Gottesman-Kitaev-Preskill Encoding in Continuous Modal Variables of Single Photons

Éloi Descamps,[1] Arne Keller,[1,2] and Pérola Milman[1,*]

[1]*Laboratoire Matériaux et Phénomènes Quantiques, Université Paris Diderot, CNRS UMR 7162, 75013 Paris, France*
[2]*Department de Physique, Université Paris-Saclay, 91405 Orsay Cedex, France*

(Received 23 October 2023; accepted 28 March 2024; published 26 April 2024; corrected 17 May 2024)

GKP states, introduced by Gottesman, Kitaev, and Preskill, are continuous variable logical qubits that can be corrected for errors caused by phase space displacements. Their experimental realization is challenging, in particular, using propagating fields, where quantum information is encoded in the quadratures of the electromagnetic field. However, traveling photons are essential in many applications of GKP codes involving the long-distance transmission of quantum information. We introduce a new method for encoding GKP states in propagating fields using single photons, each occupying a distinct auxiliary mode given by the propagation direction. The GKP states are defined as highly correlated states described by collective continuous modes, as time and frequency. We analyze how the error detection and correction protocol scales with the total photon number and the spectral width. We show that the obtained code can be corrected for displacements in time-frequency phase space, which correspond to dephasing, or rotations, in the quadrature phase space and to photon losses. Most importantly, we show that generating two-photon GKP states is relatively simple, and that such states are currently produced and manipulated in several photonic platforms where frequency and time-bin biphoton entangled states can be engineered.

DOI: 10.1103/PhysRevLett.132.170601

Derived from classical error-correction protocols, quantum error correction plays a central role in quantum information theory. The counterintuitive features of quantum mechanics are inherently fragile and necessitate error correction to enable the manifestation of the quantum advantage of protocols over their classical counterparts. Originally devised for qubits and finite-dimensional discrete systems [1,2], quantum error-correcting codes rely on creating redundant states composed of multiple physical qubits to define logical qubits. By measuring well-chosen observables that do not affect the state of the logical qubit —the stabilizers—one can detect and correct errors affecting the physical qubits. Entanglement is usually an important ingredient in error correction, as exemplified by the states that withstand physical qubit flips and dephasing [3,4].

When dealing with continuous variables such as position and momentum, it is also possible to encode quantum information into states that can be corrected for errors [5–8]. One of the most successful models for error correction in continuous variables is known as the GKP code [9] (see, for instance, [10] for a review) named after its creators: Gottesman, Kitaev, and Preskill. GKP states are a direct extension of the discrete variable codes into the continuous domain [11,12] and are correctable for errors modeled as displacements in phase space. In the realm of quantum optics, one usually thinks of encoding GKP states using two orthogonal quadratures of the electromagnetic field, since the associated observables obey the same commutation relation as position and momentum. In such a system, phase space displacements can be the consequence of unwanted interference with a parasite classical field or model photon losses [13,14]. However, a main difficulty consists of the experimental production of such highly nonclassical states, which involves, for instance, the prior (nondeterministic) production of Schrödinger catlike states (or, in practice, kittens) that are made to interfere [15]. Several proposals exist [16–21], as well as a first experimental realization [22]. Of course, one can also encode GKP states using other bosonic systems as superconducting circuits [23] or the motional states of trapped ions [24,25]. However, building a robust code adapted to propagating fields is clearly of major importance if one wills to transmit quantum information, in particular, to several independent users [26].

In the present Letter, we propose a new way to encode GKP states in quantum optics using an original approach to continuous variables. We use continuous collective variables of single photons occupying distinct auxiliary modes, as the propagation direction [27] to define redundant states. Our model can apply to different single photon continuous modes, as time and frequency (that we discuss in detail), the transverse position and momentum [28], the propagation direction [29,30], and also to the collective modes of massive particles, as the normal modes of trapped ions [31,32]. Using the established formalism, we theoretically investigate how GKP code words can be defined in a system comprising n individual photons, elucidating how some known properties of the encoded states can be retrieved, as for instance, the scaling of the error rate with the number of photons, their possibility to recover from photon losses, and the effects of imperfect preparation and measurement. Notably, we show that in the time-frequency (TF) encoding scheme, these properties have a fundamentally different physical origin compared to quadrature-based (QB) encoding. Finally, we demonstrate that the generation and manipulation of TF GKP states in quantum optics is already a reality in laboratories. This is particularly evident in experimental setups where entangled photon pairs with a correlated comblike temporal or spatial structure are observed, as exemplified in Refs. [26,33–45]. Consequently, these experimental platforms and the associated quantum protocols can immediately benefit from our findings.

We define as $\mathcal{S}_n$ the subspace consisting of n single photons occupying each an auxiliary mode, as the propagation direction. Photons are characterized by a collective spectral function that also depends on the auxiliary mode. The auxiliary modes can be seen as external degrees of freedom and frequency as internal degrees of freedom [46]. Pure states are written as

$$|\psi\rangle = \int d\omega_1 \ldots d\omega_n f(\omega_1, \ldots, \omega_n)|\omega_1, \ldots, \omega_n\rangle, \quad (1)$$

where f is a normalized function (the spectral amplitude) that determines the properties of state (1) as entanglement and its mode decomposition [47,48], and $\hat{a}_i^\dagger(\omega_i)|0\rangle = |\omega_i\rangle$. In $\mathcal{S}_n$, errors are represented in the basis of time and frequency displacements, and they can affect the photons locally; i.e., they act independently on the photons of each auxiliary mode, a situation similar to the one affecting a collection of physical qubits. Such displacements are described by operators acting on a given mode $j \in \{1, \ldots, n\}$ as $\hat{D}_{\hat{\omega}_j}(\delta_{t_j}) = e^{-i\hat{\omega}_j \delta_{t_j}}$ and $\hat{D}_{\hat{t}_j}(\delta_{\omega_j}) = e^{-i\hat{t}_j \delta_{\omega_j}}$, where $\hat{\omega}_i = \int d\omega \omega \hat{a}_i^\dagger(\omega)\hat{a}_i(\omega)$ and $\hat{t}_i = \int dt t \tilde{\hat{a}}_i^\dagger(t)\tilde{\hat{a}}_i(t)$, with $\tilde{\hat{a}}_i(t) = (1/\sqrt{2\pi}) \int d\omega e^{i\omega t}\hat{a}(\omega)$, where $\hat{a}_i^\dagger(\eta)$ creates one photon at frequency η at the ith auxiliary mode and $[\hat{\omega}_k, \hat{t}_j] = i\delta_{k,j} \int d\omega \hat{a}_k^\dagger(\omega)\hat{a}_k(\omega) = \hat{n}_k i\delta_{k,j}$ ($\hat{n}_k = \mathbb{1}$ on $\mathcal{S}_n$) [27,49]. Now we show that entangled states of n photons in the TF continuous variables sharing the same properties of GKP states can be corrected for this type of error. Such states can be written in the general separable form in collective variables:

$$|\bar{k}\rangle = \int d\Omega_1 \ldots d\Omega_n F_k(\Omega_1) \Pi_{i>1}^n G_i(\Omega_i)|\omega_1, \ldots, \omega_n\rangle, \quad (2)$$

where $\Omega_j = \sum_i^n \alpha_{i,j}\omega_i$ are collective variables, $\alpha_{i,j}$ is an invertible matrix with $\alpha_{i,j} \in \{-1/\sqrt{n}, 1/\sqrt{n}\}$, and $k \in \{0, 1\}$. We consider for simplicity and without loss of generality that $\alpha_{i,1} = 1/\sqrt{n} \ \ \forall \ \ i$, and that $n = 2^m$, $m \in \mathbb{N}$. The matrix α is unitary and symmetric, hence, $\omega_i = \sum_j \alpha_{j,i}\Omega_j$.

An ideal n photon GKP state $|\bar{k}\rangle$ in $\mathcal{S}_n$ can be defined from (2) using $F_k(\Omega_1) = \sum_{s=-\infty}^{s=\infty} \delta(\Omega_1 - (2s + k)\Omega_o)$, where δ is the Dirac delta function and Ω_o is an arbitrary (constant) frequency. Hence, the logical qubits $|\bar{0}(\bar{1})\rangle$ are nonphysical states formed by an infinity of peaks localized at frequencies which are integer multiples of Ω_o, and $2\Omega_o$ is the peak interspacing in each logical qubit (the choice of $\alpha_{i,j}$ means that we have supposed that all the photons' frequencies are equally spaced [50]). States $|\bar{k}\rangle$ are defined uniquely using the collective variable Ω_1. The functions G_i are arbitrary, and their role, not crucial for the code working principles, will be discussed later in this Letter. Thus, all the relevant information for error diagnosis and correction is contained only in variable Ω_1, and we disregard the information contained in $\Omega_{i>1}$. This type of situation is current in quantum optics where different physical properties, as group and phase velocity for multimodal fields, are associated with different collective variables. For instance, in the Hong-Ou-Mandel experiment [51] (see Refs. [52,53] for its generalization to many photons), the variable $\Omega_1 = (\omega_1 - \omega_2)/\sqrt{2}$ is directly measured, while the information in variable $\Omega_2 = (\omega_1 + \omega_2)/\sqrt{2}$ is disregarded [54]. By combining different interferometric techniques [55], one can access different collective variables measuring not only frequency but other continuous modes, as the transverse position and momentum [28,39].

TF GKP states are intrinsically multimode states relying on the particle-mode nonseparability, so they are fundamentally different from optical combs in single-mode states using spectral engineering of classical (coherent) states or single photons [33,56]. Thanks to the encoding in collective variables (modes) of individual photons, TF GKP states reveal their multiphotonic properties as the scaling of the error tolerance with the number of photons n. Hence, they are also fundamentally distinct from QB GKP states—that can be defined in single modes—even in their multidimensional version [57].

An example of a possible $|\bar{k}\rangle$ state with $G_i(\Omega_i) = \delta(\Omega_i) \ \ \forall \ \ i$ in (2) is [58]

$$|\bar{k}\rangle = \sum_{s=-\infty}^{\infty} \left|(2s + k)\frac{\Omega_o}{\sqrt{n}}\right\rangle_1 \ldots \left|(2s + k)\frac{\Omega_o}{\sqrt{n}}\right\rangle_n. \quad (3)$$

Analogous to the QB GKP states, we can identify non-Hermitian operators that act in $|\bar{k}\rangle$ as Pauli matrices [33]. One way to see this is using displacements in the collective TF variables $\hat{D}_{\hat{T}_1}(\Delta_\omega) = e^{-i\hat{T}_1 \Delta_\omega}$ and $\hat{D}_{\hat{\Omega}_1}(\Delta_t) = e^{-i\hat{\Omega}_1 \Delta_t}$, where $\Delta_{t(\omega)} \in \mathbb{R}$, $\hat{\Omega}_1 = \sum_i^n \hat{\omega}_i/\sqrt{n}$, and $\hat{T}_1 = \sum_i^n \hat{t}_i/\sqrt{n}$, with $[\hat{\Omega}_1, \hat{T}_1] = \mathbb{1}i$ in $\mathcal{S}_n$ [49]. Collective operators can also be associated with variables $\Omega_{i>1}$. By an appropriate choice of $\Delta_{t(\omega)}$, we can define the Pauli-like operators in the TF GKP subspace as $\hat{X} = e^{-i\hat{\Omega}_1 T_o}$ and $\hat{Z} = e^{-i\hat{T}_1 \Omega_o}$, with $T_o = \pi/\Omega_o$ and $\hat{Y} = i\hat{Z}\hat{X}$ so $|\bar{1}\rangle = \hat{X}|\bar{0}\rangle$. Also, a combination of the universal time-frequency gates defined and physically described in [49] can be used to complete the universal gate set in the TF GKP space [59]. A key aspect of the proposed encoding is that the construction of the logical operators $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ (and consequently of the TF GKP universal gate set) is not unique. Nevertheless, the formal construction of the TF GKP code is identical to the QB one, so all the properties of the latter can be retrieved here, but now associated with different continuous variables. States (3) enable correcting for collective time and frequency errors corresponding to TF displacements such that $|\Delta_\omega| < \Omega_o/2$ and $|\Delta_t| < \pi/(2\Omega_o)$, delimiting a TF phase space area of correctable errors satisfying $4|\Delta_\omega||\Delta_t| < \pi$. Physically, displacement errors can correspond to imperfect aligning of an interferometer and the effects of a nonlinear device as an optical fiber.

