

## Entanglement in Interacting Majorana Chains and Transitions of von Neumann Algebras

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We consider Majorana lattices with two-site interactions consisting of a general function of the fermion bilinear. The models are exactly solvable in the limit of a large number of on-site fermions. The four-site chain exhibits a quantum phase transition controlled by the hopping parameters and manifests itself in a discontinuous entanglement entropy, obtained by constraining the one-sided modular Hamiltonian. Inspired by recent work within the AdS/CFT correspondence, we identify transitions between types of von Neumann operator algebras throughout the phase diagram. We find transitions of the form  $\text{II}_1 \leftrightarrow \text{III} \leftrightarrow \text{I}_\infty$  that reduce to  $\text{II}_1 \leftrightarrow \text{I}_\infty$  in the strongly interacting limit, where they connect nonfactorized and factorized ground states. Our results provide novel realizations of such transitions in a controlled many-body model.

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*Introduction.*—Entanglement in many-body systems and quantum field theories (QFTs) has recently been explored via a novel take on local operator algebras and axiomatic QFT [1–6] (see [7,8] for a review). A fruitful platform for these analyses is the anti-de Sitter/conformal field theory correspondence [9–11] (also known as holography), which relates strongly coupled QFTs in  $d$  dimensions with gravity theories on negatively curved spacetimes in  $d + 1$  dimensions. In this context, operator algebras, and especially *von Neumann* algebras, have recently been leveraged to rigorously describe the entanglement structure of holographic systems [12–20]. A direct consequence of these investigations is the algebra typification of the two phases in the Hawking-Page transition [12,13], characterized by factorized and nonfactorized Hilbert spaces, respectively. In addition to these new developments, the study of algebraic properties emerging in quantum systems in the limit of infinitely many degrees of freedom, which enables phenomena such phase transitions and facilitates the study of entanglement, is a long-standing line of investigation [21].

This state of the art motivates us to study transitions of operator algebras arising in interacting many-body quantum systems whose Hilbert space structure allows for a controlled analysis of entanglement. A useful object which helps in classifying types of algebras is the *one-sided modular Hamiltonian* associated to a given subregion [6]. For free fermionic systems, this object is uniquely

determined by the two-point correlation functions restricted to the subregion [22]. We extend these results to interacting fermionic systems by constraining the form of the one-sided modular Hamiltonian in the limit of a large number of on-site fermions. For a wide class of interacting Majorana lattices, we exploit this extension to identify the operator algebras underlying our models, together with their transitions between different regimes of the phase diagram. This paves the way for addressing the classification of algebras, and possible transitions thereof, in previously suggested models for discrete holography, such as  $O(N)$ -invariant aperiodic spin chains [23].

More precisely, in this Letter, we introduce a lattice model with  $N$  Majorana fermions on each site, interacting via a general potential involving multibody hoppings. In the large- $N$  limit, all higher-point functions factorize. Thus, we solve the system exactly by obtaining the two-point correlation function for generic interaction potentials. We showcase the wide applicability of our techniques by considering instances of the model including both finite and infinite chains with nearest-neighbor hopping.

We report three main results: First, we derive the entropy of a two-site chain with generic interaction potential. This entropy is fully determined by the correlations between the two sites, which are dictated by the interaction potential via self-consistency. This result can be interpreted both as the thermal entropy of the chain at a finite temperature or as the entanglement entropy of a two-site subregion in a larger chain. Remarkably, we find that the entropy itself does not depend explicitly on the chosen potential. Second, for a four-site chain, we identify two phases of the system characterized by strong and weak correlations within any two-site subsystem of the chain relative to all other correlations in the system. In particular, we identify a

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regime where the correlation structure indicates the factorization of the ground state. For a vast class of potentials, we show that these phases are connected by a quantum phase transition. Moreover, we determine the entanglement entropy of the half-chain by imposing a constraint on the one-sided modular Hamiltonian. In correspondence with the phase transition, the entanglement entropy exhibits a discontinuity above a critical value of the interaction strength. Third, exploiting the exact solvability of our model, we identify transitions between the local operator algebras underlying the four-site chain. When the correlations within a given subregion are the most relevant in the system, we find a von Neumann algebra of type  $I_\infty$ , which, in general, encodes finite entanglement. In our model, entanglement vanishes, thus signaling factorization. In the opposite regime, the algebra is of type  $II_1$ , associated with infinite entanglement entropy, while still allowing for the definition of a trace functional. The intermediate domain is described by type III algebras, where the entanglement is infinite and a trace functional cannot be defined. Strikingly, in the strongly interacting limit and for an exponential potential in the fermion bilinear, we find that the transition between types  $II_1$  and  $I_\infty$  algebras occurs in correspondence with the identified quantum phase transition. This transition then connects a nonfactorized ground state to a ground state factorized into a product state.

