Nature of the Volcano Transition in the Fully Disordered Kuramoto Model

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Randomly coupled phase oscillators may synchronize into disordered patterns of collective motion. We analyze this transition in a large, fully connected Kuramoto model with symmetric but otherwise independent random interactions. Using the dynamical cavity method, we reduce the dynamics to a stochastic single-oscillator problem with self-consistent correlation and response functions that we study analytically and numerically. We clarify the nature of the volcano transition and elucidate its relation to the existence of an oscillator glass phase.

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Synchronization is ubiquitous in nature and science [1]. Periodic degrees of freedom with different individual frequencies tend to lock into phase coherent motion when weakly coupled to one another. Physics, chemistry, biology, engineering, and even sociology abound in examples [1–3]. Quantitative studies flourished after 1984 when Kuramoto introduced a mean-field model of phase oscillators with random frequencies of spread σ globally coupled with strength J [4]. For small ratios J/σ the time evolution of the oscillators remains incoherent, whereas for large J/σ they continue to change in unison. More precisely, the incoherent state remains stable up to a threshold value of J/σ [5] followed by a sharp transition into an ordered state with gradually increasing degree of synchronization [4]. Over the years, the original Kuramoto model with identical couplings between all oscillators has been investigated thoroughly, and a wealth of information on it is by now available [3,6,7].

Much less is known about the inhomogeneous situation in which the oscillators are connected by couplings J_{ij} of different strength and sign [7]. The asymmetric case with no relation between J_{ij} and J_{ji} is dominated by chaotic dynamics with little tendency to synchronization [8]. For symmetric couplings, $J_{ij} = J_{ji}$, synchronization prevails and, due to frustration, new phenomena emerge. Some of these have been discussed in [9] and in a series of recent papers by Strogatz and collaborators investigating different types of symmetric but disordered coupling matrices J_{ij} [10–13]. Partial synchronization, mixed phases, antiphase synchronization, traveling-wave states, and other patterns were found.

The system of central interest in this field, however, is the analog of the Sherrington-Kirkpatrick (SK) model of spin glasses [14] with symmetric but otherwise independent couplings drawn from a Gaussian distribution. It was studied numerically in pioneering work by Daido more than 30 years ago [15]. At $J/\sigma \simeq 1.3$, he found a peculiar "volcano" transition in the distribution of the order parameter that gave rise to speculations about the existence of an oscillator glass state and remained mysterious ever since.

So far, no analytical theory is available for this system. An attempt made in [16] remained essentially restricted to the case of asymmetric couplings. The methods developed in [13] when applied to the SK setting result in an exponential number of order parameters rendering the approach unfeasible [13,17].

In the present Letter, we analyze the Kuramoto model of phase oscillators with normally distributed frequencies and random interactions of SK type. Using the dynamic cavity approach, we reduce the dynamics of the N-oscillator system to a self-consistent stochastic differential equation for a single oscillator that we study analytically and numerically. We reproduce the volcano transition found by Daido, provide a physical picture of its nature, and discuss its relation to a possible oscillator glass phase.

The model is defined by the set of equations

$$\partial_t \theta_j(t) = \omega_j + h_j(t) + \sum_k J_{jk} \sin\big(\theta_k(t) - \theta_j(t)\big) \quad (1)$$

for the time evolution of the phases θ_j of N oscillators $j = 1, \ldots, N$. The frequencies ω_j are drawn at random from a normal distribution with zero mean and unit variance, and the initial conditions $\theta_j(0) = \theta_j^{(0)}$ derive from a flat distribution over $[-\pi, \pi)$. The couplings are symmetric, $J_{jk} = J_{kj}$, and otherwise independent Gaussian random variables with zero mean and variance J^2/N. The local fields $h_j(t)$ are introduced solely to define the susceptibilities $\chi_{jk}(t,t') \coloneqq \delta\theta_j(t)/\delta h_k(t')$ and will be put equal to zero otherwise. We are interested in the long-time dynamics of the system in the thermodynamic limit $N \to \infty$.

