Editors' Suggestion

# Bell Nonlocality in Classical Systems Coexisting with Other System Types

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The realistic interpretation of classical theory assumes that every classical system has well-defined properties, which may be unknown to the observer but are nevertheless part of reality and can, in principle, be revealed by measurements. Here we show that this interpretation can, in principle, be falsified if classical systems coexist with other types of physical systems. To make this point, we construct a toy theory that (i) includes classical theory as a subtheory and (ii) allows classical systems to be entangled with another type of system, called anticlassical. We show that our toy theory allows for the violation of Bell inequalities in two-party scenarios where one of the settings corresponds to a local measurement performed on a classical system alone. Building on this fact, we show that measurement outcomes in classical theory cannot, in general, be regarded as predetermined by the state of an underlying reality.

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*Introduction.*—Since the early days of Galileo and Newton, classical theory has been regarded as the golden standard of a physical theory that describes reality without any fundamental uncertainty. In this view, every classical system is assumed to be in a well-defined state, which may be unknown to the observer, but is nevertheless part of the physical reality. Statistical mixtures only arise from the observer's ignorance about the true state of the system, and in principle, this ignorance can always be overcome by performing measurements. In modern terminology, the view that classical systems are fundamentally in well-defined (pure) states can be summarized by the statement that classical pure states are "ontic," while classical mixed states are "epistemic" [1–3]. This statement, combined with the idea that classical measurements reveal some preexisting properties of the measured systems, lies at the core of the realistic interpretation of classical theory.

In this Letter we show that, contrary to widespread belief, a realistic interpretation of classical theory is not always logically possible: while such interpretation is consistent with all experiments involving only classical systems, it can become, in principle, falsifiable if classical systems are considered alongside other types of physical systems. To make this point, we construct a toy theory that includes classical theory as a subtheory, meaning that it coincides with classical theory when restricted to a subset of the possible physical systems. In addition to all classical systems, the toy theory includes another type of systems, called anticlassical, as illustrated in Fig. 1. An observer who has access only to classical systems cannot see any difference between classical theory and our toy theory: all measurements are, in principle, compatible, all pure states are perfectly distinguishable through measurements, and all

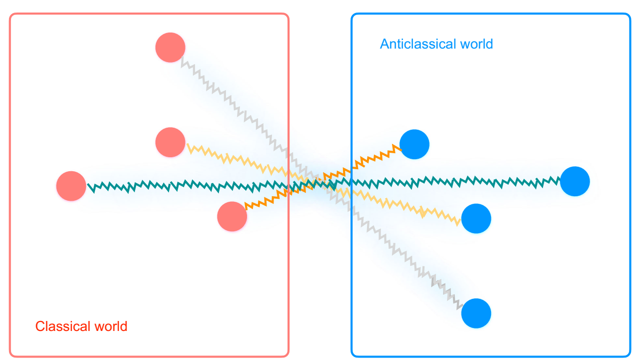


FIG. 1. In a universe described by our toy theory, an observer who has access only to classical systems (represented by red disks on the left) would see a world described by classical theory. The same situation applies to an observer with access only to anticlassical systems (blue disks on the right). In contrast, observers with access to both types of systems can observe Bell nonlocality and other nonclassical features.

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the states of all composite systems are separable. In contrast, we show that observers with joint access to both types of systems can, in principle, observe nonclassical features such as Bell nonlocality [4].

Crucially, we show that our toy theory allows for a maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [5,6] in scenarios where one of the settings corresponds to a local measurement performed on a classical system. Building on this result, we prove that measurement outcomes in classical theory cannot, in general, be regarded as predetermined. Finally, we show that, under mild assumptions, the predictions of our toy theory cannot be reproduced by any deeper theory that describes reality as a list of individual properties of classical and anticlassical systems. This result indicates that, no matter whether the properties of classical systems are accessible through measurements or not, their full specification is not sufficient, in general, to account for the correlations between classical systems and other types of physical systems.

While our toy theory is not meant to be a description of the world, it makes an important conceptual point: the realistic interpretation of classical theory can, in principle, be falsified if classical systems exist alongside other types of physical systems. Notably, our toy theory cannot be ruled out from within classical theory: every classical phenomenon is, in principle, compatible with the existence of some yet-unobserved type of system that prevents the assignment of definite values to classical variables prior to measurement.

