Anomalous Dimensions from the $\mathcal{N}=4$ Supersymmetric Yang-Mills Hexagon

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We consider the correlator $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}}\rangle$ of the Lagrange operator of $\mathcal{N}=4$ super Yang-Mills theory and two conjugate two-excitation operators in an su(2) sector. We recover the planar one-loop anomalous dimension of the renormalized operators from this hexagon computation.

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Introduction.—The on-shell Lagrangian density of $\mathcal{N}=4$ super Yang-Mills theory (SYM) can be written as [1]

$$\mathcal{L} = \frac{1}{g_{\rm YM}^2} \mathrm{tr}\bigg(-\frac{1}{2} F_{\alpha\beta}F^{\alpha\beta} + \sqrt{2}\Psi^{\alpha I}[\Phi_{IJ}, \Psi^J_\alpha] - \frac{1}{8}[\Phi^{IJ}, \Phi^{KL}][\Phi_{IJ}, \Phi_{KL}]\bigg), \tag{1}$$

which contains the field strength $F^{\alpha\beta}$, four fermions $\Psi^{\alpha I}$, and the antisymmetric scalar fields Φ^{IJ}, where $I, J, K, L \in \{1\ldots4\}$.

Correlation functions as defined by the path integral

$$\langle \mathcal{O}_1\ldots\mathcal{O}_n\rangle = \int \mathcal{D}(\Phi,\Psi,\bar\Psi,A)\mathcal{O}_1\ldots\mathcal{O}_n e^{i\int d^4x_0 \mathcal{L}_0} \tag{2}$$

should therefore obey the identity

$$g^2 \frac{\mathrm{d}}{\mathrm{d}(g^2)}\langle \mathcal{O}_1\ldots\mathcal{O}_n\rangle = \frac{-i}{g^2}\int d^4x_0 \langle \mathcal{L}_0\mathcal{O}_1\ldots\mathcal{O}_n\rangle. \tag{3}$$

Initially intended as a criticism [2], this observation has been instrumental in constructing correlators of half-BPS operators at the integrated level at one- and two-loop order, and as integrands to very high order in the coupling $g_{\rm YM}$ [3–6].

In the present work we want to study equation (3) as a relation between the two-point function of a renormalized scalar primary operator $\mathcal{K}$ and its conjugate $\tilde{\mathcal{K}}$, and the three-point function obtained by inserting $\mathcal{L}$ into it. At *Born level*

$$\langle \mathcal{L}_0\mathcal{K}_1\tilde{\mathcal{K}}_2\rangle = \frac{c(N)}{(x_{01}^2)^2(x_{02}^2)^2(x_{12}^2)^{\Delta_0-2}}. \tag{4}$$

Configuration space Feynman integrals yield the functional dependence stated in the last formula. Here Δ_0 is the naive scaling dimension of $\mathcal{K}$, and so the computation serves to determine the constant $c(N)$, where N is the rank of the gauge group.

The space time integral over the insertion point

$$\int \frac{d^4x_0}{x_{01}^4 x_{02}^4}$$

is divergent in this situation. It reproduces the one-loop divergence of the two-point function on the left of (3) so that upon regularizing we must have

$$c(N) \propto \gamma_1 \tag{5}$$

if the planar scaling dimension of $\mathcal{K}$ has the coupling constant expansion $\Delta_{\mathcal{K}} = \Delta_0 + g^2\gamma_1 + g^4\gamma_2 + \cdots$ in terms of the 't Hooft coupling $g^2 = g_{\rm YM}^2 N/(8\pi^2)$.

The spectrum problem of $\mathcal{N}=4$ SYM is related to finding the energy eigenstates of the Heisenberg spin chain [7,8]. More recently, structure constants became accessible to such *integrability* methods via the hexagon formalism of [9]. In this Letter we present a first computation verifying (5) entirely within the Bethe ansatz framework.

In the original spin chain approach to $\mathcal{N}=4$ theory the scalar $Z=\Phi^{34}$ is regarded as a vacuum and other fields can travel over a chain of such sites [7]. Unfortunately for our endeavor, the complex conjugate vacuum $\bar Z = \Phi^{12}$, half of the fermions, and the field strength $F^{\alpha\beta}$, $\dot F^{\dot\alpha\dot\beta}$ are excluded from the set of excitations, and hence it would appear that none of the terms of the on-shell Lagrangian (1) is captured.

On the other hand, for the so(6) sector comprising all scalar fields it has been known for a long time how to realize the missing conjugate of the vacuum as a *double excitation*, by placing two scalars on the same site. The missing fermions and the two parts of the field strength tensor are provided by further types of double excitations [10].

