Turbulent Flows Are Not *Uniformly* Multifractal

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Understanding turbulence rests delicately on the conflict between Kolmogorov's 1941 theory of nonintermittent, space-filling energy dissipation characterized by a unique scaling exponent and the overwhelming evidence to the contrary of intermittency, multiscaling, and multifractality. Strangely, multifractality is not typically envisioned as a *local* flow property, variations in which might be clues exposing inroads into the fundamental unsolved issues of anomalous dissipation and finite time blowup. We present a simple construction of local multifractality and find that much of the dissipation field remains surprisingly monofractal *à la* Kolmogorov. Multifractality appears as small islands in this *calm* sea, its strength growing logarithmically with the local fluctuations in energy dissipation—a seemingly universal feature. These results suggest new ways to understand how singularities could arise and provide a fresh perspective on anomalous dissipation and intermittency. The simplicity and adaptability of our approach also holds great promise in applications ranging from climate sciences to medical data analysis.

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As a central scaffold for interpreting and describing out-of-equilibrium pattern forming processes [1–4], multifractality has inevitably woven itself into the most ubiquitous of natural phenomena: turbulence [5–8]. And so, unsurprisingly, the Frisch-Parisi multifractal model [9–11] remains the key theoretical justification for the most baffling observations which go beyond the Kolmogorov picture of turbulence. These include the anomalous scaling of correlation functions of velocity differences [10,12–14], strongly non-Gaussian distributions of velocity gradients [15–17], and fluid accelerations [18]. Is there, however, an obvious way which makes the multifractal formalism more potent in attacking fundamental questions such as intermittency, anomalous dissipation, and finite-time blowup? We show this is possible through spatially local measures of multifractality which in turn leads to a surprising shift in perspective: *Most* of turbulence is essentially monofractal *à la* Kolmogorov with few patches of multifractality correlated with regions of sharply varying dissipation. This complements a recent analysis [19–21] bridging the multifractal formalism to weak solutions of the Navier-Stokes equation and possibilities of finding disconnected, spatial patches of intense high frequency fluctuations. Whence, the inevitable conclusion that fully developed turbulence is best described as intermittent, multifractal islands on a vast and calm Kolmogorovean sea.

When applied to the intermittent, energy dissipation field $\epsilon(\mathbf{x})$ (for instance, a 2D slice of dissipation [22–24] shown in Fig. 1), a multifractal analysis leads [25] to the total dissipation in d-dimensional "boxes" of size r, denoted $\mathcal{E}_r$, scaling as a *fractal* power law with a variable scaling exponent $\alpha = 3h$ (where h is the standard Hölder exponent) as $\mathcal{E}_r \sim r^{\alpha-1+d}$, and their order q moments $Z_q \equiv \sum_{N_r} \mathcal{E}_r^q \sim r^{(q-1)D_q}$ lead to generalized dimensions D_q [10,26–29] (shown in Fig. 1 inset (A), consistent with earlier measurements [10,26]). This is a direct corollary of the multifractal interpretation [9] that despite the three-dimensional embedding dimension, the energy dissipation—which is a culmination of the energy cascading process—accumulates in different, entangled fractal subsets with unique dimensions [39]. It is then possible to associate the fractal dimension f_α of these subsets with exponents lying between α and $\alpha + d\alpha$ yielding the well-known singularity or multifractal spectrum $f_\alpha - \alpha$ (as seen in Fig. 1, inset B): A broad f_α curve is the clinching measurement showing turbulence admits a range of scaling exponents and not just the mean field Kolmogorov exponent $\alpha = 1$. Obtaining these exponents and corrections to the Kolmogorov prediction, directly from the Navier-Stokes equation, still remains elusive.

We here arrive at an impasse, for the (global) multifractal analysis tends to suppress an unremarked symptom of its construction: Even if a measure is multifractal *only* in isolated patches, one is *bound* to find a broad f_α curve representing the entire data [25]. Therefore, it is not possible to conclude from the usual analysis whether turbulence is uniformly multifractal. Indeed, while statistical homogeneity and isotropy underlie (small-scale) turbulence theories [43,44], the dissipation field is decidedly inhomogeneous. We now uncover the precise relation between such *local* variations in dissipation and their corresponding degree of multifractality.