Our results complement recent works by Gisin and Del Santo [7,8], who challenged the determinism of classical physics on the grounds of the impossibility to specify real-valued variables like position and momentum with infinite precision. In our work, the impossibility to assign a predefined value to classical variables arises from correlations with some other physical systems, rather than precision limits in the definition of real numbers. As such, our results apply also to classical bits and other discrete classical variables. It is also worth mentioning that physical arguments in favor of classical indeterminism could also be put forward by setting up a dynamical interaction between classical and quantum systems (see, e.g., [9–11].) In the existing frameworks, however, classical and quantum systems cannot be entangled, and therefore there cannot be any CHSH violation when one of the settings corresponds to a measurement on a classical system alone. In this respect, our toy theory exhibits a stronger form of indeterminism.

*Classical and anticlassical systems.*—To formulate our toy theory, we adopt the framework of general probabilistic theories [12–17], in the specific version known as operational probabilistic theories (OPTs) [19–24]. An OPT describes a set of physical systems, closed under composition, and a set of transformations thereof, closed under parallel and sequential composition. Mathematically, the compositional structure is underpinned by the graphical language of process theories [25–28].

Classical theory can be regarded as a special case of an OPT [15,29]: precisely, it is the largest OPT where (i) the pure states of every given system are perfectly distinguishable through a single measurement, (ii) the pure states of every composite system are the products of pure states of the component systems, and (iii) all permutations of the set of pure states are valid physical transformations. For simplicity, we will focus on the classical theory of discrete systems such as bits and their generalizations.

We now construct a toy theory that includes classical theory as a subtheory, meaning that our toy theory coincides with classical theory when restricted to a subset of physical systems that includes all discrete classical systems. A classical system with $d$ perfectly distinguishable pure states, conventionally denoted by $0, 1, \ldots, d-1$, will be called a dit (or a bit in the special case $d = 2$.) The mixed states of a dit are probability distributions of the form $(p_i)_{i=0}^{d-1}$, with $p_i \geq 0$, $\forall\ i$ and $\sum_{i=0}^{d-1} p_i = 1$. The reversible processes acting on the dit are permutations of its pure states, while general noisy processes are described by transition probabilities $p(j|i)$. Similarly, a (generally noisy) measurement with outcomes in a set $\mathsf{A}$ can be represented by transition probabilities $p(a|i)$, yielding the probability of the outcome $a$ when the dit is in the state $i$.

An equivalent way to represent classical states, processes, and measurements, commonly used in the quantum information literature (see, e.g., [30]), is provided by diagonal matrices. Specifically, probability distributions $(p_i)_{i=0}^{d-1}$ can be equivalently represented by $d \times d$ diagonal matrices of the form $\rho = \sum_{i=0}^{d-1} p_i |i\rangle\langle i|$, where $\{|i\rangle\}_{i=0}^{d-1}$ is the canonical orthonormal basis for $\mathbb{C}^d$. A general process with transition probabilities $p(j|i)$ is described by a linear map of the form $\mathcal{M}(\rho) = \sum_{i,j} p(j|i) |j\rangle\langle j| \langle i|\rho|i\rangle$. Finally, a measurement with outcomes in the set $\mathsf{A}$ is described by a positive operator-valued measure (POVM) $(P_a)_{a\in\mathsf{A}}$ of the form $P_a = \sum_{i=0}^{d-1} p(a|i) |i\rangle\langle i|$, and the outcome probabilities can be computed with the Born rule $p(a|i) = \langle i|P_a|i\rangle$.

In our toy theory, classical systems coexist with another type of systems, called anticlassical. The anticlassical systems can be viewed as a mirror image of the classical systems: for every classical system type, there exists a corresponding anticlassical system type with exactly the same state space, the same set of physical transformations, and the same set of measurements. To help intuition, one can think of the distinction between classical and anticlassical systems as analogous to the distinction between particles and antiparticles, which have the same state spaces and yet are distinguishable by some external property, such as their charge.

While classical and anticlassical systems are described by classical probability theory when considered separately, composite systems including both types of systems exhibit nonclassical features. In the following, we present the simplest version of our toy theory, which describes arbitrary composite systems made of $m$ bits and $n$ antibits, hereafter called $(m,n)$ composites. $(m,0)$ and $(0,n)$ composites will be described by classical theory, while the nonclassical behaviors will emerge when both $m$ and $n$ are nonzero. The generalization to basic systems of arbitrary dimension, as well as the full specification of the allowed states, measurements, and processes, is provided in Supplemental Material [31].