Customarily, Bethe equations are diagonalized by the *nested Bethe ansatz* [11]. Having found a solution, its primary roots can be used also in the original Bethe ansatz for the $\mathcal{N}=4$ spectral problem, in which the (restricted set of) excitations scatter on the chain by a $\mathrm{psu}(2|2)$-invariant S matrix [12]. Operators in higher-rank sectors are then described by a multicomponent wave function.

The operator (1) is a supersymmetry descendent of the half-BPS operator $\mathcal{O}^{\mathbf{20}}$. Acting with all four generators $Q^{\alpha}_{a'}$ yields

$$\mathcal{L}=\frac{1}{4}Q^1_3Q^1_4Q^2_3Q^2_4\mathrm{tr}(\Phi^{34}\Phi^{34}). \tag{6}$$

These generators transform the spin chain vacuum as

$$Q^{\alpha}_3\Phi^{34}=\Psi^{\alpha 4}, \qquad Q^{\alpha}_4\Phi^{34}=-\Psi^{\alpha 3}, \tag{7}$$

allowing us to identify the Lagrange operator with the four excitations [10]

$$\{\Psi^{13},\Psi^{23},\Psi^{14},\Psi^{24}\} \tag{8}$$

on a length two chain. At lowest order in the coupling the Bethe wave function exists because the fermions may occupy the two sites in pairs

$$-Q^{\alpha}_3Q^{\beta}_4\Phi^{34}=\left|\begin{matrix}\Psi^{\alpha4}\\ \Psi^{\beta3}\end{matrix}\right\rangle=F^{\alpha\beta}. \tag{9}$$

The stacked state in the middle denotes a double excitation at one site. On the other hand, the level one wave function on which the four excitations scatter with the S operator of [12] dissolves the concept of spin chain length. However, the local structure of the wave function is of no further relevance in spectrum computations. Even if the length two Bethe solution with four excitations only exists owing to double excitations, the latter will not play any further rôle in calculations. Moreover, the hexagon [9] does also not depend on the local wave function. In Refs. [10,13] it is illustrated on a series of examples that it does correctly compute structure constants for operators requiring the presence of double excitations.

These calculations necessarily involve multicomponent wave functions. *A priori*, we have to write one wave function for every initial ordering of the four fermions in (8) and of the cases

$$\{\Psi^{13},\Psi^{13},\Psi^{24},\Psi^{24}\}, \qquad \{\Psi^{23},\Psi^{23},\Psi^{14},\Psi^{14}\}. \tag{10}$$

Each of the resulting 36 wave functions comes with a coefficient. Off shell, these amplitudes are uniquely determined matching the entire state on the nested Bethe ansatz [13,14].

On shell, the situation may be degenerate for descendent states, i.e., when there are infinite Bethe rapidities. One of our aims is to decide whether all 36 wave functions are necessary to recover (5). Our answer will be no: on a gauge invariant state, the four distinct supersymmetry generators in (6) anticommute. Consequently, the sequence of taking the supersymmetry variations can alter the result only by an overall sign, and for the choices in (10) the variation vanishes. Excitingly, the hexagon computation presented below has the very same features: the initial ordering of the four magnons only results in an overall sign, and the 12 cases from (10) rather nontrivially yield zero.

The computation.—The su(2) sector of the $\mathcal{N}=4$ spin chain model has n magnons $X=\Phi^{24}$ traveling over a closed chain of L sites, so there are $L-n$ further fields $Z=\Phi^{34}$. Each magnon moves with a quasimomentum p_j, or, equivalently, the *rapidity*

$$u_j=\frac{1}{2}\cot\frac{p_j}{2}. \tag{11}$$

In the planar approximation, the one-loop conformal eigenstates for every L are exactly given by the wave functions of the coordinate Bethe ansatz. Here we are chiefly interested in their eigenenergies viz anomalous dimensions. These can be found from the *Bethe equations*

$$e^{ip_jL}\prod_{k\neq j}S_{jk}=1, \qquad S_{jk}=\frac{u_j-u_k-i}{u_j-u_k+i}, \tag{12}$$

built from the shift operator $e^{ip_j}=(u_j+i/2)/(u_j-i/2)$ and the *scattering matrix* S. Importantly, there is *factorized scattering*: multiparticle scattering factorizes into two-particle processes. Finally, translation invariance along the chain implies the zero momentum constraint $\sum_j p_j=0$.