*Hamiltonian and Schwinger-Dyson equations.*—We consider a lattice of  $L$  sites with  $N$  Majorana fermions  $\psi_x^j$  at each site, with anticommutation relations  $\{\psi_x^j, \psi_y^k\} = \delta_{jk}\delta_{xy}$ . The microscopic Hamiltonian is given by

$$H = \frac{N}{2} \sum_{x,y=1}^L h_{xy} \left( \frac{2}{iN} \sum_{j=1}^N \psi_x^j \psi_y^j \right), \quad (1)$$

with a general interaction potential  $h_{xy}$ , which, without loss of generality, obeys  $h_{xy}(\xi) = h_{yx}(-\xi)$  and  $h_{xx}(\xi) = 0$ . The theory is invariant under a global  $O(N)$  rotation  $\psi_x^j \rightarrow \sum_k O^{jk} \psi_x^k$ , where  $O$  is an orthogonal matrix. It is a lattice counterpart of the Gross-Neveu model [24] with a general interaction potential, and only the bubble diagrams contribute to two-point functions at leading order in  $1/N$ ; see Supplemental Material [25]. When  $h_{xy}(\xi) \propto \xi^q$ , the model is equivalent to the replicated Brownian Sachdev-Ye-Kitaev (SYK<sub>q</sub>) model in disorder averaging [26]. In this work,  $h_{xy}(\xi)$  always includes a linear term in  $\xi$ .

We solve the model (1) by introducing the effective action of two auxiliary bilocal fields, the Green's function  $G_{xy}(\tau_1, \tau_2) = 1/N \sum_j \psi_x^j(\tau_1) \psi_y^j(\tau_2)$  and the self-energy  $\Sigma_{xy}(\tau_1, \tau_2)$ , introduced as a Lagrange multiplier, in the spirit of [27–29]. Here,  $\tau$  denotes Euclidean time. We consider the canonical ensemble at temperature  $1/\beta$  and write the thermal partition function  $Z = \int \mathcal{D}\tilde{G} \mathcal{D}\tilde{\Sigma} e^{-S_E[\tilde{G}, \tilde{\Sigma}]}$  with the effective action

$$\begin{aligned} -S_E/N &= \log \text{PF}(\partial_\tau \delta_{xy} - \Sigma_{xy}) \\ &\quad - \frac{1}{2} \sum_{x,y} \int_0^\beta d\tau_1 d\tau_2 G_{xy}(\tau_1 - \tau_2) \Sigma_{xy}(\tau_1 - \tau_2) \\ &\quad - \frac{1}{2} \sum_{x,y} \int_0^\beta d\tau h_{xy}(-2iG_{xy}(0)), \end{aligned} \quad (2)$$

where  $G_{xy}(\tau) = -G_{yx}(-\tau)$  and we have assumed time-translational invariance. From  $Z$ , we may derive the properties of the ground state (when  $\beta \rightarrow \infty$ ) or the thermodynamics of the system at finite temperature. In the large- $N$  limit, which we take first in all following computations, the saddle point approximation of the path integral leads to the *Schwinger-Dyson* (SD) equations

$$G'_{xy}(\tau_{12}) - \sum_z \int d\tau_3 \Sigma_{xz}(\tau_{13}) G_{zy}(\tau_{32}) = \delta_{xy} \delta(\tau_{12}), \quad (3)$$

$$\Sigma_{xy}(\tau_{12}) = 2i h'_{xy}(-2iG_{xy}(0)) \delta(\tau_{12}), \quad (4)$$

with  $\tau_{ij} \equiv \tau_i - \tau_j$  and  $\Sigma_{xy} = -\Sigma_{yx}$  due to the conditions on  $h_{xy}$  mentioned above. This model is exactly solvable in the large- $N$  limit, in the sense that all higher-point functions factorize into products of two-point functions; i.e., we have large- $N$  factorization.

We solve Eqs. (3) and (4) for general  $\tau$  by leveraging the fact that the self-energy is solely determined by  $-2iG_{xy}(0)$ . We can, thus, solve for  $G(0)$  by means of *self-consistency* (SC) conditions. Using the form of (3) in Fourier space, SC imposes

$$G_{xy}(0) = \frac{1}{\beta} \sum_n [[-i\omega_n - \Sigma(\omega_n)]^{-1}]_{xy}, \quad (5)$$

with  $\omega_n = 2\pi(n + 1/2)/\beta$  and  $\Sigma$  given by (4). The Green's function  $G(\tau)$  is then obtained by inserting  $G(0)$  back into (4) and (3).