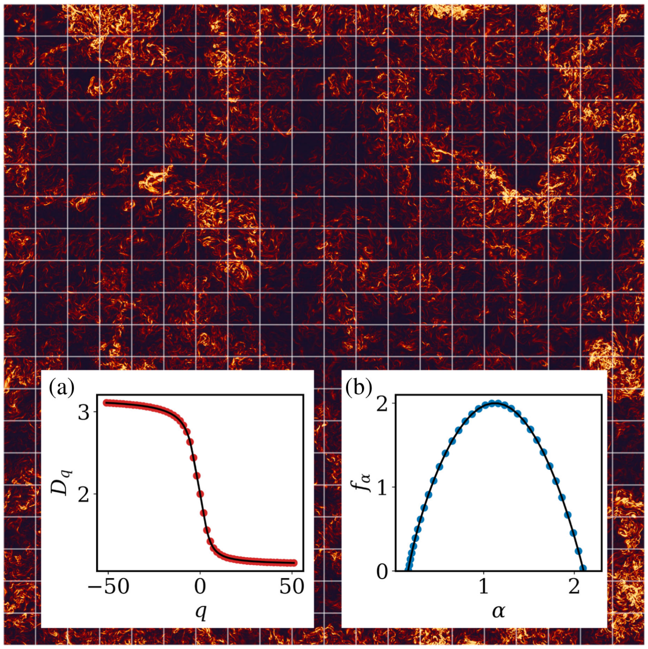

FIG. 1. Spatially varying dissipation ϵ at a representative cross section from the Johns Hopkins Turbulence Database for a flow with $\mathrm{Re}_\lambda \approx 610$. Superimposed is a regular tiling as a pictorial guide to show the subdivisions used for local multifractal analysis (made with Processing [41,42]). Generalized dimensions D_q [inset (A)] and the singularity spectrum $f_\alpha - \alpha$ [inset (B)] calculated over the full 2D cross section show the essential global multifractal nature of the dissipation field, as is usually reported.

In fact, while the first observations of intermittency [45–48] were already suggestive of spatial variations of the singularity spectrum, more recent work [49] on estimates from the Navier-Stokes equation is not only consistent with the measured non-Gaussian character of turbulence [45] but also admits the possibility of isolated, anomalously active regions of dissipation. Remarkably, investigations of weak solutions also provide key insights into the success of the Frisch-Parisi multifractal model with rigorous estimates for a range of fractal dimensions where dissipation occurs [19–21].

It seems reasonable to wonder why should the fractal sets, the singularity spectrum, and the Hölder exponents not have a local character when everything else in turbulence like the velocity, vorticity, and dissipation strongly vary in space? This would be revealed if it were possible to measure the generalized dimensions $D_q(\mathbf{x})$ and $f_\alpha(\mathbf{x}) - \alpha(\mathbf{x})$ locally. Beyond fundamentally expanding our notion of multifractality from a global to a local flow feature, these variations would assist the detection of (possible) singular $h < 1/3$ regions of anomalous dissipation [50,51]. Closest, so far, have been wavelet techniques and local energy transfer concepts [52–55] to characterize fields similar to the intractable $h(\mathbf{x})$ and multiscale urban morphology studies [56]. Our work exploits the crucial underlying variation in multifractality, allowing us to discover the local distributions of the Hölder exponents.

We develop a tiling approach [25], illustrated in Fig. 1 where a (exaggerated) white grid is superimposed on a 2D slice of the dissipation field, leading to square data tiles with edge $\mathcal{L}_T$ (or cubes, for 3D analysis). Each tile, centered at $\mathbf{x}$, is treated independently, and multifractality is measured within every tile following the usual approach typically adopted for the full dataset. We tested tiles in the range $2\eta < \mathcal{L}_T < 16\eta$, where η is the Kolmogorov dissipation scale. Larger tiles (still smaller than the inertial range) give a wider range of r values over which to construct $Z_q(\mathbf{x})$ while, for a truly localized measurement, the tiles should be as small as possible. The lower end of $\mathcal{L}_T$, however, is dictated by the need for *enough* points to measure the scaling of $Z_q(\mathbf{x})$ unambiguously [57]. We report results for $\mathcal{L}_T \approx 10\eta$ and have checked that they are robust across the range of $\mathcal{L}_T$.