A set $\{u_j\}$ solving the Bethe equations are called *Bethe roots*. In terms of these rapidities the energy of a Bethe state is

$$\gamma_1=\sum_{j=1}^{n}\frac{1}{u_j^2+\frac{1}{4}}. \tag{13}$$

Specializing to the two-excitation case, the zero momentum constraint yields $u_1=-u_2$ and therefore the remaining Bethe equation simplifies to

$$e^{ip_j(L-1)}=1. \tag{14}$$

The lowest two-excitation states are then characterized by the roots and energy eigenvalues given in Table I.

TABLE I. Bethe roots $u_2 = -u_1$ and energy eigenvalues γ_1 for su(2) primary operators of lengths $L = 4, \ldots, 9$.

L	u_2	γ_1
4	$(1/\sqrt{12})$	6
5	$\frac{1}{2}$	4
6	$\frac{1}{2}\sqrt{1 \pm (2/\sqrt{5})}$	$5 \mp \sqrt{5}$
7	$(\sqrt{3}/2), (1/\sqrt{12})$	2, 6
8	1.038 26	1.506 04
	0.398 737	4.890 08
	0.114 122	7.603 88
9	$\frac{1}{2}(\sqrt{2} \pm 1), \frac{1}{2}$	$4 \mp 2\sqrt{2}, 4$

The spin chain model for the su(2) sector has been extended to the more complete set of excitations [15]

$$\{\Phi^{ab'}, \Psi^{\alpha b'}, \bar{\Psi}^{a\dot{\beta}}, D^{\alpha\dot{\beta}}\}, \tag{15}$$

with $a, \alpha, \dot{\beta} \in \{1, 2\}$, $b' \in \{3, 4\}$. These can be written as a tensor product of two su(2|2) excitations:

$$\chi^{AB'} = \chi^A \otimes \bar{\chi}^{B'}, \quad A = \{a, \alpha\}, \quad B' = \{b', \dot{\beta}\}. \tag{16}$$

We will henceforth refer to the magnons χ^A, $\bar{\chi}^{B'}$ as the *left* and the *right chain*, respectively. The phase called S above now becomes a true *matrix*. Its action is given by two identical copies of the same su(2|2) invariant S matrix—one acting on the A index, and the other one acting on the B' one—normalized by a common factor S as given in (12). Further, the effect of higher-loop Feynman diagrams can be included changing to the so-called Zhukowsky variable $x^{\pm}(u, g^2)$ and introducing a certain phase factor [8,16,17].

Now, the full set of fields of the $\mathcal{N} = 4$ model also comprises $\{\Phi^{12}, \Psi^{\alpha a}, \bar{\Psi}^{a'\dot{\beta}}, F^{\alpha\beta}, \dot{F}^{\dot{\alpha}\dot{\beta}}\}$, which apparently do not fit into the $A \otimes B'$ tensor structure as each of them carries two indices from the same representation. Yet, as mentioned above, the missing fields are all secretly present as double excitations [10,13]. While they are invisible in the level-one wave function, their existence allows us to trust the Bethe equations where we otherwise would not by the quantum numbers—for instance, how could four fermions fit on a chain of length two without double occupation?

In Fig. 1 we depict a three-point function of spin chains, or equivalently a three-string vertex. An efficient integrable systems approach to three-point computations has been devised in [9]. The three-vertex is split into its back and front surface yielding hexagonal patches. In the figure, the *virtual edges* are colored. These correspond to bunches of propagators stretching between the operators. The *physical edges* representing spin chains are marked in black.

The spin chains have to be split as well, taking into account all distributions of excitations or *magnons*:

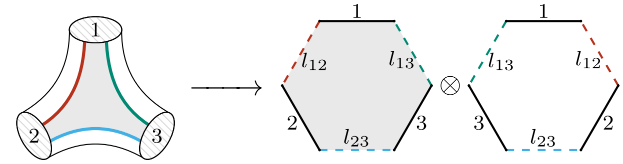

FIG. 1. Splitting a three-point function into two hexagons.

$$\begin{aligned}\psi^{l_1+l_2}_{\{X_1,X_2\}} \equiv\; & \psi^{l_1}_{\{X_1,X_2\}}\psi^{l_2}_{\{\}} - e^{ip_2 l_1}\psi^{l_1}_{\{X_1\}}\psi^{l_2}_{\{X_2\}} \\ & - e^{ip_1 l_1} S_{12}\psi^{l_1}_{\{X_2\}}\psi^{l_2}_{\{X_1\}} \\ & + e^{i(p_1+p_2)l_1}\psi^{l_1}_{\{\}}\psi^{l_2}_{\{X_1,X_2\}},\end{aligned} \tag{17}$$

where the symbol $\psi^{l}_{\{\ldots\}}$ denotes a Bethe wave function of length l with a given set of magnons. Shift operator and S matrix are as in (12). The split wave function was baptized an *entangled state* in [18].