Because of the global coupling, the model is of mean-field type and may be reduced to an effective, self-consistent single-site problem. To accomplish this reduction, we employ the dynamical cavity method that rests on the assumption of stochastic stability of the thermodynamic limit [18]. We consider system (1) for one particular realization of frequencies, initial conditions, and couplings and *add* one new oscillator, $j = 0$, with phase $\theta_0(t)$ and new quenched random variables ω_0, $\theta_0^{(0)}$, and J_{0k}.

The presence of the new oscillator induces slight deviations $\theta_j \to \tilde{\theta}_j = \theta_j + \delta\theta_j$ in the dynamics of the previous ones. Since $J_{0k} = O(1/\sqrt{N})$, they are small and may be treated in linear response:

$$\delta\theta_j(t) = \sum_l \int_0^t dt' \chi_{jl}(t,t') J_{l0} \sin\big(\theta_0(t') - \theta_l(t')\big). \qquad (2)$$

To leading order in N, the equation of motion of the new oscillator then assumes the form

$$\begin{aligned}\partial_t\theta_0(t) = \omega_0 + h_0(t) + \sum_k J_{0k} \sin\big(\theta_k(t) - \theta_0(t)\big) \\ + \sum_{k,l} J_{0k} J_{l0} \int_0^t dt' \chi_{kl}(t,t') \cos\big(\theta_k(t) - \theta_0(t)\big) \\ \times \sin\big(\theta_0(t') - \theta_l(t')\big). \qquad (3)\end{aligned}$$

The new couplings J_{0k} are statistically independent of the $\theta_i(t)$, $i = 1, \ldots, N$, and, therefore, the third and fourth term on the rhs of this equation are large sums of independent random contributions with finite first and second moments. By the central limit theorem, they are, therefore, Gaussian random functions and may be characterized by their respective first and second moments. As it turns out, the third term gives rise to a Gaussian noise with zero mean, whereas the fourth term is characterized by negligible fluctuations for $N \to \infty$ and may be replaced by its average [19]. Denoting by $\langle \cdots \rangle$ the combined average over all couplings (including the new ones) as well as over the frequencies and initial conditions for $i \geq 1$, Eq. (3) acquires the form [19]

$$\begin{aligned}\partial_t\theta_0(t) = \omega_0 + h_0(t) + J \cos\theta_0(t)\xi_2(t) - J \sin\theta_0(t)\xi_1(t) \\ - J^2 \int_0^t dt' R(t,t') \sin\big(\theta_0(t) - \theta_0(t')\big). \qquad (4)\end{aligned}$$

Here ξ_i, $i = 1, 2$, are two independent Gaussian noise sources with

$$\langle \xi_i(t) \rangle \equiv 0, \qquad \langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij} C(t,t'). \qquad (5)$$

Moreover,

$$C(t,t') \coloneqq \frac{1}{2N} \sum_j \big\langle \cos\big(\theta_j(t) - \theta_j(t')\big) \big\rangle, \qquad (6)$$

$$R(t,t') \coloneqq \frac{1}{2N} \sum_j \big\langle \chi_{jj}(t,t') \cos\big(\theta_j(t) - \theta_j(t')\big) \big\rangle. \qquad (7)$$

The noise terms derive from the disordered crosstalk of the θ_j on θ_0 and are, hence, proportional to J. The response term stems from the feedback of the polarization of the θ_j due to θ_0 on θ_0 itself and is proportional to J^2.

Consequently, the phase $\theta_0(t)$ of the new oscillator obeys a Langevin equation with multiplicative colored noise and memory kernel with correlation and response functions determined by the dynamics of the original N-oscillator system in the *absence* of θ_0. We now close the argument by stipulating that in the averaged $(N+1)$-oscillator system θ_0 should in no way be special. Therefore, the average in (6) and (7) may be replaced by one over the dynamics of $\theta_0(t)$ itself.