The dissipation field of course varies within these tiles, and we find it useful to keep track of, for each tile, the maximum $\epsilon_{\max}(\mathbf{x})$, minimum $\epsilon_{\min}(\mathbf{x})$, and mean $\bar{\epsilon}(\mathbf{x})$ dissipation, as well as $\Delta\epsilon(\mathbf{x}) \equiv \epsilon_{\max}(\mathbf{x}) - \epsilon_{\min}(\mathbf{x})$ as a measure of the fluctuation of the field, all normalized by the global mean dissipation $\langle\epsilon\rangle$. We use data from four different direct numerical simulations (DNSs)—both in house [25] and publicly available data from the Johns Hopkins Turbulence Database (JHTDB) [22–24]—with Taylor-scale based Reynolds numbers $200 \leq \mathrm{Re}_\lambda \leq 1300$. Our results are consistent across this wide range of Reynolds numbers and independent of simulations; in what follows, we present results from a $4096^2 \times 192$ subset of the 4096^3 dataset. We quantify the reliability of our local multifractal measurements (see the Supplemental Material [25]) with the Pearson correlation coefficient ρ for the linear-regression fits used to obtain $D_q(\mathbf{x})$. We find $\rho > 0.98$ for more than 99.98% of the 3D tiles.

Before stepping into spatially varying multifractal spectra, we pause to look at the special case of D_2 ($q = 2$), the correlation dimension, that gives a measure of the inhomogeneity in a fractal set [29,58]. Figure 2 shows a planar cross section of the $D_2(\mathbf{x})$ field [59], starkly varying in space, with sizeable coherent regions of similarly valued D_2. This shows that the field is far from random and reflective of the structures in the dissipation field. We wish to underline that our method allows one, for the first time, to visualize this field. This opens up new directions to study the structure of these intrinsic, and as yet elusive, fields of generalized dimensions underlying the fractal skeleton of turbulence.

We are now equipped to calculate *local* measures of multifractality—$f_\alpha(\mathbf{x}) - \alpha(\mathbf{x})$—and uncover whether turbulence is indeed *uniformly* multifractal. We know that, even locally, the singularity spectrum should satisfy $\alpha \geqslant \alpha_{\min} = -2$, $f_\alpha \leqslant \alpha + 2$, and $f_\alpha \geqslant 0$ [20]. The maximum of the spectrum in three dimensions is $f_\alpha \leq 3$; within

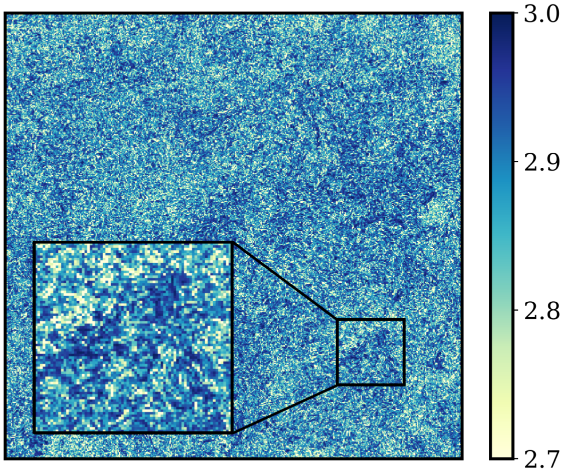

FIG. 2. A cross section of the D_2 (correlation dimension) field, obtained from a local multifractal analysis, which coarsens a $4096 \times 4096 \times 8$ data slice to 512×512 tiles. The stark variation in D_2 over space, which remained hitherto unseen, highlights coherent regions of D_2 nestling in a fluctuating (and nonrandom) field.

the monofractal, nonintermittent, Kolmogorov view of turbulence, the singularity spectrum is single valued with $\alpha = 1$ and $f_\alpha = 3$. While there is no bound (in real turbulence) for $\alpha_{\max}$, it is reasonable to assume $\alpha_{\max} \approx 3$ for a region with no singular structures.

It therefore follows that an intuitive but precise measure of the degree of local multifractality in the flow, exploiting the spread of singularity strengths, is $\Phi(\mathbf{x}) = \mathrm{std}[\alpha(\mathbf{x})] = \sqrt{\langle\alpha^2\rangle - \langle\alpha\rangle^2}$, where $\langle\cdot\rangle$ denotes an average over all values of α. It follows therefore that monofractal regions show $\Phi \approx 0$, and larger values of Φ correspond to high degrees of local multifractality. Theoretically, it is easy to show that for α uniformly ranging from -2 to 3, the largest values of Φ, corresponding to highly multifractal regions, are $\Phi \approx 1.7 \sim \mathcal{O}(1)$. Thence, our approach ought to yield $0 \lesssim \Phi(\mathbf{x}) \lesssim \mathcal{O}(1)$, with the lower and upper bound corresponding to monofractal and multifractal statistics, respectively.