Restricting to collective errors is missing our main goal, which is creating states which are robust against local displacements in the TF variables of each photon. Physically, this corresponds to a situation where each photon occupying a different propagation mode is distributed throughout different channels, and frequency and time displacements occur independently in each mode (and consequently, each photon). Such TF displacements are rotations in the quadrature phase space, a type of error against which the usual QB GKP is not very efficient [10] and require using rotation-symmetric states in the quadrature space [61]. As for states (3), they are robust against global and local rotations by construction. We now study the effect of local noise in the TF GKP states (2). We have

$$\hat{D}_{\hat{\omega}_j}(\delta_{t_j})|\bar{k}\rangle = \int d\Omega_1 \ldots d\Omega_n e^{-i\frac{\Omega_1 \delta_{t_j}}{\sqrt{n}}} F_k(\Omega_1) \times \prod_{k>1}^{n} \tilde{G}_k(\Omega_k)|\omega_1, \ldots, \omega_n\rangle, \tag{4}$$

where $\tilde{G}_k(\Omega_k) = e^{-i\frac{\alpha_{j,k}\Omega_k \delta_{t_j}}{\sqrt{n}}} G_k(\Omega_k)$. In addition,

$$\hat{D}_{\hat{t}_j}(\delta_{\omega_j})|\bar{k}\rangle = \int d\Omega_1 \ldots d\Omega_n F_k\left(\Omega_1 - \frac{\delta_{\omega_j}}{\sqrt{n}}\right) \times \prod_{k>1}^{n} G_k(\Omega_k - \bar{\Omega}_k)|\omega_1, \ldots, \omega_n\rangle, \tag{5}$$

where $\bar{\Omega}_k = \alpha_{j,k}\delta_{\omega_j}$. Equations (4) and (5) lead to an important result: The code words (3) protect against shifts in local variables ω_j in a way that scales with $\sqrt{n}$ with the number of photons n. Some examples of correctable errors are then a single photon in mode j that is displaced by $|\delta_{t_j}| < \sqrt{n}\pi/(2\Omega_o)$, up to $\approx\sqrt{n}$ photons in different modes j that are each displaced by $|\delta_{t_j}| < \pi/(2\Omega_o)$, and n photons that are each equally displaced by $|\delta_{t_j}| < \pi/(2\Omega_o\sqrt{n})$. Thus, if one focuses on local errors, the code words (3) can protect them provided that they lie in a phase space area of size $4|\sum_j^n \delta_{\omega_j}||\sum_j^n \delta_{t_j}| < n\pi$ [62]. We can see this as a rescaling of the code, since the overall protection corresponds to the one of a single-photon GKP state formed by peaks that are distant of $2\sqrt{n}\Omega_o$ and $\sqrt{n}/(2\Omega_o)$ in time and frequency variables, respectively. Interestingly, contrary to what one would observe in a classical time-frequency Fourier relation where the dilatation of the frequency space is accompanied by the shrinking of the time space and vice versa, the observed effective phase space dilatation is a geometric consequence of encoding information in collective variables while errors occur in local ones. This rescaling leads to a photon number dependency of the probability error rate analogous to the one observed for QB GKP encoding, as we will see later.

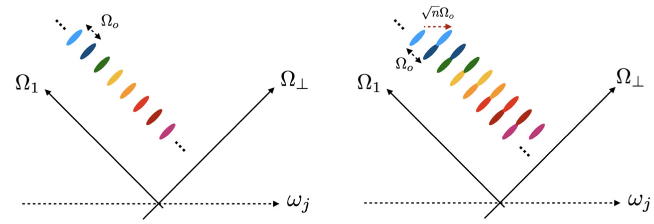

FIG. 1. Left: physical TF GKP state with peak spacing $2\Omega_o$ in the collective variable Ω_1. $\Omega_\perp$ is a collective variable orthogonal to Ω_1. Right: a displacement of $2\sqrt{n}\Omega_o$ in the local variable ω_j also displaces variable Ω_1. The displacement in the orthogonal direction $\Omega_\perp$ is not relevant since we ignore the information it contains.

Operators $\hat{X}$ and $\hat{Z}$ are not unique. We can define $\hat{X}_j = e^{-i\hat{\omega}_j T_o \sqrt{n}}$ and check by computing $\hat{X}_j|\bar{k}\rangle$ that $\hat{X}_j$ acts in variables Ω_1 in the same way as $\hat{X}$ does (see Fig. 1 and [59]). Using this, we can detect the loss of one photon in mode j' (unknown) and adapt to its effects by measuring the time and frequency displacements only. If a photon is lost, the TF GKP state becomes $|\bar{k}\rangle_{-1} = \int d\omega \hat{a}_j(\omega)|\bar{k}\rangle = \hat{\mathcal{E}}_j|\bar{k}\rangle$ (we considered that the photon loss rate is independent of the frequency [63]). Defining $\hat{S}_j = e^{-i\eta_j \Omega_o \hat{t}_j} \hat{X}_j^2 e^{i\eta_j \Omega_o \hat{t}_j} = e^{-i2(\hat{\omega}_j - \hat{n}_j \eta_j \Omega_o) T_o \sqrt{n}}$, $\eta_j \in \mathbb{R}$, and only considering the information in Ω_1, we have that $\hat{S}_j \hat{S}_{j+1}$ stabilizes $|\bar{k}\rangle$ in $\mathcal{S}_n$ if $(\eta_j + \eta_{j+1})\sqrt{n} = m$, $m \in \mathbb{Z}$, for all j, and it stabilizes $\hat{\mathcal{E}}_{j'}|\bar{k}\rangle = |\bar{k}\rangle_{-1}$ if $j' \neq j, j+1$. In addition, $\hat{S}_j \hat{S}_{j+1} \hat{\mathcal{E}}_j |\bar{k}\rangle = \hat{\mathcal{E}}_j \hat{S}_{j+1} |\bar{k}\rangle = e^{-2i\eta_j \pi \sqrt{n}} \hat{\mathcal{E}}_j |\bar{k}\rangle$, so by judiciously choosing η_j we can detect a photon loss and the mode from which it was lost (j or $j+1$ here). We can also define $\hat{S} = \Pi_j \hat{S}_j$, which is a stabilizer of $|\bar{k}\rangle$. Using that $\hat{S}\hat{\mathcal{E}}_j|\bar{k}\rangle = e^{-i\eta_j \pi \sqrt{n}} \hat{\mathcal{E}}_j |\bar{k}\rangle$, we can detect in a single shot that a photon has been lost and from which mode, by judiciously choosing η_j's and if $\sum_{j=1}^n \eta_j \sqrt{n} = m$ [59]. The stabilizer measurement provides essential information about the mode that lost a photon, permitting us to adapt the operations and measurements to an $n-1$ photon configuration: If a photon is lost, the collective effects in displacements are smaller, and in order to have $e^{2i\hat{\Omega}_1 T'_o}|\bar{k}\rangle_{-1} = |\bar{k}\rangle_{-1}$, we must use $T'_o = T_o n/(n-1)$. One can also reinsert the lost photon by applying a two-photon conditional gate involving mode j and an arbitrary mode i' in the code, and a displacement, so that $\hat{D}_{\hat{T}_1}(-\omega_j)e^{i\hat{t}_j\hat{\omega}_{i'}}\hat{a}_j^\dagger(\omega_j)|\bar{k}\rangle_{-1} = |\bar{k}\rangle$ [49,59,64].