In this way, we end up with our self-consistent single-oscillator problem defined by (4) and (5) together with

$$C(t,t') \coloneqq \frac{1}{2} \big\langle \cos\big(\theta_0(t) - \theta_0(t')\big) \big\rangle, \qquad (8)$$

$$R(t,t') \coloneqq \frac{1}{2} \big\langle \chi_{00}(t,t') \cos\big(\theta_0(t) - \theta_0(t')\big) \big\rangle, \qquad (9)$$

$$\chi_{00}(t,t') \coloneqq \frac{\delta\theta_0(t)}{\delta h_0(t')}. \qquad (10)$$

The average $\langle \cdots \rangle$ is now over the realizations of the ξ_i as well as over ω_0 and $\theta_0^{(0)}$. In the following, the by now superfluous index 0 will be omitted. For large values of t and t', the functions C, R, and χ will depend only on $\tau \coloneqq t - t'$. Equations equivalent to (4), (5), and (8)–(10) may also be derived with the help of the Martin-Siggia-Rose formalism [20] as applied to systems with quenched disorder [21–23].

The self-consistent stochastic single-site problem derived above represents a difficult system of equations. Analytical progress is possible by perturbation theory in J which we worked out up to fourth order [19]:

$$C(\tau) = C_0(\tau) + J^2 C_2(\tau) + J^4 C_4(\tau) + O(J^6), \qquad (11)$$

$$R(\tau) = R_0(\tau) + J^2 R_2(\tau) + J^4 R_4(\tau) + O(J^6). \qquad (12)$$

The complete expressions are rather long, and we quote here only the asymptotic behavior for large τ:

$$\begin{aligned}C_0(\tau) &\sim R_0(\tau) \sim \frac{1}{2} e^{-\tau^2/2}, \\ C_2(\tau) &\sim 3R_2(\tau) \sim \frac{3\sqrt{\pi}}{16} \tau e^{-\tau^2/4}, \\ C_4(\tau) &\sim 13R_4(\tau) \sim \frac{13\pi\sqrt{3}}{864} \tau^2 e^{-\tau^2/6}. \qquad (13)\end{aligned}$$

As can be seen, the decay of both correlation and response functions becomes slower with increasing order, and one may speculate about the general asymptotic behavior

$$C_{2n}(\tau) \sim R_{2n}(\tau) \sim \tau^n e^{-[\tau^2/2(n+1)]}.$$

The single-oscillator dynamics may also be studied by numerical simulations [24,25]. Starting with initial guesses for $C(t,t')$ and $R(t,t')$ and generating a family of trajectories $\{\theta(t)\}$ according to (4), one determines refined approximations for $C(t,t')$ and $R(t,t')$ via (8) and (9) and iterates until convergence is reached [19]. In this procedure, the functional derivative $\delta/\delta h(t')$ of Eq. (4) yields an equation of motion for $\chi(t,t')$ which is solved in parallel with the one for $\theta(t)$ [25]. This numerical analysis is rather different from and hopefully complementary to the direct simulation of the original N-oscillator problem (1) as performed by Daido. In particular, no finite-size effects have to be taken into account: The solution of (4) is expected to display the statistical properties of a typical oscillator in an $N = \infty$ system [24]. Finite-size effects are a notorious nuisance in the simulation of glassy systems [26,27], and their absence is quite advantageous.

To validate our approach, we display in Fig. 1 the correlation function $C(\tau)$ as obtained from the original N-oscillator system and (6), the one determined self-consistently according to (8), and analytical results from perturbation theory. For $J = 0.80$, i.e., below the volcano transition, the agreement is very good. At $J = 1.75$, well above $J_c \simeq 1.3$, both simulations show a markedly increased correlation range as can be inferred from the comparison with $C_0(\tau)$. Unfortunately, the validity of our fourth-order perturbation theory does not extend up to $J = 1.75$.