Before tackling the correlator $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}} \rangle$ we ultimately want to evaluate, we review the hexagon approach on the example of the simpler but related three-point function $\langle \mathbb{1}\mathcal{K}\tilde{\mathcal{K}} \rangle$, which computes the norm of the operator $\mathcal{K}$ [9]. To decrease the number of terms it is advisable to focus on *transverse excitations*, which do not mix with the vacuum under the *twisted translation* [9] used to put the outer points to the standard positions 0, 1, ∞.

For transverse scalars X on $\mathcal{K}$ we have the rapidities $u_5 = -u_6$, while the operator $\tilde{\mathcal{K}}$ must carry the conjugate excitations $\bar{X}$ with rapidities $u_7 = -u_8$. The corresponding spin chain is split as in (17), which allows us to express the correlator in terms of hexagons as

$$\mathcal{A} = \sum_{\substack{\alpha\cup\bar{\alpha}=\{u_5,u_6\} \\ \beta\cup\bar{\beta}=\{u_7,u_8\}}} \begin{matrix}(-1)^{|\alpha|+|\beta|}\omega(l_{23},\alpha,\bar{\alpha})\omega(l_{23},\beta,\bar{\beta})* \\ \mathfrak{h}(\{\},\alpha,\bar{\beta})\mathfrak{h}(\{\},\beta,\bar{\alpha}).\end{matrix} \tag{18}$$

Here $\alpha, \bar{\alpha}$ and $\beta, \bar{\beta}$ denote the partitions of the sets of magnons into two subsets, and the *splitting factors* ω can be inferred from (17). In the Born approximation we do not require coupling constant corrections to the splitting factors.

In order to evaluate the hexagon form factor we first need to move all magnons to the same physical edge by using *crossing* transformations [9]. This sends

$$x^{\pm} \xrightarrow{\pm 2\gamma} \frac{g^2}{2x^{\pm}}, \tag{19}$$

so clearly the operation does not commute over the g expansion. Therefore, the Zhukowsky variables $x^{\pm}$ can only be restricted to the Born approximation after crossing. Second, on the hexagon the su(2|2) invariant S matrix [12] is multiplied by the scalar factor h [9] that has a monodromy under crossing, since it contains the BES dressing phase [17]. In the conventions of [9] we associate 0γ crossing to edge 1 of both hexagons, on the back hexagon (white hexagon in Fig. 1) we assume -2γ for magnons of operator 2 and -4γ for magnons of operator 3. On the front (gray) -2γ for operator 3 and -4γ for operator 2. Note also that crossing sends $X \to -X$ and $\bar{X} \to -\bar{X}$.

Further, as in the spectral problem the excitations are represented as $\chi^{AB'} = \chi^A \otimes \bar{\chi}^{B'}$. Again, we arrange all χ's on a *left* and all $\bar{\chi}$'s on a *right chain*. Next, the hexagon scatters only one of the chains using the S matrix [12] and finally the left and right chain are contracted employing the rule

$$\langle \phi_j^a | \bar{\phi}_j^{b'} \rangle = \epsilon^{ab'}, \qquad \langle \psi_j^\alpha | \bar{\psi}_j^{\dot{\beta}} \rangle = \epsilon^{\alpha\dot{\beta}}. \tag{20}$$

This assignment is valid in the *spin chain frame* [9,19].

In the norm computation, only hexagons with equal numbers of X and $\bar{X}$ magnons can be nonvanishing because the transverse scalars cannot be contracted on the twisted vacuum. The only distinct amplitudes are

$$\begin{aligned}
\langle \mathfrak{h} | \{\}\{\}\{\} \rangle &= 1, \\
\langle \mathfrak{h} | \{\}\{X_5\}\{\bar{X}_7\} \rangle &= \frac{i}{u_5 - u_7} e^{ip_7}, \\
\langle \mathfrak{h} | \{\}\{X_5 X_6\}\{\bar{X}_7 \bar{X}_8\} \rangle &= \frac{4u_5 u_7 \cdot e^{2ip_5}}{(2u_5 + i)(2u_7 + i)} * \\
&\quad \frac{2u_5^2 + 2u_7^2 + 1}{(u_5 - u_7)^2 (u_5 + u_7)^2}.
\end{aligned} \tag{21}$$