First, in Fig. 3(a) we show local $D_q(\mathbf{x})$ vs q curves measured at different spatial positions, corresponding to different values of $\bar{\epsilon}$. Clearly, while the shape of each curve resembles the global statistics (Fig. 1, inset A), a very strong spatial dependence on where we measure the generalized dimensions is unmistakable. Furthermore, the spread in $D_q(\mathbf{x})$ is not trivially related to mean dissipation $\bar{\epsilon}$ (or even the maximum $\epsilon_{\max}$) around $\mathbf{x}$, as one would expect from naïve intuition; the secret to such variation, as we shall demonstrate, lies in how locally fluctuating (within each tile) the dissipation field is.

All this brings us to the central message of this work. In Fig. 3(b), we show representative plots of local $f_\alpha(\mathbf{x}) - \alpha(\mathbf{x})$ for the same locations (see legend for panel a). Quite clearly—and contrary to what one sees in the conventional global measurements of the singularity spectrum [see, e.g., Refs. [10,26] and Fig. 1, inset]—there are several regions where the flow is essentially monofractal (the f_α spectrum being very narrow) and hence consistent with the ideas of Kolmogorov, while other highly multifractal regions lead to broad f_α curves. These results already hint that multifractality can be considered as a *local* property of a field.

In Fig. 3(c) we show a pseudocolor plot of Φ. Quite remarkably, much of the flow is Kolmogorov-like with $\Phi \ll 1$; the highly multifractal regions—$\Phi(\mathbf{x}) \sim \mathcal{O}(1)$—are isolated patches which, as we shall see, correlate completely with the extreme (singular) regions of energy dissipation. This result is remarkable. It illustrates that, surprisingly, turbulent flows are *not* uniformly multifractal; indeed on the contrary, much of the turbulent flow seems to respect, locally, Kolmogorov's ideas of an exact, self-similar cascade. We also note that the range of $\Phi(\mathbf{x})$ is within the theoretical range discussed above.

What determines the magnitude and variation of $\Phi(\mathbf{x})$? We find that the probability distribution function (PDF) of Φ, conditioned on $\Delta\epsilon(\mathbf{x})$, is revealing. In the inset of Fig. 3(d) we show this PDF for three different values of $\Delta\epsilon$. Clearly, as evident from the previous measurements, the distribution is sharply peaked at values of $\Phi \gtrsim 0$ with an (likely) exponential tail for $\Phi \sim \mathcal{O}(1)$. We also find that the probability of having a higher degree of multifractality increases, albeit marginally, when there is a greater variation of $\epsilon(\mathbf{x})$ within a tile. The mean value $\bar{\Phi}$, for a given $\Delta\epsilon$ (sampled in windows of $\Delta\epsilon \pm 0.25$), grows as $\bar{\Phi} \sim \log(\Delta\epsilon)$ [Fig. 3(d)]. This logarithmic dependence marks an important relation between a local dissipative structure characterized by fluctuations in ϵ and its multifractal nature. We recall that the log-normal theory (K62) [60] also leads to a logarithmic relation, albeit globally [8,26]. Our results in fact reduce to K62 if Φ does not fluctuate significantly. However, Figs. 3(b) and 3(c) show that such an assumption cannot hold, with the relation between ϵ and f_α calling for greater nuance. Further, we highlight that this logarithmic dependence seems universal and is found even for a measure composed of pure noise.