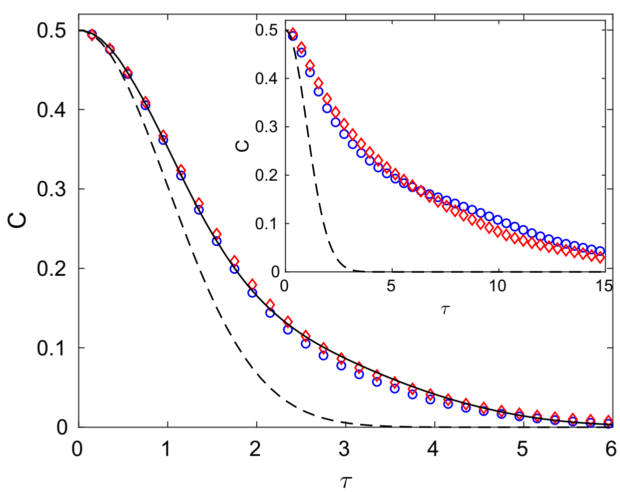

FIG. 1. Correlation function $C(\tau)$ as obtained via (6) from simulations of an $N = 10^3$ oscillator network averaged over 50 disorder realizations (red diamonds) and as resulting from (8) and 5×10^4 trajectories of the single-oscillator dynamics (blue circles). Statistical errors are smaller than the symbol size. The main figure is for $J = 0.80$, the inset for $J = 1.75$. Also shown are the lowest-order perturbative result $C_0(\tau)$ (black dashed line) and, for $J = 0.80$, the fourth-order result (11) (black full line).

We now turn to the analysis of the volcano transition. As order parameters that are able to signal synchronization into *disordered* phase patterns, Daido introduced the so-called complex local fields [15]:

$$\Psi_j(t) \coloneqq r_j(t)e^{i\phi_j(t)} \coloneqq \frac{1}{J}\sum_k J_{jk}e^{i\theta_k(t)}. \tag{14}$$

With their help, Eq. (1) may be rewritten as

$$\partial_t\theta_j(t) = \omega_j + h_j(t) + Jr_j(t)\sin\left(\phi_j(t) - \theta_j(t)\right).$$

From this form, it is apparent that the distribution $P(r)$ of the moduli r_j is crucial for the overall tendency of the system to synchronize. The ϕ_j merely determine the specific pattern on which the phases lock. In fact, Daido detected the volcano transition in histograms of

$$\tilde{P}(r) \coloneqq \frac{P(r)}{2\pi r}$$

compiled from time series $r_j(t)$ for different j. For small coupling strength, $\tilde{P}$ has its maximum at $r = 0$. With increasing J, the height of this maximum shrinks, and at $J = J_c$ it turns into a minimum with another maximum developing at nonzero value of r.

In view of (14), the order parameter for the newly added oscillator is given by

$$\Psi(t) \coloneqq r(t)e^{i\phi(t)} \coloneqq \frac{1}{J}\sum_j J_{0j}e^{i\tilde{\theta}_j(t)}. \tag{15}$$

Using $\tilde{\theta}_j = \theta_j + \delta\theta_j$ together with (2), it may be split into two parts:

$$\Psi(t) = \frac{1}{J}\sum_j J_{0j}e^{i\theta_j(t)} + \frac{i}{J}\sum_j J_{0j}e^{i\theta_j(t)}\delta\theta_j(t) + \cdots$$
$$\eqqcolon \xi(t) + \zeta(t). \tag{16}$$

Here, $\xi = \xi_1 + i\xi_2$ is a complex Gaussian noise defined by (5). Arguments similar to those that carried us from (3) to (4) reveal that the fluctuations of ζ as induced by the randomness in the couplings and the $\theta_j, j \geq 1$ are negligible for $N \to \infty$. As a result, we get [19]

$$\zeta(t) \coloneqq \rho(t)e^{i\varphi(t)} \coloneqq J\int_0^t dt' R(t,t')e^{i\theta_0(t')}, \tag{17}$$

with

$$\rho(t) = J\Big|\int_0^t dt' R(t,t') e^{i\theta_0(t')}\Big|. \tag{18}$$

The decomposition (16) of Ψ now clearly reveals the mechanism behind the volcano transition. For $J = 0$, the order parameter is equal to ξ, and, therefore, it is a Gaussian random variable with zero mean. Accordingly, the maximum of $\tilde{P}(r)$ is at $r = 0$. With increasing J, the Gaussian distribution for Ψ is systematically shifted such that its maximum coincides with ζ. This reduces $\tilde{P}(0)$, and for sufficiently large $|\zeta|$ the maximum of $\tilde{P}$ at $r = 0$ turns into a minimum; see also Fig. 3 in [19]. Note that the crucial prefactor J in (17) results from the different J scaling of noise and response terms in (4).