In the third row of the last equation we have identified $u_6 = -u_5$, $u_8 = -u_7$ to obtain a concise formula. Note that both nontrivial amplitudes contain *particle creation poles* that occur when conjugate magnons occupy different physical edges. Since both $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are characterized by the same rapidities we cannot identify $u_7 = u_5$ without factoring out or regularizing [20] the real poles. Fortunately, they do factor out upon adding up all the partitions. Yet, this can only happen upon omitting the explicit momentum factors from Eq. (21). The need to drop these can be demonstrated deriving the amplitudes in the string frame and switching to the spin chain picture employing the formulas in [9,22]. Here we use the rule

$$2u - i \to e^{-ip}(2u + i). \tag{22}$$

since denominator factors of the form $(2u_5 + i)$ can, e.g., arise from the difference $x_5^+ - x_6^-$ in S matrix elements. The need to rescale by momentum factors has been realized in the original paper [9] and was extended for fermions in [22].

After factorization we can identify $u_7 = u_5$. By way of example, the length 4 and 5 results are

$$\begin{aligned}
\langle \mathbb{1}\mathcal{K}^4\tilde{\mathcal{K}}^4 \rangle &= \frac{512 u_5^2 e^{ip_5}}{(1+4u_5^2)^7} (23 - 128u_5^2 + 992u_5^4 \\
&\quad - 512u_5^6 + 768u_5^8), \\
\langle \mathbb{1}\mathcal{K}^5\tilde{\mathcal{K}}^5 \rangle &= \frac{512 u_2^5 e^{ip_5}}{(1+4u_5^2)^9} (65 - 952u_5^2 + 12\,336u_5^4 \\
&\quad - 38\,144u_5^6 + 87\,808u_5^8 \\
&\quad - 30\,720u_5^{10} + 20\,480u_5^{12}).
\end{aligned} \tag{23}$$

Substituting the rapidities for $L = 4, 5$ from Table I we find the values $108(-1)^{4/3}, -80i$, respectively, confirming [9,23]

$$\langle \mathbb{1}\mathcal{K}\tilde{\mathcal{K}} \rangle = \mathcal{G} S_{12}, \tag{24}$$

where $\mathcal{G}$ is the *Gaudin norm* [24].

The $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}} \rangle$ computation additionally requires an entangled state at point 1 distributing the four magnons of $\mathcal{L}$ with rapidities $u_1, \ldots, u_4$ over the two hexagons. This considerably augments the number of possible hexagon amplitudes. In the Born approximation there is no e^{ip} rescaling for the magnons of $\mathcal{L}$ because of their vanishing momenta (viz infinite rapidities). Thus the rescaling is tied solely to the occupation by scalar excitations from $\mathcal{K}, \tilde{\mathcal{K}}$ at given momentum and crossing. We can infer it from (21) where an inverse image in $\langle \mathbb{1}\mathcal{K}\tilde{\mathcal{K}} \rangle$ exists. However, a new feature of $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}} \rangle$ is the occurrence of nonvanishing hexagon amplitudes with one or three of the scalars from $\mathcal{K}, \tilde{\mathcal{K}}$ and also one or three of the magnons of $\mathcal{L}$. Fortunately, the rescaling should not depend on the flavor of the scalars in question so that we can compare to a norm calculation with longitudinal scalars where necessary. In Table II we list the additional hexagon amplitudes [25] and the momentum factors which need to be removed.

According to (6), the Lagrange operator is a fourfold supersymmetry descendent of the vacuum. We therefore expect four infinite Bethe roots. In order to find a regulated Bethe solution for the Yang-Mills term of the Lagrange operator as derived in [10] a twist regulator in a nested Bethe-ansatz for an su(2|2) sector was employed. To this end a factor $e^{im_j\beta}$ is introduced into the level-j Bethe equation, with a universal order parameter $\beta \ll 1$. The Bethe roots are assumed to have an expansion

TABLE II. Additional momentum factors for amplitudes containing magnons from $\mathcal{K}, \tilde{\mathcal{K}}$.