What then is the role of the average dissipation $\bar{\epsilon}(\mathbf{x})$ in determining the spatial nonuniformity of $\Phi(\mathbf{x})$? A joint distribution [Fig. 3(e)] shows that the answer is fairly nontrivial. Clearly, the most likely value of $\Phi(\mathbf{x})$ grows logarithmically with $\bar{\epsilon}(\mathbf{x})$ as shown through the dashed white line. For low values of $\bar{\epsilon}(\mathbf{x})$, it is far more likely to have $\Phi(\mathbf{x}) \ll 1$; although, surprisingly, the less likely extreme values of $\Phi(\mathbf{x})$ also coincide with regions of low $\bar{\epsilon}(\mathbf{x})$. This reflects that it is not the mean dissipation in a region, but the *variation* of dissipation, that manifests multifractality [as shown in Fig. 3(d)]. At higher $\bar{\epsilon}(\mathbf{x})$, the smallest admissible values of $\Phi(\mathbf{x})$ slightly increase with $\bar{\epsilon}(\mathbf{x})$, while the largest values of $\Phi(\mathbf{x})$ also dip. While this result might appear contrary to our notion that extreme

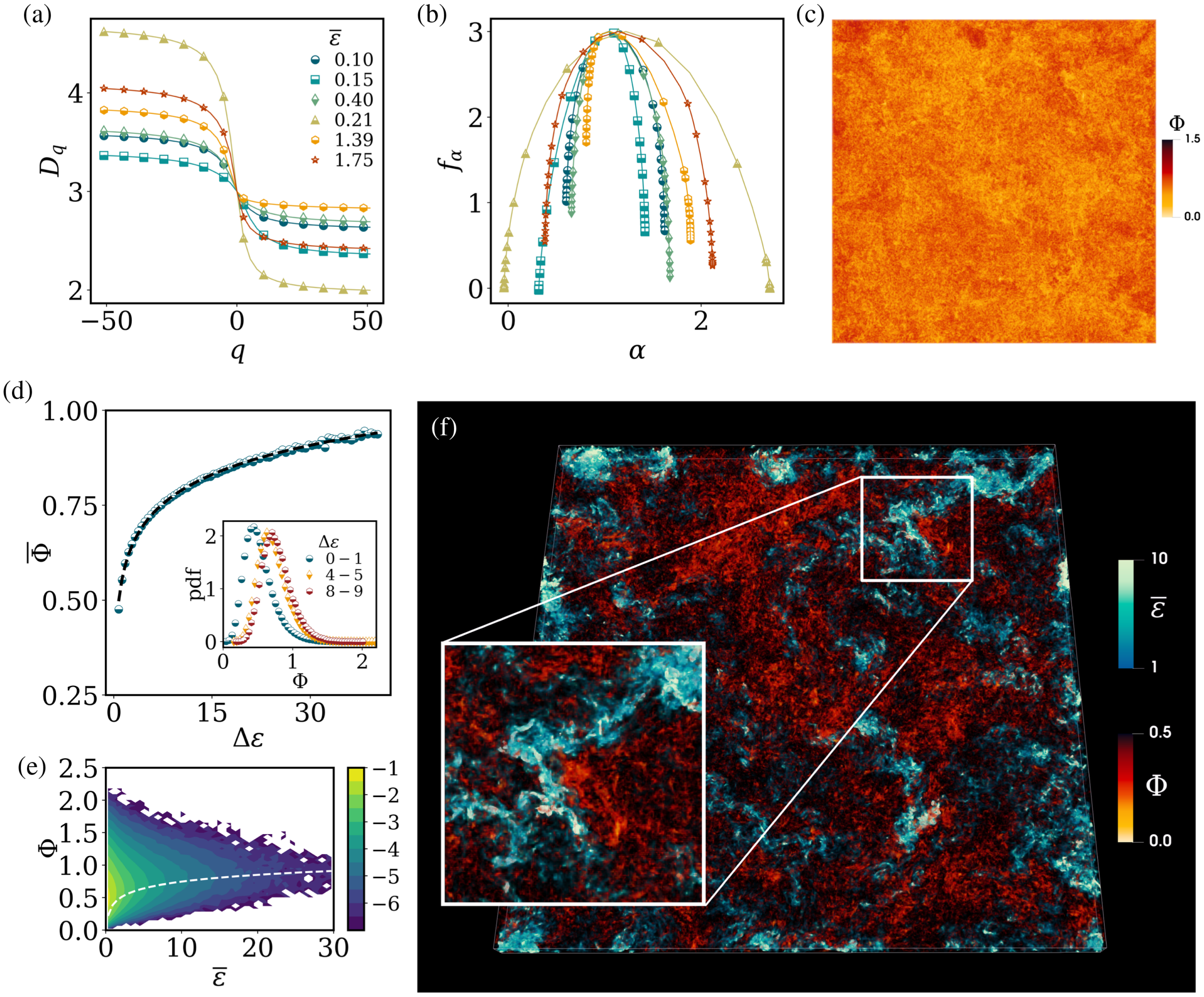

FIG. 3. (a) Local generalized dimensions D_q vs q, for randomly sampled tiles, and (b) the corresponding local $f_\alpha - \alpha$ spectra of singularity strengths, show strong variation in the multifractal properties over space. (c) The resultant Φ field shows large regions of the flow are almost monofractal with $\Phi \approx 0$, with pockets of $\Phi \sim \mathcal{O}(1)$ with its PDF [(d), inset)] shifting toward higher Φ when sampled in regions of higher $\Delta\epsilon = \epsilon_{\max} - \epsilon_{\min}$. (d) The mean value of Φ is shown to grow as the logarithm of $\Delta\epsilon$. (e) The joint distribution of Φ and the mean dissipation $\bar{\epsilon}$ in each tile show that most likely Φ values grow logarithmically (dashed white line) with $\bar{\epsilon}$. (f) A volume rendering of $\bar{\epsilon} \geq 1$ is superimposed with the $\Phi \leq 0.5$ field, which being spatially exclusive, clearly illustrates that the most *monofractal* flow regions are coincident with mild dissipation.

dissipation *alone* begets multifractality, it finds parallel in a recent study showing local Hölder exponents, measured by proxy, also do not trivially correlate with inertial dissipation [53]. In fact, experiments have shown that the *most* dissipative structures locally resemble Burgers vortices [61]. While these intense spots make the entire field highly intermittent and contribute to broadening the global multifractal spectrum, the *local* multifractal picture can be different. We have tested this idea [25] on a more tractable curdling model and found the conclusions surprisingly robust.

