Arbitrary Amplification of Quantum Coherence in Asymptotic and Catalytic Transformation

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Quantum coherence is one of the fundamental aspects distinguishing classical and quantum theories. Coherence between different energy eigenstates is particularly important, as it serves as a valuable resource under the law of energy conservation. A fundamental question in this setting is how well one can prepare good coherent states from low coherent states and whether a given coherent state is convertible to another one. Here, we show that any low coherent state is convertible to any high coherent state arbitrarily well in two operational settings: asymptotic and catalytic transformations. For a variant of asymptotic coherence manipulation where one aims to prepare desired states in local subsystems, the rate of transformation becomes unbounded regardless of how weak the initial coherence is. In a non-asymptotic transformation with a catalyst, a helper state that locally remains in the original form after the transformation, we show that an arbitrary state can be obtained from any low coherent state. Applying this to the standard asymptotic setting, we find that a catalyst can increase the coherence distillation rate significantly—from zero to infinite rate. We also prove that such anomalous transformation requires small but nonzero coherence in relevant modes, establishing the condition under which a sharp transition of the operational capability occurs. Our results provide a general characterization of the coherence transformability in these operational settings and showcase their peculiar properties compared to other common resource theories such as entanglement and quantum thermodynamics.

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Quantum coherence between different energy eigenstates is a valuable resource inevitable for quantum clocks [1], metrology [2], and work extraction [3]. Under the law of energy conservation, coherence in the above sense is easily lost due to decoherence, while it is impossible to create and inflate coherence without any help. In this regard, coherence is a precious quantum resource that should be utilized as efficiently as possible.

A central problem concerning quantum coherence as an operational resource is to characterize its manipulability with energy-conserving unitary [4,5]. This physical setting comes with a fundamental constraint that the total amount of quantum coherence cannot be increased by energy-conserving operations. To understand its manipulation power, two formalisms of state transformations—asymptotic and catalytic transformations—have been actively investigated.

One standard setting for resource manipulation is the *asymptotic transformation*, where one aims to convert many copies of the initial quantum state to many copies of another target state [6]. The key performance quantifier of the asymptotic manipulation is its transformation rate, the ratio of the number of copies of the final state to those of the initial one. On the asymptotic coherence manipulation, it has been shown that there is a strong limitation that the transformation rate from generic mixed states to pure coherent states is zero [7].

Another standard setting for resource manipulation is *catalytic transformation*, where one is allowed to borrow the help of another auxiliary system called catalyst—an ancillary system that should return to its own state at the end of the process. In particular, *correlated catalyst*, which could have a correlation with the main system after the transformation, has been shown to be effective in enhancing the resource manipulability for several physical settings [8–20]. However, similarly to the asymptotic transformation, fundamental limitations on catalytic enhancement have been observed. A notable result is the coherence no-broadcasting theorem [21,22], showing that no coherence could be created with a correlated catalyst if the input state is exactly incoherent. These previous studies, both on asymptotic and catalytic transformations, indicate the potential difficulty of manipulating quantum coherence.

Contrarily to these suggestions, we here show that an arbitrary coherence manipulation is enabled in asymptotic and catalytic coherence transformation. We consider a variant of the asymptotic transformation where one aims to prepare a target state on each subsystem [23] and show that the transformation rate becomes unbounded if the initial state has nonzero coherence. In the correlated-catalytic transformation, we prove that arbitrary state transformation becomes possible as long as the initial state has nonzero coherence. This shows that the observation from the coherence no-broadcasting theorem is unstable

about the perturbation of the initial state in the following sense: As long as the initial state contains even a tiny amount of coherence, every coherent state suddenly becomes reachable. In addition, for target states besides measure-zero exceptions, *exact* transformation is possible, which is a much stronger claim than the conventional resource-theoretic results allowing a small error in the final state.

As a direct consequence of our result, we show that the standard asymptotic transformation rate becomes infinite with the help of correlated catalysts. This resolves the open problem proposed in Ref. [11], asking whether correlated catalysts could improve the asymptotic rate at all, in the most drastic manner—catalysts can make undistillable coherent states infinitely distillable.