Hexagon	Factor
$\langle \mathfrak{h} \| \{\ldots\}\{Y_5\}\{\} \rangle, \langle \mathfrak{h} \| \{\ldots\}\{\}\{Y_7\} \rangle$	1
$\langle \mathfrak{h} \| \{\ldots\}\{Y_5 Y_6\}\{\} \rangle$	e^{2ip_5}
$\langle \mathfrak{h} \| \{\ldots\}\{Y_5 Y_6\}\{Y_7\} \rangle$	$e^{2ip_5} e^{2ip_7}$
$\langle \mathfrak{h} \| \{\ldots\}\{\}\{Y_7 Y_8\} \rangle, \langle \mathfrak{h} \| \{\ldots\}\{Y_5\}\{Y_7 Y_8\} \rangle$	1

$$u_j \to \frac{u_{j,-1}}{\beta} + u_{j,0} + u_{j,1}\beta + \ldots \tag{25}$$

of which we will only need the leading order in β in this Letter. The hexagon calculation only depends on the level one roots, so we just need

$$u_{j,-1} \in \left\{\frac{+1+i}{\sqrt{2}}, \frac{+1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}\right\} \tag{26}$$

with $j = 1, \ldots, 4$. We will henceforth drop the -1 label on these roots. Note that they satisfy $\prod_j u_j = 1$.

Since twist is needed on the supersymmetry and on an auxiliary R symmetry node one might wonder whether supersymmetry and R invariance are manifest in every partial wave function of the multicomponent ansatz or rather only in the sum over all 36 parts. For the Yang-Mills term of the Lagrangian the individual wave functions are degenerate—will we find the same in our three-point calculation?

To form the entangled state of $\mathcal{L}$ we generalize (17) by scattering the four fermions with the full $\mathrm{su}(2|2)^2$ S matrix [12]. *Born level* means restricting to leading order in g and β. It is advisable to do so from the very beginning, resulting in the simple scattering

$$\begin{aligned}
&\phi_1^a\psi_2^\gamma \to \psi_2^\gamma\phi_1^a,\\
&\psi_1^\alpha\phi_2^c \to \phi_2^c\psi_1^\alpha,\\
&\phi_1^a\phi_2^c \to \phi_2^c\phi_1^a + \frac{\tilde{g}}{u_1u_2}\epsilon^{ac}\epsilon_{\alpha\gamma}\psi_2^\alpha\psi_1^\gamma\mathcal{Z}^-,\\
&\psi_1^\alpha\psi_2^\gamma \to \psi_2^\gamma\psi_1^\alpha + \frac{\tilde{g}}{u_1u_2}\epsilon^{\alpha\gamma}\epsilon_{ac}\phi_2^a\phi_1^c\mathcal{Z}^+
\end{aligned} \tag{27}$$

on both chains. Here we introduced $\tilde{g} = (g\beta^2)/\sqrt{2}$. Thus scattering is diagonal up to the remnants of the C_{12} and F_{12} elements of the S matrix [12], where the $\mathcal{Z}^\pm$ indicate length changing effects. In [10] the calculation

$$\langle \mathcal{L}\mathcal{O}^L\mathcal{O}^L\rangle = 0 \tag{28}$$

with two half-BPS states $\mathcal{O}$ of length L was presented. This computation is chiral in the sense that four fermions can only be self-contracted under certain conditions. Namely, the scattering processes to form the entangled state as well as those in the evaluation of the hexagons need to produce in total C^2, CF, F^2 (or higher) for every term. Thereby, all the leading contributions come with the factor

$$\frac{\tilde{g}^2}{\prod_j u_j} \to \tilde{g}^2 \tag{29}$$

if we go on shell. In fact, the actual Bethe solution is never needed because the rapidity dependence is global. Furthermore, the length-changing effects denoted by $\mathcal{Z}^\pm$ referring to the insertion or deletion of a site in (27) are irrelevant because all magnons have infinite rapidities.

On the other hand, when evaluating $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}}\rangle$ length changing is of the utmost importance since there are magnons with finite rapidities. Recall that we can scatter on either chain when computing hexagons. For instance, scattering on the right chain produces C elements so that only spin chain shortening operations $\mathcal{Z}^-$ appear. We can then move the Z markers to the left using [12]

$$\chi^{B'}\mathcal{Z}^\pm \to e^{\pm ip}\mathcal{Z}^\pm\chi^{B'}, \tag{30}$$

and remove them from the form factor as [9]

$$\begin{aligned}
&\langle \mathfrak{h}|\mathcal{Z}^\pm\chi^{B_1'}(p_1)\ldots\chi^{B_n'}(p_n)\rangle\\
&\quad = \prod_{j=1}^n e^{\mp ip_j/2}\langle \mathfrak{h}|\chi^{B_1'}(p_1)\ldots\chi^{B_n'}(p_n)\rangle.
\end{aligned} \tag{31}$$

