] Y. Ema, R. Jinno, K. Mukaida, and K. Nakayama, Violent preheating in inflation with nonminimal coupling, J. Cosmol. Astropart. Phys. 02 (2017) 045.

[25] M. P. Hertzberg, On inflation with non-minimal coupling, J. High Energy Phys. 11 (2010) 023.

[26] F. Bezrukov, A. Magnin, M. Shaposhnikov, and S. Sibiryakov, Higgs inflation: Consistency and generalisations, J. High Energy Phys. 01 (2011) 016.

[27] F. Bezrukov, D. Gorbunov, and M. Shaposhnikov, Late and early time phenomenology of Higgs-dependent cutoff, J. Cosmol. Astropart. Phys. 10 (2011) 001.

[28] J. Ren, Z.-Z. Xianyu, and H.-J. He, Higgs gravitational interaction, weak boson scattering, and Higgs inflation in Jordan and Einstein frames, J. Cosmol. Astropart. Phys. 06 (2014) 032.

[29] Y. Ema, Dynamical emergence of scalaron in Higgs inflation, J. Cosmol. Astropart. Phys. 09 (2019) 027.

[30] C. T. Hill and G. G. Ross, Gravitational contact interactions and the physical equivalence of Weyl transformations in effective field theory, Phys. Rev. D **102**, 125014 (2020).

[31] A. Ito, W. Khater, and S. Rasanen, Tree-level unitarity in Higgs inflation in the metric and the Palatini formulation, J. High Energy Phys. 06 (2022) 164.

[32] G. K. Karananas, M. Shaposhnikov, and S. Zell, Field redefinitions, perturbative unitarity and Higgs inflation, J. High Energy Phys. 06 (2022) 132.

[33] C. P. Burgess, H. M. Lee, and M. Trott, Power-counting and the validity of the classical approximation during inflation, J. High Energy Phys. 09 (2009) 103.

[34] J. L. F. Barbon and J. R. Espinosa, On the naturalness of Higgs inflation, Phys. Rev. D **79**, 081302(R) (2009).

[35] Y. Ema, K. Mukaida, and J. van de Vis, Higgs inflation as nonlinear sigma model and scalaron as its σ-meson, J. High Energy Phys. 11 (2020) 011.

[36] Y. Ema, K. Mukaida, and J. van de Vis, Renormalization group equations of Higgs-R^2 inflation, J. High Energy Phys. 02 (2021) 109.

[37] C. P. Burgess, H. M. Lee, and M. Trott, Comment on Higgs inflation and naturalness, J. High Energy Phys. 07 (2010) 007.

[38] D. Gorbunov and A. Tokareva, Scalaron the healer: Removing the strong-coupling in the Higgs- and Higgs-dilaton inflations, Phys. Lett. B **788**, 37 (2019).

[39] Here, we focus on the discussion of frame independence so only the conformal mode is sufficient. One might need to include the full spacetime metric as in Ref. [40] if more complicated transformation is involved such as the disformal transformation that mixes the graviton components and scalar fields.

[40] K. Finn, S. Karamitsos, and A. Pilaftsis, Frame covariance in quantum gravity, Phys. Rev. D **102**, 045014 (2020).

[41] Y. Mikura and Y. Tada, On UV-completion of Palatini-Higgs inflation, J. Cosmol. Astropart. Phys. 05 (2022) 035.

[42] F. Bauer and D. A. Demir, Higgs-Palatini inflation and unitarity, Phys. Lett. B **698**, 425 (2011).

[43] S. Rasanen, Higgs inflation in the Palatini formulation with kinetic terms for the metric, Open J. Astrophys. **2**, 1 (2019).

[44] H. T. Nieh and M. L. Yan, An identity in Riemann-Cartan geometry, J. Math. Phys. (N.Y.) **23**, 373 (1982).

[45] M. Shaposhnikov, A. Shkerin, I. Timiryasov, and S. Zell, Higgs inflation in Einstein-Cartan gravity, J. Cosmol. Astropart. Phys. 02 (2021) 008.

[46] M. He, K. Kamada, and K. Mukaida, Quantum corrections to Higgs inflation in Einstein-Cartan gravity, J. High Energy Phys. 01 (2024) 014.

[47] We do not consider an arbitrary function $F(\phi^i, R)$ in the gravity part simply to avoid the complication from a new scalar degree of freedom which generically appears in $f(R)$ theories in metric formalism and is beyond the purpose of this Letter.

[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.191501 for some explicit calculations in a general $\mathrm{O}(N)$ nonlinear σ model to demonstrate the statements on its properties in the main text.