Our protocols require a nonzero amount of coherence—even if extremely small—in the initial state to implement arbitrary state conversions. To fully characterize this requirement, we formalize no-go theorems on state conversions by introducing the notion of *resonant coherent modes*. These no-go theorems reveal that initial coherence, even if it is negligibly small, is inevitable for arbitrary state conversions, and exactly zero coherence must result in zero coherence. Together with the feasible transformations described above, these characterize state transformability both in asymptotic and catalytic settings, revealing that the distinction between zero and nonzero coherence is an extremely sharp threshold.

We remark that these "amplification" effects do not contradict the physical requirements that the total amount of coherence should not increase. Our results rest on the fact that coherence can *locally* increase, as observed in several settings previously [24,25]. Our results extend these observations in the context of asymptotic and catalytic coherence manipulation and provide general characterizations of the anomalous coherence amplification phenomena observed in each operational setting.

Coherence transformation.—Superposition between energy eigenstates is manifested in time evolution. For a system with Hamiltonian H, a state ρ is called *coherent* if $\mathcal{U}_t(\rho) \neq \rho$ for some time t, where $\mathcal{U}_t(\rho) \coloneqq e^{-iHt}\rho e^{iHt}$ is the unitary time evolution. A state is called *incoherent* if it is not coherent. We remark that the coherence we consider in this work is what is so-called *unspeakable coherence* [26]. (Not to be confused with another type known as *speakable coherence* [27].)

Available operations in manipulating quantum coherence should not create coherence from incoherent states, as respecting the law of energy conservation. In reflecting this restriction, a natural set of available operations for the coherence manipulation is the *covariant operations* with time translation [28], i.e., the action of a channel $\Lambda\colon S \to S'$, where S and S' are input and output systems, commutes with the unitary time evolution as $\Lambda \circ \mathcal{U}_t^S = \mathcal{U}_t^{S'} \circ \Lambda$ for all t [4,5,29]. From the operational perspective, any covariant operation Λ can be equivalently written by an energy-conserving unitary U and an incoherent state η as $\Lambda(\rho) = \mathrm{Tr}_A[U(\rho \otimes \eta)U^\dagger]$, where A is some auxiliary system [5,30]. In other words, covariant operations are operations which can be implemented by an energy-conserving unitary with incoherent states.

Coherent modes.—Our findings clarify that whether relevant modes have (maybe tiny but) nonzero coherence leads to a drastic change. To formalize this, we introduce the notion of resonant coherent modes. A mode for Δ is a pair of two energy levels with energy difference Δ, and a state ρ has a coherent mode Δ if $\rho_{ij} \neq 0$ with $E_i - E_j = \Delta$ is satisfied for some i, j, where $\rho_{ij} \coloneqq \langle i|\rho|j\rangle$ and $|i\rangle$ is an energy eigenstate with energy E_i for the given Hamiltonian H. We then define the set $\mathcal{C}(\rho)$ of resonant coherent modes of state ρ as all linear combinations of nonzero coherent modes with integer coefficients, i.e.,

$$\mathcal{C}(\rho) \coloneqq \left\{ x | x = \sum_{i,j(\rho_{ij}\neq 0)} n_{ij}\Delta_{ij}, n_{ij} \in \mathbb{Z} \right\} \tag{1}$$

for an energy interval $\Delta_{ij} = E_i - E_j$ and a nonzero off-diagonal entry ρ_{ij} of a density matrix ρ for energies E_i and E_j. Notably, in the asymptotic and catalytic coherence manipulation, one can create a coherence on mode $\Delta = \Delta_1 + \Delta_2$ if the initial state has coherence on modes Δ_1 and Δ_2 [31].

Asymptotic manipulation.—We first consider the asymptotic manipulation. Suppose ρ is an initial state and ρ' is a target state. In the standard framework of asymptotic transformation, one considers a series $\{\Lambda_n\}_n$ of available operations that transforms $\rho^{\otimes n}$ to $\rho'^{\otimes \lfloor rn \rfloor}$ with vanishing error at the limit of $n \to \infty$. The asymptotic transformation rate $R(\rho \to \rho')$ is the supremum over all achievable rates r. When ρ' is a pure state ϕ, it is particularly called asymptotic distillation. For coherence distillation with covariant operations, the distillation rate $R(\rho \to \phi)$ is known to be zero for an arbitrary full-rank state ρ and an arbitrary coherent pure state ϕ [7], which puts a fundamental restriction on the tangibility of coherence as an operational resource.