States (3) are not physical. We can define their normalizable version $|\tilde{0}(\tilde{1})\rangle$ using $\tilde{F}_k(\Omega_1) = \sum_{s=-\infty}^{s=\infty} e^{-\kappa^2(2s+k)^2\Omega_o^2} e^{-\frac{[\Omega_1-(2s+k)\Omega_o]^2}{\Delta^2}}$, $k \in \{0,1\}$, where each peak of the TF GKP code has a Gaussian spectrum of width $\Delta(\ll \Omega_o)$ in variable Ω_1, and the comb of peaks distribution is modulated by a Gaussian envelope of width $\kappa^{-1} \ll T_o/\pi$ [9]. We will consider for simplicity that $\Delta = \kappa$. A finite width provides an intrinsic error probability to states $|\tilde{0}(\tilde{1})\rangle$, seen as perfect states (3) that have been subjected to a distribution of displacements (errors). Hence, an error probability $\mathcal{E}(\Delta/\Omega_o)$ is associated with the error-correction protocol through the definition of nonperfectly orthogonal states as code words. The finite spectral width can be modeled as independent displacements of individual photons with a Gaussian amplitude distribution of width Δ_j. Each photon j in (3) is described by state

$$\left|\widetilde{(2s+k)}\frac{\Omega_o}{\sqrt{n}}\right\rangle_j = \int d\omega e^{-\Delta_j^2(2s+k)^2\left(\frac{\Omega_o}{\sqrt{n}}\right)^2} e^{-\frac{\left(\omega-\frac{(2s+k)\Omega_o}{\sqrt{n}}\right)^2}{\Delta_j^2}}|\omega\rangle_j. \quad (6)$$

The local variables ω_i behave as independent random variables, and we can suppose $\Delta_j = \Delta$. Each n photon peak has the form $\Pi_j^n|\widetilde{(2s+k)(\Omega_o/\sqrt{n})}\rangle_j$, and the total state's temporal envelope is Δ^{-1}. By changing to the collective variables Ω_i, each peak is described by a Gaussian spectral distribution of width Δ in all variables Ω_i, leading to $\tilde{F}_k(\Omega_1)$ shown above. Hence, different from the QB GKP encoding, the spectral width does not depend on the average photon number, and the probability of mistaking $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ is given by $\mathcal{E}(\Delta/\Omega_o) = (\Delta/\pi\Omega_o)e^{\frac{-\pi\Omega_o^2}{4\Delta^2}}$ [9]. We recall however that in the present encoding the effective displacement on the collective variable Ω_1 decreases with $\sqrt{n}$, increasing the effective distance between peaks. Consequently, the region of potential overlap between the two supposedly orthogonal QB GKP qubit states is modified [65].

By analyzing the effects of photon losses discussed above for physical TF GKP states, we see that the loss of m photons will not significantly affect the code if the cumulated effective peak interspacing modification $T_o m/n$ [59] in the whole state [$\approx 1/(2\Delta T_o)$ peaks] is within each peak's half-width $\Delta/2$, leading to the condition $2T_o m/n \times 1/(2\Delta T_o) = m/(\Delta n) \leq \Delta$, or $n/m \geq 1/\Delta^2 \gg \pi/(\Omega_o T_o) = 1$. These errors propagate with the number of gates, and a detailed analysis should be carried out, but displacement-based correction strategies can be devised based on this scaling [59].

We now discuss the role of a finite frequency width in the collective variables $\Omega_{j>1}$. For simplicity, we will consider $\Omega_\perp$ to be one of these variables and ignore all the others, considering a state with spectral amplitude $F_k(\Omega_1)G(\Omega_\perp)$. The width of the spectral distributions F_k and G in (2) can be independent and related to different physical constraints, as, for instance, energy conservation and the phase matching condition in spontaneous parametric down-conversion. If the spectral function of state (2) is separable in variables Ω_1 and $\Omega_\perp$ and all the measurements performed on variable Ω_1, the spectral width σ or the particular shape of G have no importance. However, state preparation may be imperfect, leading to a state that is still separable but in the variables $\Omega'_1 = \cos\theta\Omega_1 + \sin\theta\Omega_\perp$ and $\Omega'_\perp = \cos\theta\Omega_\perp - \sin\theta\Omega_1$. This model also describes the situation of imperfect measurements, where variable Ω'_1 is measured instead of Ω_1. In these cases, the peaks' width in the measured variable is broadened by an additive factor $\sigma\sin\theta$, and the peak spacing is rescaled to $2\Omega_o\cos\theta$ (for details and a figure, see Ref. [59]). This effective width and peak spacing can be seen as errors that do not significantly affect the code if the cumulated change in peak spacing $\Omega_o(1-\cos\theta)[1/(2\Omega_o\Delta)]$ lies within the peak's half-width $\Delta/2$, or $(1-\cos\theta) \lesssim \Delta^2 \ll \Omega_o T_o/\pi = 1$. Otherwise, we can adapt the code to the modified interspacing $2\Omega_o\cos\theta$ (where θ is known), with an associated error probability $\mathcal{E}(\sigma\tan\theta/\Omega_o)$ in frequency and $\mathcal{E}[\tan\theta/(\sigma T_o)]$ in time, leading to $\tan\theta \ll \min\{\pi\sigma/(2\Omega_o), \Omega_o/(2\sigma)\}$ [59].