Note that $\pm 2\gamma$ on the second edge inverts momentum factors. Further, crossing can create entire powers of e^{ip} when acting on the Zhukowsky variables in the usual combinations like $x^+ - y^-, 1 - g^2/(2x^-y^+), \ldots$ in the S matrix elements. Half integer powers may arise from the relative normalization between bosons and fermions:

$$\sqrt{i(x^- - x^+)}|_{u^{\mp 2\gamma}} = \mp\, e^{ip/2}\frac{i\sqrt{i(x^- - x^+)}}{x^+}\Big|_{u^{0\gamma}} \tag{32}$$

as well as from (31). Finally we have to deal with the Z markers arising from scattering the four fermions according to (27) when building the entangled state of $\mathcal{L}$. As the Lagrangian is inserted at point 1 and its magnons have vanishing momentum, we can move these markers to the very left without picking up any additional factors. Then we utilize (31) once again. Using our assumptions on the rapidities (vanishing momenta $p_1\ldots p_4$ for the magnons of $\mathcal{L}$ and $u_6 = -u_5$, $u_8 = -u_7$ on $\mathcal{K}$, $\tilde{\mathcal{K}}$), we tabulate the extra momentum factors for all contributing partitions and notice that they are equal on the front and back hexagon. We are thus free to choose on which hexagon to act with the markers from the entangled state for $\mathcal{L}$. Applying them to both would overcount their effect. In a tessellation [23,26] with more hexagon tiles around the Lagrangian insertion one will presumably need some averaging prescription [27]. Last, for these three-point functions a curious observation with respect to the total Z marker action is the symmetry under $\mathcal{Z}^+ \leftrightarrow \mathcal{Z}^-$. This can be traced to the similarity of the C and F elements.

Given all these precautions the miracle happens: there are only integer powers of e^{ip} and the particle creation poles factor out as in the norm computation. Despite the fact that 2286 of 35 664 possible hexagon amplitudes are nontrivial, there are maximally 180 contributing partitions in any partial wave function so that factorization can again

TABLE III. Rational functions obtained from the hexagon evaluation for bridge lengths l_{23}.

l_{23}	Rational function
2	0
3	$[8192u_5^2e^{ip_5}/(1+4u_5^2)^5]$
4	$[8192u_5^2e^{ip_5}/(1+4u_5^2)^7]\cdot(1-12u_5^2)^2$
5	$[8192u_5^2e^{ip_5}/(1+4u_5^2)^9]\cdot 8(1-4u_5^2)^2(1-8u_5^2+80u_5^4)$
6	$[8192u_5^2e^{ip_5}/(1+4u_5^2)^{11}]\cdot(1-40u_5^2+80u_5^4)^2(5-24u_5^2+80u_5^4)$
7	$[8192u_5^2e^{ip_5}/((1+4u_5^2)^{13})]\cdot(3-4u_5^2)^2(1-12u_5^2)^2(3-80u_5^2+1824u_5^4-5376u_5^6+8960u_5^8)$
8	$[8192u_5^2e^{ip_5}/(1+4u_5^2)^{15}]\cdot 2(1-84u_5^2+560u_5^4-448u_5^6)^2(7-112u_5^2+928u_5^4-1792u_5^6+1792u_5^8)$

be achieved analytically. More precisely, the 24 permutations of four distinct fermions (8) each yield 104 partitions, and the results are equal up to the obvious signs. The 12 cases with degenerate fermions (10) all nontrivially vanish. Here 144, 168, or 180 partitions sum to zero.

In $\langle \mathbb{1}\mathcal{K}^L\tilde{\mathcal{K}}^L\rangle$ the width of the edge connecting the operators at points 2, 3 is unambiguously $l_{23}=L$ because the identity has length 0 [9,28]. In general, the bridge length reads $l_{23}=(L_2+L_3-L_1)/2$ and for $\langle \mathcal{L}\mathcal{K}^L\tilde{\mathcal{K}}^L\rangle$ we obtain $l_{23}=L-L_1/2$. The leading $\mathrm{tr}(F^2)$ term surely has $L_1=2$. In our factorization exercise there are always two Z markers in any term. To the combinations $\mathcal{Z}^-\mathcal{Z}^-, \mathcal{Z}^+\mathcal{Z}^-, \mathcal{Z}^+\mathcal{Z}^+$ we could assign operator length 0, 2, 4, respectively. If we do not want to destroy the cancellation of the particle creation pole, every partition will have to have the same starting value for l_{23} while we rely on the markers to correctly incorporate the length changing effects.