However, the necessity of obtaining the state close to $\phi^{\otimes \lfloor rn \rfloor}$ can be reasonably relaxed for many operational settings. For instance, consider the scenario where multiple parties are separated from each other and would like to consume a good coherent state locally. In such a setting, the quality of the resource state is determined by how close the local marginal state is to the maximally coherent state. The framework that fits this operational setting was considered previously and called *asymptotic marginal transformation* [9,23]. Suppose ρ and ρ' are the states on the systems S and S' respectively. The state ρ can be converted to ρ' with an asymptotic marginal transformation with rate r if there exists a series of available operations $\{\Lambda_n\}_n$ from $S^{\otimes n}$ to $S'^{\otimes \lfloor rn \rfloor}$ such that the reduced state of

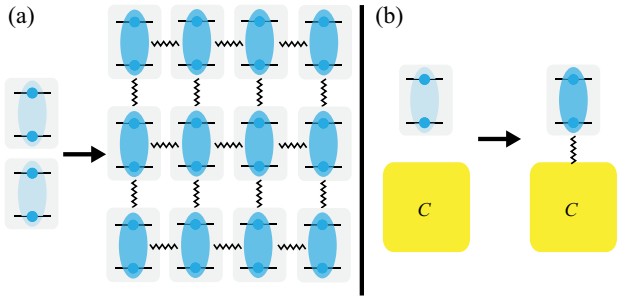

FIG. 1. (a): An asymptotic marginal transformation maps n copies of ρ into m copies of ρ' with correlation among copies. Theorem 1 states that for almost all ρ and ρ', the rate of asymptotic marginal transformation of $\rho \to \rho'$ defined as $\lim_{n\to\infty}\max_m(m/n)$ is unbounded. (b): A correlated-catalytic transformation maps a product state of system S and catalyst C written as $\rho \otimes c$ to τ such that $\mathrm{Tr}_S[\tau] = c$ and $\mathrm{Tr}_C[\tau] = \rho'$. Theorem 2 states that for almost all ρ and ρ', ρ is convertible to ρ' with a correlated catalyst. In addition, the strength of the correlation between the main and catalytic systems can be made arbitrarily small.

$\Lambda_n(\rho^{\otimes n})$ on every subsystem approaches ρ' with a vanishing error at the $n \to \infty$ limit. If the reduced state of any single subsystem exactly coincides with the target state ρ' for some finite n, we say that this asymptotic marginal transformation is *exact*. Although the asymptotic marginal transformation rate $\tilde{R}(\rho \to \rho')$, which is defined as the highest achievable rate in the marginal asymptotic conversion, serves as an upper bound of the standard asymptotic transformation $\tilde{R}(\rho \to \rho') \geq R(\rho \to \rho')$, these two rates coincide in many settings such as entanglement, quantum thermodynamics, and nonclassicality [23], suggesting that this relaxation may not realize a significant improvement in the ability of transformation (see also Sec. II A in the Supplemental Material [32]).

Despite these previous observations, we prove that any low coherent state can be transformed to any high coherent state with an arbitrarily high transformation rate. Namely, there is no restriction on coherence transformation, and all states without measure-zero exceptions admit infinite asymptotic marginal distillation rates [Fig. 1(a)].

Theorem 1.—For arbitrary states ρ and ρ', $\tilde{R}(\rho \to \rho')$ diverges if ρ has nonzero coherence in the sense of $\mathcal{C}(\rho') \subseteq \mathcal{C}(\rho)$. Moreover, the asymptotic marginal transformation can be made exact if ρ' is full rank. In both cases, the correlation between one subsystem and the others can be made arbitrarily small. On the other hand, if $\mathcal{C}(\rho') \not\subseteq \mathcal{C}(\rho)$, even a single copy of ρ' cannot be prepared from any number of copies of ρ with arbitrarily small error.