Finally, two n photon TF GKP states can be entangled by applying frequency controlled-NOT (CNOT) gates $\hat{C}_{i,j} = e^{i\hat{\omega}_i\otimes\hat{t}_j}$ [33,49], implementing $\hat{C}_{i,j}|\omega_i\rangle_i|\omega_j\rangle_j = |\omega_i\rangle_i|\omega_i + \omega_j\rangle_j$. We define $\hat{\mathcal{D}}_{1,2} = \otimes_{i=1}^n \hat{C}_{(i,1),(i,2)}$ [59], where (i,j) denotes the ith spatial mode of the jth TF GKP qubit, with $j = 1(2)$ for the control (target) qubit. Hence, $\hat{\mathcal{D}}_{1,2}|\bar{k}_1\rangle_1|\bar{k}_2\rangle_2 = |\bar{k}_1\rangle_1|\overline{(k_2+k_1) \bmod 2}\rangle_2$, and using $|\mp\rangle_i = (1/\sqrt{2})(|\bar{0}\rangle_i + |\bar{1}\rangle_i)$, we obtain $\hat{\mathcal{D}}_{1,2}|\mp\rangle_1|\bar{0}\rangle_2 = 1/\sqrt{2}(|\bar{0}\rangle_1|\bar{0}\rangle_2 + |\bar{1}\rangle_1|\bar{1}\rangle_2)$. Interestingly, it is also possible to implement the CNOT gate between two TF GKP qubits by coupling only one photon from the target qubit to one photon of the control qubit: Since information is encoded in the collective variables Ω_1 of each qubit, for qubit states as (3), for instance, we also have that $e^{in\hat{\omega}_{(i,1)}\otimes\hat{t}_{(i,2)}}|\bar{k}_1\rangle_1|\bar{k}_2\rangle_2 = |\bar{k}_1\rangle_1|\overline{(k_2+k_1) \bmod 2}\rangle_2$ (see Ref. [59] for details). In [64,66], frequency controlled two-photon gates were experimentally implemented, and promising proposals exist for cavity QED platforms; e.g., [67].

We can compare our results to other encodings based on GKP-like states within the continuous modes of single photons. In the TF domain, it is possible to define GKP qubits using a frequency comb spectral distribution in single photons. In that case, one photon corresponds to one qubit, and the spectral function of each photon defines a two-level-like system that exhibits local robustness against TF displacements [33,68]. This is a single-mode classical-like effect independent of the number of photons involved, and if the photon's peaks are separated by $2\Omega_o/\sqrt{n}$ [as in (3)], the protection against displacement errors is limited to amplitudes $\delta_{\omega_j} \sim \Omega_o/(2\sqrt{n})$ per photon. Of course, it is possible to enhance protection against local (single-photon) quibit flipping and dephasing by using entangled photons. However, the so-constructed codes operate similarly to discrete ones [1–4], and correction for qubit flip and dephasing requires using entangled states of at least five photons. In contrast, the encoding proposed here demonstrates enhanced protection starting from $n = 2$.

Based on our analysis, we can reinterpret the results of [33] as the production of a two-photon GKP state. While it remains a GKP state on a small scale, the existing techniques for producing QB GKP states involve the manipulation of Schrödinger kittens with a mean photon number of the order of 1. Therefore, TF GKP states hold significant promise: Techniques to directly generate large entangled states [42,69–75] and high-dimensional combs [76] to manipulate photons using nonlinear devices so as to implement single mode [77–79] or two-mode controlled [64,66,67] time-frequency operators in the universal set rapidly develop, together with high-performance frequency (or mode) resolved [80–82] and nondestructive single-photon detectors [83,84].

In conclusion, we have introduced and conducted an extensive study of a novel quantum optical encoding method for GKP states that can be implemented in small scale in many laboratories using current technology. Using the presently available TF GKP states, we can already envision applications in various domains as quantum communications [26,85], quantum computation [86], and quantum metrology [87]. Moreover, we can broaden the scope of potential applications by applying the proposed GKP encoding from the flourishing domain of TF-based quantum photonics [88] to other continuous degrees of freedom of mode entangled photons, such as their transverse position and momentum [28,39], the propagation direction [29,30], and even to individual electrons in distinguishable modes with an underlying bosonic structure [89]. An interesting perspective involves adapting recent theoretical and experimental advancements related to QB GKP states in quantum information to the modal domain [16,90–96]. Finally, the tools we have provided for defining quantum continuous variables using collective variables of single photons can also be extended to other error-correcting codes, such as cat codes [97–101]. However, this remains a subject for future investigation.

We acknowledge funding from the Plan France 2030 through the project ANR-22-PETQ-0006 and N. Fabre, F. Baboux, and S. Ducci for fruitful discussions.

*Corresponding author: perola.milman@u-paris.fr

[1] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, Phys. Rev. A **52**, R2493 (1995).

[2] A. M. Steane, Error correcting codes in quantum theory, Phys. Rev. Lett. **77**, 793 (1996).

[3] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Perfect quantum error correcting code, Phys. Rev. Lett. **77**, 198 (1996).

[4] D. Gottesman, Stabilizer codes and quantum error correction, arXiv:quant-ph/9705052.

[5] I. L. Chuang, D. W. Leung, and Y. Yamamoto, Bosonic quantum codes for amplitude damping, Phys. Rev. A **56**, 1114 (1997).

[6] P. T. Cochrane, G. J. Milburn, and W. J. Munro, Macroscopically distinct quantum-superposition states as a bosonic code for amplitude damping, Phys. Rev. A **59**, 2631 (1999).

[7] S. L. Braunstein, Error correction for continuous quantum variables, Phys. Rev. Lett. **80**, 4084 (1998).

[8] S. Lloyd and Jean-Jacques E. Slotine, Analog quantum error correction, Phys. Rev. Lett. **80**, 4088 (1998).

[9] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).

[10] A. L. Grimsmo and S. Puri, Quantum error correction with the Gottesman-Kitaev-Preskill code, PRX Quantum **2**, 020101 (2021).

[11] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, Quantum error correction and orthogonal geometry, Phys. Rev. Lett. **78**, 405 (1997).

[12] D. Gottesman, Class of quantum error-correcting codes saturating the quantum Hamming bound, Phys. Rev. A **54**, 1862 (1996).

[13] V. V. Albert, K. Noh, K. Duivenvoorden, D. J. Young, R. T. Brierley, P. Reinhold, C. Vuillot, L. Li, C. Shen, S. M. Girvin, B. M. Terhal, and L. Jiang, Performance and structure of single-mode bosonic codes, Phys. Rev. A **97**, 032346 (2018).

[14] A. Joshi, K. Noh, and Y. Y. Gao, Quantum information processing with bosonic qubits in circuit QED, Quantum Sci. Technol. **6**, 033001 (2021).

[15] I. Tzitrin, J. E. Bourassa, N. C. Menicucci, and K. K. Sabapathy, Progress towards practical qubit computation using approximate Gottesman-Kitaev-Preskill codes, Phys. Rev. A **101**, 032315 (2020).

[16] B. W. Walshe, B. Q. Baragiola, R. N. Alexander, and N. C. Menicucci, Continuous-variable gate teleportation and bosonic-code error correction, Phys. Rev. A **102**, 062411 (2020).

[17] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais, K. K. Sabapathy, N. C. Menicucci, and I. Dhand, Blueprint for a scalable photonic fault-tolerant quantum computer, Quantum **5**, 392 (2021).