More quantitatively, we have

$$P(r(t)) \coloneqq \left\langle \int \mathcal{D}\xi_1(\cdot)\mathcal{D}\xi_2(\cdot)\delta(r(t) - |\Psi(t)|)\right\rangle, \tag{19}$$

where the outer average is over ω and $\theta^{(0)}$. Further analytical progress is now hampered by the fact that due to its dependence on $\theta_0(t')$ the order parameter $\Psi(t)$ is a complicated functional of the $\xi_i(\cdot)$. Both our perturbative results and our numerical simulations, however, indicate [19] that typically $\rho(t)$ is very near to its upper bound

$$\rho_u \coloneqq J\int_0^t dt' \big|R(t,t')e^{i\theta_0(t')}\big| \to J\int_0^\infty d\tau |R(\tau)|, \tag{20}$$

where in the last step the limit $t \to \infty$ was taken.

Using ρ_u instead of ρ in (19), no dependence on ω and $\theta^{(0)}$ remains, and, moreover, the path integrals over the noise histories simplify to Gaussian averages over $\xi_i(t)$. Performing these, we end up with [19]

$$\tilde{P}(r) \simeq \tilde{P}_u(r) = \frac{1}{\pi} e^{-r^2-\rho_u^2} I_0(2r\rho_u), \tag{21}$$

where I_0 denotes a modified Bessel function [28]. From

$$\frac{\partial^2 \tilde{P}_u}{\partial r^2}\bigg|_{r=0} = \frac{2}{\pi} e^{-\rho_u^2}(\rho_u^2 - 1),$$

the corresponding approximation J_{cu} for J_c is implicitly given by $\rho_u(J_{cu}) = 1$. Using the lowest-order perturbative result for $R(\tau)$ [19], we find

$$J_{cu} = \frac{4}{\sqrt{2\pi}} \simeq 1.6.$$

Figure 2 compares (21) with numerical simulations. For small J, the agreement is fine. With increasing J, the nonlocal correlations between $\zeta(t)$ and $\xi(t')$ that were neglected in the derivation of (21) become more and more important. Nevertheless, the qualitative agreement between approximate analytical theory and numerical simulations remains valid. In particular, the volcano transition is clearly described by (21). The histograms from both numerical simulations are consistent with those of Daido [15] and, hence, also reproduce his result $J_c \simeq 1.3$, which is somewhat smaller than J_{cu}.

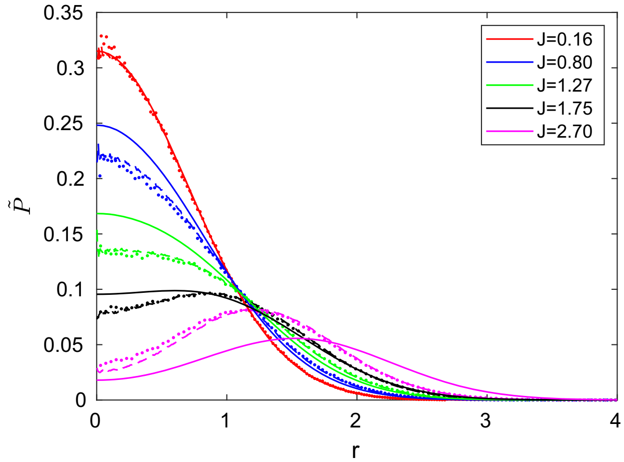

FIG. 2. Comparison between $\tilde{P}_u(r)$ as determined by (21) (full lines) with numerical simulations of 10^3 trajectories of the effective single-oscillator problem (4) (dotted lines) and of the original N-oscillator problem (1) with $N = 10^3$ averaged over 20 disorder realizations (dashed lines).

We finally discuss the relation of our findings to the possible existence of an oscillator glass phase. Our perturbation results (13) for the correlation and response functions show a tendency to long-range autocorrelations and slowly decaying memory kernel with increasing values of J. Although this may be taken as precursors of a freezing transition, these asymptotics give no hint on a power law decay that is generally believed to characterize a glass phase [29]. Moreover, in the analysis of (4) for large J, we do not find any indication for a persistent part in the correlation function similar to the case of the SK spin glass [22]. This is consistent with the phase diffusion observed by Daido for very long times [15] and may originate in the peculiar symmetries of the model that show up also in the linear stability analysis of the incoherent state [5]. Also, the mechanism behind the volcano transition described by the decomposition (16) of the order parameter is rather general, and it seems likely that it may be found in several models [13,17] irrespective of whether these possess a glass phase or not. Finally, from the investigation of so-called "noise-induced" transitions, it is well known that it may be misleading to link a thermodynamic transition to the topological change of a probability distribution [30]. In line with [13,17], we, therefore, believe that the volcano transition found in [15] does not immediately lead into a glass phase.