*Nonclassical composites.*—The simplest nonclassical composite is the (1,1) composite, consisting of a bit and an antibit. In this case, the pure states are represented by rank-one projectors onto unit vectors $|\Psi\rangle$ with well-defined parity, that is, unit vectors satisfying either the condition $\Pi_0|\Psi\rangle = |\Psi\rangle$ or the condition $\Pi_1|\Psi\rangle = |\Psi\rangle$, where $\Pi_0$ ($\Pi_1$) is the projector on the subspace spanned by the vectors $\{|0\rangle|0\rangle, |1\rangle|1\rangle\}$ ($\{|0\rangle|1\rangle, |1\rangle|0\rangle\}$). The mixed states of a bit and an antibit are described by density matrices of the form $\rho = \sum_j q_j |\Psi_j\rangle\langle\Psi_j|$, where $(|\Psi_j\rangle)_j$ are pure states and $(q_j)_j$ is a probability distribution. For an $(m,m)$ composite, the most general pure state is a unit vector of the form $|\Psi\rangle = [I_{(B_1\ldots B_m)} \otimes U^{(A_1\ldots A_m)}]|\Psi'\rangle$, where $U^{(A_1\ldots A_m)}$ is a unitary operator that permutes $m$ bits (antibits) and $|\Psi'\rangle$ is a unit vector satisfying the condition

$$|\Psi'\rangle = \left(\Pi_{k_1}^{B_1A_1} \otimes \cdots \otimes \Pi_{k_m}^{B_mA_m}\right)|\Psi'\rangle, \tag{1}$$

for given vector $(k_1, \ldots, k_m) \in \{0,1\}^{\times m}$, where we used the notation $\Pi_{k_i}^{B_iA_i}$ for the projector onto the subspace of the composite system of the $i$th bit and $i$th antibit with fixed parity $k_i \in \{0,1\}$.

The pure states of arbitrary $(m,n)$ composites are defined in Supplemental Material [31]. General mixed states are defined as density matrices that are convex combinations of rank-one density matrices associated with the above pure states. Measurements on system $S$ are defined as POVMs $\{P_i\}_{i=1}^k$ whose operators are linear combinations, with positive coefficients, of the allowed states and satisfy the normalization property $\sum_{i=1}^k P_i = I_S$. The outcome probabilities are then given by the Born rule $p_i = \mathrm{Tr}[P_i\rho]$. With these definitions, states and measurements satisfy a fundamental consistency condition: when a subset of the systems is measured, the conditional state of the remaining systems is still a valid state allowed by our toy theory. We call this condition consistency of the conditional states and prove it in Supplemental Material [31], where we also show that similar consistency properties hold for all processes in our toy theory. In particular, all the multipartite states, processes, and measurements allowed by our toy theory coincide with the states, processes, and measurements of classical theory once all the anticlassical systems are eliminated.

It is worth noting that, unlike classical theory and standard quantum theory on the complex field, our toy theory does not satisfy local tomography [12,13,44,48–51], the property that the states of composite systems are completely characterized by the correlations of local measurements. While this property holds separately for all classical systems and for all anticlassical systems, it fails to hold when classical and anticlassical systems are combined together.

The violation of local tomography is not an accident, but rather a necessary condition for obtaining nonclassical composites out of systems with classical state spaces [52] (see also the no-go theorem in [53] where local tomography is implicit in the choice of possible tensor products). Nevertheless, we show that our toy theory satisfies a weaker locality property, known as bilocal tomography [40], for all multipartite systems consisting of bits and antibits: any arbitrary state of $m$ bits and $n$ antibits can be fully characterized by the correlations of measurements performed on pairs of bits and antibits. A proof of this fact is provided in Supplemental Material [31]. Other examples of physical theories that violate local tomography but satisfy bilocal tomography are quantum theory on real vector spaces [48,49], fermionic quantum theory [32,33], and doubled quantum theory [24,34].

*Classical mixtures from entanglement.*—It is immediate to see that every mixed state of a classical bit can be obtained from a pure state of the composite system by discarding the antibit. For example, the generic mixed state $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ can be obtained from the pure entangled state $|\Psi\rangle = \sqrt{p}|0\rangle|0\rangle + \sqrt{1-p}|1\rangle|1\rangle$. In other words, every mixed state of a classical bit admits a *purification* [21,44].