[18] M. V. Larsen, C. Chamberland, K. Noh, J. S. Neergaard-Nielsen, and U. L. Andersen, Fault-tolerant continuous-variable measurement-based quantum computation architecture, PRX Quantum **2**, 030325 (2021).

[19] Y. Zheng, A. Ferraro, A. F. Kockum, and G. Ferrini, Gaussian conversion protocol for heralded generation of

generalized Gottesman-Kitaev-Preskill states, Phys. Rev. A **108**, 012603 (2023).
[20] M. Eaton, R. Nehra, and O. Pfister, Non-Gaussian and Gottesman-Kitaev-Preskill state preparation by photon catalysis, New J. Phys. **21**, 113034 (2019).
[21] R. Dahan, G. Baranes, A. Gorlach, R. Ruimy, N. Rivera, and I. Kaminer, Creation of optical cat and GKP states using shaped free electrons, Phys. Rev. X **13**, 031001 (2023).
[22] S. Konno, W. Asavanant, F. Hanamura, H. Nagayoshi, K. Fukui, A. Sakaguchi, R. Ide, F. China, M. Yabuno, S. Miki, H. Terai, K. Takase, M. Endo, P. Marek, R. Filip, P. van Loock, and A. Furusawa, Propagating Gottesman-Kitaev-Preskill states encoded in an optical oscillator, Science **383**, 6680 (2024).
[23] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard *et al.*, Quantum error correction of a qubit encoded in grid states of an oscillator, Nature (London) **584**, 368 (2020).
[24] C. Flühmann, T. L. Nguyen, M. Marinelli, V. Negnevitsky, K. Mehta, and J. P. Home, Encoding a qubit in a trapped-ion mechanical oscillator, Nature (London) **566**, 513 (2019).
[25] C. Flühmann, V. Negnevitsky, M. Marinelli, and J. P. Home, Sequential modular position and momentum measurements of a trapped ion mechanical oscillator, Phys. Rev. X **8**, 021001 (2018).
[26] F. Appas, F. Baboux, M. Amanti, A. Lemaître, F. Boitier, E. Diamanti, and S. Ducci, Flexible entanglement-distribution network with an AlGaAs chip for secure communications, npj Quantum Inf. **7**, 118 (2021).
[27] E. Descamps, N. Fabre, A. Keller, and P. Milman, Quantum metrology using time-frequency as quantum continuous variables: Resources, sub-shot-noise precision and phase space representation, Phys. Rev. Lett. **131**, 030801 (2023).
[28] D. S. Tasca, R. M. Gomes, F. Toscano, P. H. Souto Ribeiro, and S. P. Walborn, Continuous variable quantum computation with spatial degrees of freedom of photons, Phys. Rev. A **83**, 052325 (2011).
[29] A. S. Solntsev, F. Setzpfandt, A. S. Clark, C. W. Wu, M. J. Collins, C. Xiong, A. Schreiber, F. Katzschmann, F. Eilenberger, R. Schiek, W. Sohler, A. Mitchell, C. Silberhorn, B. J. Eggleton, T. Pertsch, A. A. Sukhorukov, D. N. Neshev, and Y. S. Kivshar, Generation of nonclassical biphoton states through cascaded quantum walks on a nonlinear chip, Phys. Rev. X **4**, 031007 (2014).
[30] A. Raymond, S. Francesconi, J. Palomo, P. Filloux, M. Morassi, A. Lemaître, F. Raineri, M. Amanti, S. Ducci, and F. Baboux, in *Proceedings of the Conference on Lasers and Electro-Optics/Europe (CLEO/Europe 2023) and European Quantum Electronics Conference (EQEC 2023)*, https://opg.optica.org/abstract.cfm?URI=EQEC-2023-eb_6_6.
[31] A. Steane, The ion trap quantum information processor, Appl. Phys. B **64**, 623 (1997).
[32] D. F. V. James, Quantum dynamics of cold trapped ions with application to quantum computation, Appl. Phys. B **66**, 181 (1998).
[33] N. Fabre, G. Maltese, F. Appas, S. Felicetti, A. Ketterer, A. Keller, T. Coudreau, F. Baboux, M. I. Amanti, S. Ducci, and P. Milman, Generation of a time-frequency grid state with integrated biphoton frequency combs, Phys. Rev. A **102**, 012607 (2020).
[34] G. Maltese, M. I. Amanti, F. Appas, G. Sinnl, A. Lemaître, P. Milman, F. Baboux, and S. Ducci, Generation and symmetry control of quantum frequency combs, npj Quantum Inf. **6**, 13 (2020).
[35] M. Kues, C. Reimer, P. Roztocki, L. Cortés, S. Sciara, B. Wetzel, Y. Zhang, A. Cino, S. Chu, B. Little, D. Moss, L. Caspani, J. Azaña, and R. Morandotti, On-chip generation of high-dimensional entangled quantum states and their coherent control, Nature (London) **546**, 622 (2017).
[36] P. Imany, J. A. Jaramillo-Villegas, O. D. Odele, K. Han, D. E. Leaird, J. M. Lukens, P. Lougovski, M. Qi, and A. M. Weiner, 50-GHz-spaced comb of high-dimensional frequency-bin entangled photons from an on-chip silicon nitride microresonator, Opt. Express **26**, 1825 (2018).
[37] L. Olislager, J. Cussey, A. T. Nguyen, P. Emplit, S. Massar, J.-M. Merolla, and K. P. Huy, Frequency-bin entangled photons, Phys. Rev. A **82**, 013804 (2010).
[38] Y. Tomohiro, R. Ikuta, T. Kobayashi, S. Miki, F. China, H. Terai, N. Imoto, and T. Yamamoto, Massive-mode polarization entangled biphoton frequency comb, Sci. Rep. **12**, 8964 (2022).
[39] M. R. Barros, A. Ketterer, O. J. Farías, and S. P. Walborn, Free-space entangled quantum carpets, Phys. Rev. A **95**, 042311 (2017).
[40] C. L. Morrison, F. Graffitti, P. Barrow, A. Pickston, J. Ho, and A. Fedrizzi, Frequency-bin entanglement from domain-engineered down-conversion, APL Photonics **7**, 066102 (2022).
[41] J. M. Lukens and P. Lougovski, Frequency-encoded photonic qubits for scalable quantum information processing, Optica **4**, 8 (2017).
[42] F. A. Sabattoli, L. Gianini, A. Simbula, M. Clementi, A. Fincato, F. Boeuf, M. Liscidini, M. Galli, and D. Bajoni, A silicon source of frequency-bin entangled photons, Opt. Lett. **47**, 6201 (2022).
[43] X. Zhang, B. A. Bell, A. Mahendra, C. Xiong, P. H. W. Leong, and B. J. Eggleton, Integrated silicon nitride time-bin entanglement circuits, Opt. Lett. **43**, 3469 (2018).
[44] M. Clementi, F. A. Sabattoli, M. Borghi, L. Gianini, N. Tagliavacche, H. el Dirani, L. Youssef, N. Bergamasco, C. Petit-Etienne, E. Pargon, J. Sipe, M. Liscidini, C. Sciancalepore, M. Galli, and D. Bajoni, Programmable frequency-bin quantum states in a nano-engineered silicon device, Nat. Commun. **14** (2023).
[45] F. Kaneda, H. Suzuki, R. Shimizu, and K. Edamatsu, Direct generation of frequency-bin entangled photons via two-period quasi-phase-matched parametric downconversion, Opt. Express **27**, 1416 (2019).
[46] C. Dittel, G. Dufour, G. Weihs, and A. Buchleitner, Wave-particle duality of many-body quantum states, Phys. Rev. X **11**, 031041 (2021).
[47] C. Fabre and N. Treps, Modes and states in quantum optics, Rev. Mod. Phys. **92**, 035005 (2020).
[48] C. K. Law, I. A. Walmsley, and J. H. Eberly, Continuous frequency entanglement: Effective finite Hilbert space and entropy control, Phys. Rev. Lett. **84**, 5304 (2000).