In conclusion, we have provided an intuitive as well as an analytical understanding of the volcano transition characteristic for synchronization processes in disordered networks of coupled phase oscillators. A key ingredient is the decomposition (16) of the local order parameter in a random and a systematic part. Our methods should be relevant for systems beyond networks of simple phase oscillators. Examples include optomechanical arrays [31], chemical systems [32], and colonies of bacteria [33] as well as the very recently discussed networks of Lohe [34] and Stuart-Landau oscillators [35].

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[1] S. H. Strogatz, *Sync: How Order Emerges from Chaos in the Universe, Nature, and Daily Life* (Hyperion, New York, 2003).
[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambrigde University Press, Cambridge, England, 2001).
[3] A. Pikovsky and M. Rosenblum, Chaos **25**, 097616 (2015).
[4] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, New York, 1984).
[5] S. H. Strogatz and R. E. Mirollo, J. Stat. Phys. **63**, 613 (1991).
[6] S. H. Strogatz, Physica (Amsterdam) **143D**, 1 (2000).
[7] J. A. Acebron, L. L. Bonilla, C. J. Perez Vicente, and F. Ritort, Rev. Mod. Phys. **77**, 137 (2005).
[8] H. Sompolinsky, A. Crisanti, and H. J. Sommers, Phys. Rev. Lett. **61**, 259 (1988).
[9] L. L. Bonilla, C. J. Perez Vicente, and J. M. Rubi, J. Stat. Phys. **70**, 921 (1992).
[10] H. Hong and S. H. Strogatz, Phys. Rev. Lett. **106**, 054102 (2011).
[11] H. Hong and S. H. Strogatz, Phys. Rev. E **84**, 046202 (2011).
[12] I. M. Kloumann, I. M. Lizarraga, and S. H. Strogatz, Phys. Rev. E **89**, 012904 (2014).
[13] B. Ottino-Löffler and S. H. Strogatz, Phys. Rev. Lett. **120**, 264102 (2018).
[14] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[15] H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).
[16] J. C. Stiller and G. Radons, Phys. Rev. E **58**, 1789 (1998).
[17] D. Pazó and R. Gallego, Phys. Rev. E **108**, 014202 (2023).
[18] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific Publishing Company, Singapore, 1987).
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.187201 for details.
[20] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[21] C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[22] H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).
[23] A. Crisanti and H. Sompolinsky, Phys. Rev. E **98**, 062120 (2018).
[24] H. Eissfeller and M. Opper, Phys. Rev. Lett. **68**, 2094 (1992).
[25] F. Roy, G. Biroli, G. Bunin, and C. Cammarota, J. Phys. A **52**, 484001 (2019).
[26] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[27] H. Daido, Phys. Rev. E **61**, 2145 (2000).
[28] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Gaithersburg, MD, 1964).
[29] This may, of course, also be due to our weak-coupling perturbation theory failing to describe the frozen phase.
[30] H. Brand, A. Schenzle, and G. Schröder, Phys. Rev. A **25**, 2324 (1982).
[31] G. Heinrich, M. Ludwig, J. Qian, B. Kubala, and F. Marquardt, Phys. Rev. Lett. **107**, 043603 (2011).
[32] J. F. Totz, J. Rode, M. R. Tinsley, K. Showalter, and H. Engel, Nat. Phys. **14**, 282 (2018).
[33] A. Prindle, P. Samayoa, I. Razinkov, T. Danino, L. S. Tsimring, and J. Hasty, Nature (London) **481**, 39 (2012).
[34] S. Lee, Y. Jeong, S.-W. Son, and K. Krischer, J. Phys. A **57**, 085702 (2024).
[35] A. Pikovsky and F. Bagnoli, New J. Phys. **26**, 023054 (2024).