We finally tie up these ideas with a visual illustration of where the Kolmogorov-like regions are embedded, in Fig. 3(f), restricting the dissipation field to large values $\bar{\epsilon}(\mathbf{x}) \geq 1$, superimposed with $\Phi(\mathbf{x}) \leq 0.5$. Unlike the sparsely populated high $\bar{\epsilon}(\mathbf{x})$ regions, the more frequent low $\bar{\epsilon}(\mathbf{x})$ regions (hidden from view here) remain largely occupied by low $\Phi(\mathbf{x})$ (these regions are also coincident with mild to low kinetic energy). Clearly, then, the regions of monofractal flow are *strongly correlated* to the more populous regions of mild dissipation, showing that the Kolmogorov-like regions locally dissipate less than the multifractal regions.

We have given first evidence that multifractality in turbulent flows is not spatially uniform, which compels a revision in our accustomed understanding of turbulence. While this may conform to a *belief* shared by some, it is important to actually *show* that this can be measured. In the absence of a robust theory, we make a final test in a Navier-Stokes-like flow—the decimated turbulence model [30]—which is *guaranteed* to be nonintermittent [25]. A confirmation of the conclusions drawn above would mean that the nonintermittent flow ought to show lower values of $\Phi(\mathbf{x})$ than what is measured in real, intermittent turbulence. Indeed, this is what we find as PDFs of $\Phi(\mathbf{x})$ consistently shift toward lower values [25] with decreasing intermittency confirming the relation between intermittency and local multifractality [62].

To summarize, our work shows that turbulence manifests strong multifractality only in localized pockets of intermittency with a quiescent Kolmogorovean background of mild dissipation. Furthermore, while local measurements of a single $h(\mathbf{x})$ remain intractable (as even *locally* turbulence seems to admit a range of h), our prescription of $\Phi(\mathbf{x})$ provides a starting point to ask where and whether singularities could occur. Our framework opens up a completely novel avenue for studying flow singularities and generalized dimension fields in tandem with structures like intense vorticity worms [63–65], nonlocally induced velocity jets [66], or precursors to singular dissipation [61,67]. The lessons learned from local multifractality will also be useful on a different front: Reduced predictive models for synthetic turbulence that rely on global multifractal statistics can now inch closer to the real nature of physical fields, by accounting for the spatial variation in multifractality [8]. Studies with classical curdling models [25] will hence be illuminating.

Moreover, this projects multifractality beyond its statistically reductive role [68] to applications in prediction and diagnostics. Therefore, this approach can be applied to data from across disciplines, where multifractality has been found emergent like in physics and chemistry [1], medicine [69], geophysics [70], climate [71], and finance [72], and is likely to be just as revealing in unpredictable ways.

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