[49] N. Fabre, A. Keller, and P. Milman, Time and frequency as quantum continuous variables, Phys. Rev. A **105**, 052429 (2022).

[50] As a matter of fact, this last point is related to the type of error (3) can correct for and the definition of the collective variables we used. We could adapt these points to states where the peak spacing contribution per photon is different from the chosen one.

[51] C. K. Hong, Z. Y. Ou, and L. Mandel, Measurement of subpicosecond time intervals between two photons by interference, Phys. Rev. Lett. **59**, 2044 (1987).

[52] Z. Y. Ou, Multi-photon interference and temporal distinguishability of photons, Int. J. Mod. Phys. B **21**, 5033 (2007).

[53] T. Douce, A. Eckstein, S. P. Walborn, A. Z. Khoury, S. Ducci, A. Keller, T. Coudreau, and P. Milman, Direct measurement of the biphoton Wigner function through two-photon interference, Sci. Rep. **3**, 3530 (2013).

[54] Notice that this is different from the arbitrary choice we made previously of setting $\alpha_{i,1} = 1/\sqrt{n}\ \forall\ i$.

[55] D. Branning, A. L. Migdall, and A. V. Sergienko, Simultaneous measurement of group and phase delay between two photons, Phys. Rev. A **62**, 063808 (2000).

[56] T. Yamazaki, T. Arizono, T. Kobayashi, R. Ikuta, and T. Yamamoto, Linear optical quantum computation with frequency-comb qubits and passive devices, Phys. Rev. Lett. **130**, 200602 (2023).

[57] B. Royer, S. Singh, and S. M. Girvin, Encoding qubits in multimode grid states, PRX Quantum **3**, 010335 (2022).

[58] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced positioning and clock synchronization, Nature (London) **412**, 417 (2001).

[59] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.170601 for details, which includes Ref. [60].

[60] K. M. Jordan, R. A. Abrahao, and J. S. Lundeen, Quantum metrology timing limits of the Hong-Ou-Mandel interferometer and of general two-photon measurements, Phys. Rev. A **106**, 063715 (2022).

[61] A. L. Grimsmo, J. Combes, and B. Q. Baragiola, Quantum computing with rotation-symmetric bosonic codes, Phys. Rev. X **10**, 011058 (2020).

[62] V. V. Albert, K. Noh, K. Duivenvoorden, D. J. Young, R. T. Brierley, P. Reinhold, C. Vuillot, L. Li, C. Shen, S. M. Girvin, B. M. Terhal, and L. Jiang, Performance and structure of single-mode bosonic codes, Phys. Rev. A **97**, 032346 (2018).

[63] C. W. Gardiner and M. J. Collett, Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation, Phys. Rev. A **31**, 3761 (1985).

[64] H. Le Jeannic, A. Tiranov, J. Carolan, T. Ramos, W. Ying, M. Appel, S. Scholz, A. Wieck, A. Ludwig, N. Rotenberg, L. Midolo, J. García-Ripoll, A. Sørensen, and P. Lodahl, Dynamical photon–photon interaction mediated by a quantum emitter, Nat. Phys. **18**, 1191 (2022).

[65] This can be seen by comparing the error probability of a single photon comblike state with interspacing $2\omega_o/\sqrt{n}$.

[66] H.-H. Lu, J. M. Lukens, B. P. Williams, P. Imany, N. A. Peters, A. M. Weiner, and P. Lougovski, A controlled-NOT gate for frequency-bin qubits, npj Quantum Inf. **5**, 24 (2019).

[67] U. Alushi, T. Ramos, J. J. García-Ripoll, R. Di Candia, and S. Felicetti, Waveguide QED with quadratic light-matter interactions, PRX Quantum **4**, 030326 (2023).

[68] Y. L. Len, T. Gefen, A. Retzker, and J. Koł odyński, Quantum metrology with imperfect measurements, Nat. Commun. **13**, 6971 (2022).

[69] P. Thomas, L. Ruscio, O. Morin, and G. Rempe, Efficient generation of entangled multiphoton graph states from a single atom, Nature (London) **608**, 677 (2022).

[70] H.-S. Zhong, Y. Li, W. Li, L.-C. Peng, Z.-E. Su, Y. Hu, Y.-M. He, X. Ding, W. Zhang, H. Li, L. Zhang, Z. Wang, L. You, X.-L. Wang, X. Jiang, L. Li, Y.-A. Chen, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, 12-photon entanglement and scalable scattershot boson sampling with optimal entangled-photon pairs from parametric down-conversion, Phys. Rev. Lett. **121**, 250505 (2018).

[71] I. Schwartz, D. Cogan, E. R. Schmidgall, Y. Don, L. Gantz, O. Kenneth, N. H. Lindner, and D. Gershoni, Deterministic generation of a cluster state of entangled photons, Science **354**, 434 (2016).

[72] C.-W. Yang, Y. Yu, J. Li, B. Jing, X.-H. Bao, and J.-W. Pan, Sequential generation of multiphoton entanglement with a Rydberg superatom, Nat. Photonics **16**, 658 (2022).