In the rest of the Letter, we discuss the implications of purification for the interpretation of classical physics. Let us first assume, for the sake of argument, that our toy theory describes nature at the fundamental (i.e., ontic) level. In this setting, the claim that every classical system must be in a pure state at the ontic level would imply that the joint states of a bit-antibit pair are always of the separable form $\Sigma = q|0\rangle\langle 0| \otimes \rho_0 + (1-q)|1\rangle\langle 1| \otimes \rho_1$, for some probability $q \in [0,1]$ and some states $\rho_0$ and $\rho_1$ of the antibit (see Supplemental Material [31]). However, this condition is manifestly in contradiction with the existence of pure entangled states. Operationally, any pure entangled state of a bit-antibit pair can be distinguished from all separable states by performing a measurement allowed by the toy theory. For example, the pure entangled state $|\Psi\rangle = \sqrt{p}|0\rangle|0\rangle + \sqrt{1-p}|1\rangle|1\rangle$, $p \in (0,1)$, can be distinguished with a guaranteed success probability of at least $\min\{p, 1-p\}$ from all separable states (see Supplemental Material [31]). Hence, we conclude that, in a world where our toy theory is fundamental, the belief that classical systems must always be in some (possibly unknown) pure states can be experimentally falsified.

*Bell nonlocality.*—We now use our toy theory to challenge the common belief that the outcomes of classical measurements reveal the values of some preexisting properties of the measured systems. The starting point of our argument is the observation that our toy theory exhibits activation of Bell nonlocality [54–58]. Suppose that a bit $B$ and an antibit $A$ are in the entangled state $|\Phi\rangle_{BA} = (|0\rangle_B|0\rangle_A + |1\rangle_B|1\rangle_A)/\sqrt{2}$. This state alone does not give rise to any Bell inequality violation: since the local measurements on a bit and antibit are classical, one can easily construct a local hidden variable model. However, Bell nonlocality arises when we consider the two-copy state $|\Phi\rangle_{B_1A_1} \otimes |\Phi\rangle_{B_2A_2}$, where $B_1B_2$ are bits, and $A_1A_2$ are antibits. Suppose that two parties, Alice and Bob, play a nonlocal game, such as the CHSH game [5,6], in the scenario where Alice has access to system $B_1A_2$, while Bob has access to system $B_2A_1$, as illustrated in Fig. 2.

We now show that the state $|\Phi\rangle_{B_1A_1} \otimes |\Phi\rangle_{B_2A_2}$ allows Alice and Bob to reproduce the correlations of arbitrary single-qubit measurements performed locally on a 2-qubit maximally entangled state. More specifically, we show that a qubit measurement that projects Alice's qubit on a given orthonormal basis $\{|v_0\rangle, |v_1\rangle\}$ with $|v_0\rangle = \alpha|0\rangle + \beta|1\rangle$ can be simulated by a measurement on the bit-antibit pair $B_1A_2$, described by two orthogonal projectors $\{P_0, P_1\}$ with

$$\begin{aligned} P_0 &= |V_0^{(0)}\rangle\langle V_0^{(0)}|_{B_1A_2} + |V_0^{(1)}\rangle\langle V_0^{(1)}|_{B_1A_2} \\ |V_0^{(0)}\rangle_{B_1A_1} &= \alpha|0\rangle_{B_1}|0\rangle_{A_2} + \beta|1\rangle_{B_1}|1\rangle_{A_2} \\ |V_0^{(1)}\rangle_{B_1A_2} &= \alpha|0\rangle_{B_1}|1\rangle_{A_2} + \beta|1\rangle_{B_1}|0\rangle_{A_2}, \end{aligned} \tag{2}$$

and $P_1 = I_{B_1} \otimes I_{A_2} - P_0$, acting on the bit-antibit pair $B_1A_2$. Similarly, a measurement that projects Bob's qubit on the orthonormal basis $\{|w_0\rangle, |w_1\rangle\}$ with $|w_0\rangle = \gamma|0\rangle + \delta|1\rangle$ can be simulated by the projective measurement $\{Q_0, Q_1\}$ defined by

$$\begin{aligned} Q_0 &= |W_0^{(0)}\rangle\langle W_0^{(0)}|_{B_2A_1} + |W_0^{(1)}\rangle\langle W_0^{(1)}|_{B_2A_1} \\ |W_0^{(0)}\rangle &= \gamma|0\rangle_{B_2}|0\rangle_{A_1} + \delta|1\rangle_{B_2}|1\rangle_{A_1} \\ |W_0^{(1)}\rangle &= \gamma|1\rangle_{B_2}|0\rangle_{A_1} + \delta|0\rangle_{B_2}|1\rangle_{A_1}, \end{aligned} \tag{3}$$

and $Q_1 = I_{B_2} \otimes I_{A_1} - Q_0$. When these measurements are performed on the state $\rho = |\Phi\rangle\langle\Phi|_{B_1A_1} \otimes |\Phi\rangle\langle\Phi|_{B_2A_2}$, Alice and Bob obtain outcomes $a$ and $b$ with probability

$$p(a,b) = \mathrm{Tr}[(P_a \otimes Q_b)\rho] \equiv |\langle v_a|\langle w_b||\Phi\rangle|^2, \tag{4}$$

equal to the outcome probability of the original single-qubit measurements performed on the 2-qubit maximally entangled state $|\Phi\rangle$ (see Supplemental Material [31] for more details). In this way, every pair of local measurements on a maximally entangled 2-qubit quantum state can be simulated by local measurements in our toy theory. In particular, Alice and Bob can simulate the optimal strategy in the CHSH game [5,6,59,60], thereby achieving a maximal violation of the CHSH inequality.