We remark that $\mathcal{C}(\rho') \subseteq \mathcal{C}(\rho)$ is quite a mild condition since any state ρ with extremely small but nonzero coherence on all modes automatically passes this requirement regardless of ρ'.

Theorem 1 provides the complete characterization of the general asymptotic marginal coherence transformation, including the case of distillation when ρ' is pure. Intuitively speaking, if the initial state contains nonzero coherence on modes that are coherent in the target state, then an arbitrary rate can be realized. The condition $\mathcal{C}(\rho') \subseteq \mathcal{C}(\rho)$, whether the state has (maybe extremely small but) nonzero coherence or has exactly zero coherence, serves as an extremely sharp and the only threshold separating infinite and zero asymptotic transformation rates.

The diverging rate for the exact transformation shown in Theorem 1 is also remarkable. In fact, asymptotic resource transformation typically comes with a severe restriction when no errors are allowed, and therefore much less is known for exact transformation compared to transformation with a vanishing error. Our result presents a rare scenario in which exact transformation realizes an outstanding performance that coincides with the performance of nonzero error transformation.

Correlated-catalytic transformation.—We now consider the correlated-catalytic transformation, where we employ an auxiliary system C called catalyst which does not change its own state between the initial and the final state but helps state conversion in system S. We say that ρ is convertible to ρ' through correlated-catalytic transformation if there exists a finite-dimensional catalytic system C with a catalyst state c, and a covariant operation Λ on SC such that $\tau = \Lambda(\rho \otimes c)$ with $\mathrm{Tr}_S[\tau] = c$ and $\mathrm{Tr}_C[\tau] = \rho'$. Our final state may have a correlation between the system and the catalyst, which reflects the name "correlated catalyst."

We investigate covariant operations with a correlated catalyst. Recent studies have revealed a severe limitation for correlated-catalytic covariant operations, called the coherence no-broadcasting theorem [21,22]. This theorem states that a fully incoherent initial state is convertible only to an incoherent state through a covariant operation even with the help of a correlated catalyst. This may suggest that a correlated catalyst offers little advantage in state convertibility with quantum coherence. However, we show exactly the opposite—correlated catalysts allow enormous operational power to most covariant state conversions, and the only exception is the case with no coherence in the initial state.

Theorem 2.—For arbitrary states ρ and ρ', ρ is convertible to ρ' with a correlated catalyst with an arbitrarily small error if $\mathcal{C}(\rho') \subseteq \mathcal{C}(\rho)$, and the transformation can be made exact if ρ' is full rank. In addition, the correlation between the system and catalyst can be made arbitrarily small.

This shows that a correlated catalyst enables an arbitrary coherence amplification—an almost incoherent state can be transformed to an almost maximally coherent state with a correlated catalyst [Fig. 1(b)], solving the conjecture in Ref. [31] in the affirmative. Similarly to the case of asymptotic transformation, the only meaningful distinction lies in whether the state has nonzero coherent modes or not.

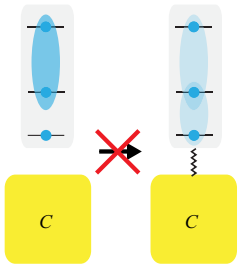

FIG. 2. Suppose that a mode has no coherence in the initial state. Then even if the initial state has coherence on other modes irrationally related to the mode in interest, a covariant operation with a correlated catalyst cannot provide coherence on this mode. This restriction is stronger than the coherence no-broadcasting theorem [21,22].

Notably, the correlation between the system and the catalyst can be made arbitrarily small, implying that the final state τ is extremely close to a product state of $\rho' \otimes c$.

By choosing the initial state as $\rho^{\otimes n}$ and the target state as $\phi^{\otimes rn}$ for a coherent state ρ and a pure coherent state ϕ, there exists a correlated catalyst that enables the transformation from $\rho^{\otimes n}$ to $\phi^{\otimes rn}$ with an arbitrarily small error for every n and r. This setup corresponds to the standard (not marginal) asymptotic distillation, i.e., the error in the final state is measured for the entire state $\phi^{\otimes rn}$, assisted by correlated catalysts. Noting that the standard asymptotic distillation rate $R(\rho \to \phi)$ without a catalyst is zero for every full-rank state ρ [7], our result gives the first example for which the catalyst improves the asymptotic transformation rate, resolving the open problem raised in Ref. [11].