[73] G. Corrielli, M. Pont, A. Fyrillas, I. Agresti, G. Carvacho, N. Maring, P. E. Emeriau, F. Ceccarelli, R. Albiero, P.-H. D. Ferreira, N. Somaschi, J. Senellart, M. Morassi, A. Lemaitre, I. Sagnes, P. Senellart, F. Sciarrino, M. Liscidini, N. Belabas, and R. Osellame, Generation of four-photon GHZ states on a laser written integrated platform, in *Proceedings of the Optica Quantum 2.0 Conference and Exhibition* (Optica Publishing Group, 2023), p. QM4A.7, 10.1364/QUANTUM.2023.QM4A.7.

[74] M. Pont, G. Corrielli, A. Fyrillas, I. Agresti, G. Carvacho, N. Maring, P.-E. Emeriau, F. Ceccarelli, R. Albiero, P. Ferreira, N. Somaschi, J. Senellart, I. Sagnes, M. Morassi, A. Lemaître, P. Senellart, F. Sciarrino, M. Liscidini, N. Belabas, and R. Osellame, High-fidelity generation of four-photon GHZ states on-chip, arXiv:2211.15626.

[75] M. Liscidini and J. E. Sipe, Scalable and efficient source of entangled frequency bins, Opt. Lett. **44**, 2625 (2019).

[76] R. Ikuta, R. Tani, M. Ishizaki, S. Miki, M. Yabuno, H. Terai, N. Imoto, and T. Yamamoto, Frequency-multiplexed photon pairs over 1000 modes from a quadratic nonlinear optical waveguide resonator with a singly resonant configuration, Phys. Rev. Lett. **123**, 193603 (2019).

[77] B. Niewelt, M. Jastrzebski, S. Kurzyna, J. Nowosielski, W. Wasilewski, M. Mazelanik, and M. Parniak, Experimental implementation of the optical fractional Fourier transform in the time-frequency domain, Phys. Rev. Lett. **130**, 240801 (2023).

[78] S. Kurzyna, M. Jastrzebski, N. Fabre, W. Wasilewski, M. Lipka, and M. Parniak, Variable electro-optic shearing interferometry for ultrafast single-photon-level pulse characterization, Opt. Express **30**, 39826 (2022).

[79] M. Lipka and M. Parniak, Ultrafast electro-optic time-frequency fractional Fourier imaging at the single-photon level, Opt. Express **32**, 9573 (2024).

[80] T. B. Propp and S. J. van Enk, How to project onto an arbitrary single-photon wave packet, Phys. Rev. A **102**, 053707 (2020).
[81] H.-H. Lu, J. M. Lukens, N. A. Peters, O. D. Odele, D. E. Leaird, A. M. Weiner, and P. Lougovski, Electro-optic frequency beam splitters and tritters for high-fidelity photonic quantum information processing, Phys. Rev. Lett. **120**, 030502 (2018).
[82] L.-W. Luo, S. Ibrahim, A. Nitkowski, Z. Ding, C. B. Poitras, S. J. B. Yoo, and M. Lipson, High bandwidth on-chip silicon photonic interleaver, Opt. Express **18**, 23079 (2010).
[83] D. Niemietz, P. Farrera, S. Langenfeld, and G. Rempe, Nondestructive detection of photonic qubits, Nature (London) **591**, 570 (2021).
[84] E. Distante, S. Daiss, S. Langenfeld, L. Hartung, P. Thomas, O. Morin, G. Rempe, and S. Welte, Detecting an itinerant optical photon twice without destroying it, Phys. Rev. Lett. **126**, 253603 (2021).
[85] K. Fukui, R. N. Alexander, and P. van Loock, All-optical long-distance quantum communication with Gottesman-Kitaev-Preskill qubits, Phys. Rev. Res. **3**, 033118 (2021).
[86] E. Knill, R. Laflamme, and G. Milburn, A scheme for efficient quantum computation with linear optics, Nature (London) **409**, 46 (2001).
[87] K. Duivenvoorden, B. M. Terhal, and D. Weigand, Single-mode displacement sensor, Phys. Rev. A **95**, 012305 (2017).
[88] H.-H. Lu, M. Liscidini, A. L. Gaeta, A. M. Weiner, and J. M. Lukens, Frequency-bin photonic quantum information, Optica **10**, 1655 (2023).
[89] B. Roussel, C. Cabart, G. Fève, and P. Degiovanni, Processing quantum signals carried by electrical currents, PRX Quantum **2**, 020314 (2021).
[90] C. Calcluth, N. Reichel, A. Ferraro, and G. Ferrini, Sufficient condition for universal quantum computation using bosonic circuits, arXiv:2309.07820.
[91] C. Calcluth, A. Ferraro, and G. Ferrini, Efficient simulation of Gottesman-Kitaev-Preskill states with Gaussian circuits, Quantum **6**, 867 (2022).
[92] U. Chabaud and M. Walschaers, Resources for bosonic quantum computational advantage, Phys. Rev. Lett. **130**, 090602 (2023).
[93] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais, K. K. Sabapathy, N. C. Menicucci, and I. Dhand, Blueprint for a scalable photonic fault-tolerant quantum computer, Quantum **5**, 392 (2021).
[94] N. C. Menicucci, Fault-tolerant measurement-based quantum computing with continuous-variable cluster states, Phys. Rev. Lett. **112**, 120504 (2014).
[95] C. Vuillot, H. Asasi, Y. Wang, L. P. Pryadko, and B. M. Terhal, Quantum error correction with the toric Gottesman-Kitaev-Preskill code, Phys. Rev. A **99**, 032344 (2019).
[96] N. Fabre, Teleportation-based error correction protocol of time and frequency qubit states, Appl. Sci. **13**, 9462 (2023).
[97] J. Guillaud and M. Mirrahimi, Repetition cat qubits for fault-tolerant quantum computation, Phys. Rev. X **9**, 041053.
[98] J. Hastrup and U. L. Andersen, All-optical cat-code quantum error correction, Phys. Rev. Res. **4**, 043065 (2022).
[99] M. Bergmann and P. van Loock, Quantum error correction against photon loss using multicomponent cat states, Phys. Rev. A **94**, 042332 (2016).
[100] L. Li, C.-L. Zou, V. V. Albert, S. Muralidharan, S. M. Girvin, and L. Jiang, Cat codes with optimal decoherence suppression for a lossy bosonic channel, Phys. Rev. Lett. **119**, 030502 (2017).
[101] Z. Leghtas, G. Kirchmair, B. Vlastakis, R. J. Schoelkopf, M. H. Devoret, and M. Mirrahimi, Hardware-efficient autonomous quantum memory protection, Phys. Rev. Lett. **111**, 120501 (2013).

Correction: An error in wording was introduced during the proof cycle in the third sentence of the fourth-to-last paragraph and has been fixed.