In analogy to (23), for $l_{23}=2\ldots 8$ we find the rational functions given in Table III for the initial ordering $\{\psi^{13},\psi^{14},\psi^{23},\psi^{24}\}$ of the $\mathcal{L}$ excitations.

Curiously, $l_{23}=L$ yields zero from the third row on upon substituting the Bethe roots from Table I. This confirms operator length two for the full Lagrangian, and indeed for all operators [29] in Table I,

$$\frac{\langle \mathcal{L}\mathcal{K}^L\tilde{\mathcal{K}}^L\rangle}{\langle \mathbb{1}\mathcal{K}^L\tilde{\mathcal{K}}^L\rangle}=\frac{\tilde{g}^2\gamma_1}{L}, \tag{33}$$

with $l_{23}=L-1$. This is our main result.

Conclusions.—We have successfully generalized the simple test in [10] to the formula $\langle \mathcal{L}\mathcal{K}\tilde{\mathcal{K}}\rangle \propto \gamma_1$ as expected from field theory, see (5). However, we do not fully understand the normalization. For a *connected* correlator, the hexagon result should be scaled up by a factor

$$\frac{\sqrt{L}}{\sqrt{\mathcal{G}\prod_{i<j}S_{ij}}} \tag{34}$$

per operator. Now, $\langle \mathbb{1}\mathcal{K}\tilde{\mathcal{K}}\rangle$ is disconnected, by which token the factor $\sqrt{L}$ should be omitted there, explaining the appearance of the explicit $1/L$ in (33).

How about the normalization of $\mathcal{L}$, though? The Gaudin determinant from the full set of Bethe equations including secondary roots equals $144^2\beta^{16}$, whereas the phase $\prod_{i<j}S_{ij}$ associated to the Lagrangian is 1. On the other hand, reading off coefficients for the individual wave functions of the multicomponent ansatz from an equivalent nested Bethe ansatz leads to the conjecture that each such coefficient is given by the complete *above level-one wave function* [10,13]. We have verified this with $L>2$ for the off-shell problem with four level-one Q_3^2 excitations and two auxiliary sl(2) (left) and su(2) (right) excitations each. For example, the coefficient of the $\{\psi^{13},\psi^{14},\psi^{23},\psi^{24}\}$ partial wave function discussed above is the nested wave function for auxiliary excitations with rapidities $\{v_1w_1,v_2,w_2,1\}$, respectively, on the four level-one vacuum sites $\psi^{24}(u_i)$. Here we have to scatter the two left wing and the two right wing magnons over each other resulting in four similar blocks. Given the degeneracy of the $24+12$ wave functions described in the last section we compute an extra factor of $-24\sqrt{3}\beta^4$ from the sum of these coefficients put on shell. For the explicit Bethe solution see [10]. The right-hand side of formula (33) should thus be rescaled as

$$\frac{\beta^4}{2}\frac{\gamma_1}{L}*\frac{-24\sqrt{3}\beta^4}{144\beta^8}*\sqrt{2LL}=-\frac{\gamma_1}{\sqrt{4!}}. \tag{35}$$

While the powers of β fall into place and we may dismiss the minus sign as somewhat accidental, we have no good explanation for the factor $\sqrt{4!}$ in the normalization of the Lagrange operator. Note, however, that differences between operator norms in integrability and field theory, respectively, are not uncommon [9].

Nevertheless, we have presented evidence for the Lagrangian insertion to work in integrability as in field theory. Our computation is perfectly stable although descendent operators can be quite intricate to handle in the hexagon approach. In fact, explicit knowledge of a Bethe solution for $\mathcal{L}$ is not required unless we are interested in the exact normalization in (5). If not, we may pick any of the wave functions without degenerate fermions and ignore the rest of the ensemble.

The obvious future applications of the technique would be an extension to other types of operators carrying anomalous dimension, and most of all nonplanar corrections to the anomalous dimension of transverse su(2) sector operators by using a tessellation as in Refs. [21,28,30].

Finally, the derivative of scaling dimensions with respect to the coupling constant was also discussed in the fishnet model [31] in [32] from a hexagon point of view and in [33] using separation of variables. The second article emphasizes how the relevant fishnet expressions derive from those of the parent $\mathcal{N} = 4$ theory, and in the third it is advocated to apply the separation of variables technique in $\mathcal{N} = 4$ SYM in order to gain access to nonperturbative physics. Our work can hopefully provide some hints in this context although it is currently aimed at the weak coupling expansion. Note that we consider the excitations forming the Lagrange operator as physical; with respect to the superpotential see also [10].

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