As for the converse, we expect that $\mathcal{C}(\rho') \subseteq \mathcal{C}(\rho)$ also gives the necessary condition for the transformation to exist. Here, we give a partial result toward the full solution to this problem. As naturally guessed, coherence in the initial state would not be helpful in creating coherence on the mode that is only irrationally related to the resonant coherent modes. To formalize this, we introduce $\mathcal{C}'$, which is an extension of $\mathcal{C}$ to rational coefficients: $\mathcal{C}'(\rho) := \{x | x = \sum_{i,j(\rho_{ij}\neq 0)} n_{ij}\Delta_{ij}, n_{ij} \in \mathbb{Q}\}$. We then obtain the necessary condition for the approximate correlated-catalytic covariant transformation (Fig. 2).

Theorem 3.—For two states ρ and ρ' such that $\mathcal{C}'(\rho') \not\subseteq \mathcal{C}'(\rho)$, there does not exist a correlated-catalytic covariant transformation from ρ to ρ'.

This result can be understood as *mode no-broadcasting*—new coherent modes cannot be created by a covariant operation with a correlated catalyst. This contains the coherence no-broadcasting theorem as a special case with $\mathcal{C}'(\rho) = \{0\}$ and extends it to the case of coherent initial states. We conjecture that the above condition is strengthened to $\mathcal{C}(\rho') \not\subseteq \mathcal{C}(\rho)$, which would provide the exact characterization of the feasible coherence transformation with a correlated catalyst together with Theorem 2.

Proof sketch.—Here, we provide a proof sketch of our main results. The complete proofs are presented in the Supplemental Material [32].

We first outline the proof of the achievable part of Theorem 1. Our protocol employs another operational framework known as marginal-catalytic transformations [31] (see also Ref. [24]). In particular, it was shown that for any full-rank state ρ' there exists a set $C_1, \ldots, C_N$ of catalytic systems with states $c_1, \ldots, c_N$ and a covariant operation Λ: $S \otimes C_1 \otimes \cdots \otimes C_N \to S' \otimes C_1 \otimes \cdots \otimes C_N$ such that $\Lambda(c_1 \otimes \cdots \otimes c_N) = \tau$ with $\mathrm{Tr}_{C_1,\ldots,C_N}[\tau] = \rho'$ and $\mathrm{Tr}_{\setminus C_i}[\tau] = c_i$ for all $i = 1, \ldots, N$. Furthermore, these catalysts are partially reusable: If we have N^k sets of catalysts $c_1 \otimes \cdots \otimes c_N$, an appropriate recombination of them allows one to prepare $(k+1)N^k$ copies of ρ' with these marginal catalysts.

We now construct our protocol, which is inspired by Ref. [9]. We first show that a set $c_1 \otimes \cdots \otimes c_N$ of catalysts can be prepared exactly from $\rho^{\otimes\mu}$ by a covariant operation for some integer μ. This allows us to transform μN^k copies of ρ into N^k sets of catalysts. Using these catalysts, we obtain $(k+1)N^k$ copies of ρ' by a marginal catalytic covariant transformation, after which we discard the catalytic systems. The transformation rate is $(k+1)/\mu$, which can be made arbitrarily large by setting sufficiently large k. This transformation can be made exact for a full-rank target state ρ' by employing the result in Ref. [18].

The converse part of Theorem 1 can be shown by utilizing the properties of the modes of asymmetry [83].

Theorem 2 can be obtained by applying the well-known technique to derive correlated-catalytic convertibility from asymptotic convertibility with vanishing error. This type of result was first shown in Ref. [16] in the context of quantum thermodynamics (cf. Refs. [84,85] for exact asymptotic transformation), and a general form of statement is explicitly shown and proven in Ref. [31]. In particular, this construction was recently used to convert asymptotic marginal transformation to correlated-catalytic transformation [18].