Let us now examine the implications of the above result for the interpretation of classical theory. A first, important consequence is that the value of Alice's classical bit cannot, in general, be regarded as predetermined. This conclusion follows from the fact that the violation of the CHSH inequality can be achieved with setup in which one of Alice's measurements is the canonical measurement on bit $B_1$. Technically, this follows from the fact that one of Alice's measurements in the original quantum scenario is a qubit measurement on the computational basis $\{|0\rangle, |1\rangle\}$. In our simulation, this measurement corresponds to the projectors $P_0 = |0\rangle\langle 0|_{B_1} \otimes I_{A_2}$ and $P_1 = I \otimes I - P_0 = |1\rangle\langle 1|_{B_1} \otimes I_{A_2}$, as one can see from Eq. (2). Operationally, this measurement is realized by discarding the antibit $A_2$ and measuring bit $B_1$ on the basis $\{|0\rangle, |1\rangle\}$. Since Alice's bit value is a measurement outcome in a setup that violates the CHSH inequality, we conclude that the bit value cannot be predetermined [41]: explicitly, in Supplemental Material we show that, if the underlying ontic state determines the value of Alice's bit up to an error $\epsilon$, then the CHSH value cannot exceed $2(1 + 2\epsilon)$ and therefore cannot reach the maximum value $2\sqrt{2}$ when $\epsilon$ is small. In Supplemental Material, we also show that the above argument applies to all pure entangled states of a dit and an antidit [31].

Another implication of Bell nonlocality is that, even if we replace our toy theory with a more fundamental description of nature, this description cannot, under reasonable assumptions, assign individual ontic states to classical systems. Two different arguments leading to this conclusion are provided in Supplemental Material [31]. In both cases, the conclusion is that classical systems in our toy theory cannot be reduced to independent and uncorrelated degrees of freedom of the underlying reality.

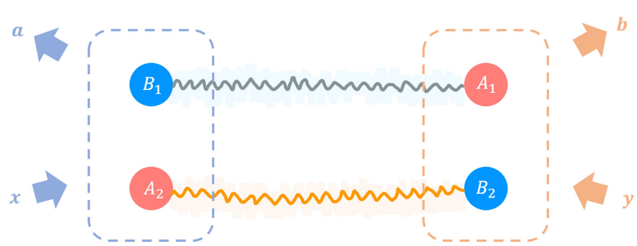


FIG. 2. Activation of Bell nonlocality with bit-antibit entangled pairs. Alice (left) and Bob (right) perform local measurements on two copies of an entangled state of a bit-antibit pair. The first copy (top) involves bit $B_1$ and antibit $A_1$, while the second copy (bottom) involves bit $B_2$ and antibit $A_2$. Alice's and Bob's laboratories (represented by dotted boxes) contain systems $B_1A_2$ and $A_2B_1$, respectively. Their measurements have settings $x$ and $y$, respectively, and produce outcomes $a$ and $b$, respectively.

*Conclusions.*—In this Letter, we have shown that the realistic interpretation of classical theory can, in principle, be falsified when classical systems coexist with other types of physical systems. We built a toy theory in which every classical system can be entangled with a dual, anticlassical system. The entanglement between classical and anticlassical systems gives rise to activation of Bell nonlocality and implies that, in general, the outcomes of measurements on classical systems cannot be interpreted as revealing the values of some preexisting properties of the measured systems.

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[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.190201 for the generalization to systems of arbitrary dimension, which includes Refs. [18,23,32–39], for the proof of bilocal tomography, which includes Ref. [40], for the proof of impossibility of assigning individual pure states to classical systems, for the simulation of local measurement on a 2-qubit maximally entangled state, for the proof of no violation of two-party Bell inequalities when all but one setting of one party have predetermined outcomes, which

includes Ref. [41], for the proof of activation of Bell nonlocality for arbitrary pure states, which includes Refs. [42,43], for the proof that the predictions of our toy theory cannot be reproduced by any hidden variable theory, which includes Refs. [1–3,19–21,44–47], and for the proof of consistency of the conditional state.
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