We finally outline the proof of Theorem 3. We suppose contrarily that the final state ρ' has coherence on a mode $\Delta E \notin \mathcal{C}'(\rho)$ and derive contradiction with the coherence no-broadcasting theorem [21,22]. Let $L(\Delta)$ be an infinite-dimensional system whose energy levels form a ladder with energy interval Δ. We embed the main system S and catalytic system C into a product of ladder systems $L(\Delta_0) \otimes L(\Delta_1) \otimes L(\Delta_2) \otimes \cdots$ such that Δ_0 is an integer multiple of ΔE ($\Delta E = m\Delta_0$ for some integer m), and the set $\{\Delta_0, \Delta_1, \Delta_2, \ldots\}$ are rational-linearly independent. By assumption, ρ' has coherence in $L(\Delta_0)$, for which ρ is incoherent. For brevity, we abbreviate the set $\Delta_1, \Delta_2, \ldots$ as $\tilde{\boldsymbol{\Delta}}$.

Our key observation is that since a covariant operation acts on each rational-linearly independent mode separately, if $\rho \otimes c$ on $L(\Delta_0) \otimes L(\tilde{\boldsymbol{\Delta}})$ can be transformed to τ by a covariant operation, the same transformation is also possible

on systems with another arbitrary set $\tilde{\boldsymbol{\Delta}}' = (\Delta_1', \Delta_2', \ldots)$. Namely, $\rho \otimes c \to \tau$ on $L(\Delta_0) \otimes L(\tilde{\boldsymbol{\Delta}})$ (the same density matrix on ladders with different energy spacings) is possible by a covariant operation. Setting $\tilde{\boldsymbol{\Delta}}' = \mathbf{0}$ in the above modification, where all states outside $L(\Delta_0)$ are degenerate, we find that ρ on $L(\Delta_0) \otimes L(\mathbf{0})$ is completely incoherent. On the other hand, the final state ρ' has coherence in $L(\Delta_0)$, which contradicts the coherence no-broadcasting theorem.

Discussion.—We showed the anomalous potential of the manipulation of quantum coherence in the asymptotic and catalytic coherence distillation. These results are highly special to quantum coherence that cannot be seen in other resource theories such as entanglement [86,87], quantum thermodynamics [19], and speakable coherence [26,27,88] (see Sec. V in the Supplemental Material [32]). Related to this, we stress that our result is different from the well-known embezzlement phenomena observed in several resource theories [89,90], admitting arbitrary state conversions by allowing a small error in a catalyst. Our framework allows no errors in the catalyst, and thus the operational capability comes from an entirely different mechanism.

Our results shed light on the power of correlation in resource manipulation. In fact, without correlation, amplification of coherence is impossible in both asymptotic and catalytic settings. The importance of correlation has already been discussed intensively in the context of quantum thermodynamics [91–94]. Quantum thermodynamics with an uncorrelated catalyst has many restrictions with Rényi entropies in state convertibility [84,89,95,96], while most of the restrictions are lifted by proper use of correlations, and only the second law of thermodynamics with the relative entropy remains [14,16]. For the coherence transformation, previous studies [24,31] showed an astonishing operational power enabled by correlations between multiple catalysts. Our results confirm that the unbounded power of coherence transformation is also present in the setting with much more operational motivation—asymptotic and correlated-catalytic coherence transformation—lifting quantum coherence as an even more tangible operational resource.

Note added.—During the completion of our manuscript, we became aware of an independent related work by Kondra *et al.* [97], which was concurrently posted to arXiv with ours. Also, an anonymous referee of the QIP conference notified us that when ρ' is pure and the period (the minimum time after which the state returns to the original one) for ρ and ρ' coincide, one can also obtain the diverging asymptotic marginal transformation rate (with an arbitrary small error) by generalizing the construction for sublinear coherence distillation in Ref. [7] [Supplementary Note 7] to the case of marginal asymptotic conversion. This approach, which is different from ours, in fact admits a larger target state, up to the size sublinear in the number of copies of ρ. Our Theorem 1, on the other hand, applies to the fully general setting and contains further insights into the possibility of exact transformation and fundamental limitations imposed by the resonant coherence modes. We thank the referee for their insightful comments.

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