*Two-site chain.*—To provide an explicit application of our general techniques, we now focus on a system governed by a general Hamiltonian  $H$  of the form (1) with  $L = 2$  sites and at a finite temperature  $1/\beta$ . Without loss of generality, we can absorb  $\beta$  into the general form of the Hamiltonian (1), i.e., set  $\beta = 1$ . Thus, the density matrix reads

$$\rho = \frac{e^{-H}}{Z}, \quad H = Nh \left( \frac{2}{iN} \sum_j \psi_1^j \psi_2^j \right), \quad (6)$$

where  $Z = \text{Tr}(e^{-H})$  and  $h(\xi) \equiv h_{12}(\xi) = h_{21}(-\xi)$ . We can exactly solve the SD equation (3) to obtain the Green's function  $G(\tau)$  [25]. At  $\tau = 0$ , the solution reads

$$G_{xy}(0) = \frac{1}{2} \begin{pmatrix} 1 & -i \tanh[h'(X)] \\ i \tanh[h'(X)] & 1 \end{pmatrix}, \quad (7)$$

with  $X \equiv -2iG_{12}(0)$ . We can read off the SC equation

$$X = -\tanh[h'(X)], \quad (8)$$

which can alternatively be derived from (5). In general, we have  $-1 < X < 1$  regardless of the form of the potential. Two relevant regimes of (8) are when  $|X| \rightarrow 0$  and  $|X| \rightarrow 1$ , corresponding to weak and strong correlations between the two sites, respectively.

From the path integral in the large- $N$  limit, we can study the thermodynamic properties of the system [25]. In particular, we find the entropy density  $S/N \equiv s$  to be

$$s(X) = -\frac{1+X}{2} \log \frac{1+X}{2} - \frac{1-X}{2} \log \frac{1-X}{2}, \quad (9)$$

where  $X$  satisfies the SC equation (8). The fact that the entropy density function (9) does not explicitly depend on the interaction potential signifies the first main result of this work. This independence is a feature of the entropy only, while other quantities, like the free energy, which is given by  $F = N[h(X) - s(X)] + \mathcal{O}(1)$ , indeed depend on the form of the potential. Also, this independence of the entropy on the interaction potential is spoiled by  $1/N$  corrections [25]. Remarkably, from (9), we see that  $S \rightarrow N \log 2 + \mathcal{O}(1)$  when  $|X| \rightarrow 0$  and  $S \rightarrow 0$  when  $|X| \rightarrow 1$  [25]. Let us emphasize that, since the large- $N$  limit is always taken first, the entropy is finite only when  $X = 1$  and is otherwise linearly divergent with  $N$ .

*Four-site chain.*—We can think of the two-site system as being part of larger chains and take advantage of the results derived above to study the entanglement structure. As an example, we consider an open chain of length  $L = 4$  at zero temperature. Although our techniques are valid for any potential, we consider here a specific instance  $h_{x,x+1}(\xi) = \mu_x(1 - e^{J\xi})/(2J) = h_{x+1,x}(-\xi)$ , where  $J > 0$  is the interaction strength and  $\mu_1 = \mu_3 \equiv \mu_a$ ,  $\mu_2 \equiv \mu_b$  denote hopping parameters. All remaining entries of  $h_{xy}$  are zero. For convenience, we introduce the hopping ratio  $r \equiv \mu_a/\mu_b$ . To access the ground state properties in the large- $N$  limit, the hierarchy of parameters  $N \gg \mu_b\beta$ ,  $\mu_a\beta \gg 1$  needs to be taken into account. We solve the SD equation (3) and obtain  $G(\tau)$  [25], which at  $\tau = 0$  reads

$$G_{xy}(0) = \frac{1}{2} \begin{pmatrix} 1 & i\sin\theta & 0 & i\cos\theta \\ -i\sin\theta & 1 & i\cos\theta & 0 \\ 0 & -i\cos\theta & 1 & i\sin\theta \\ -i\cos\theta & 0 & -i\sin\theta & 1 \end{pmatrix}, \quad (10)$$

where the parameter  $\theta$  is determined by the SC constraint derived from (5):

$$\frac{\tan\theta}{2} = \frac{G_{12}(0)}{2G_{23}(0)} = \frac{h'_{12}(-2iG_{12}(0))}{h'_{23}(-2iG_{23}(0))} = r e^{J(\sin\theta - \cos\theta)}. \quad (11)$$

This transcendental equation may be solved numerically and has a unique solution for  $J < J_c$  and three solutions for  $J > J_c$ , where the critical value can be proven analytically to be  $J_c = \sqrt{2}$  [25]. This multivaluedness of the SC equation indicates that the system exhibits a discontinuous behavior as a function of  $r$ , now to be seen as a control

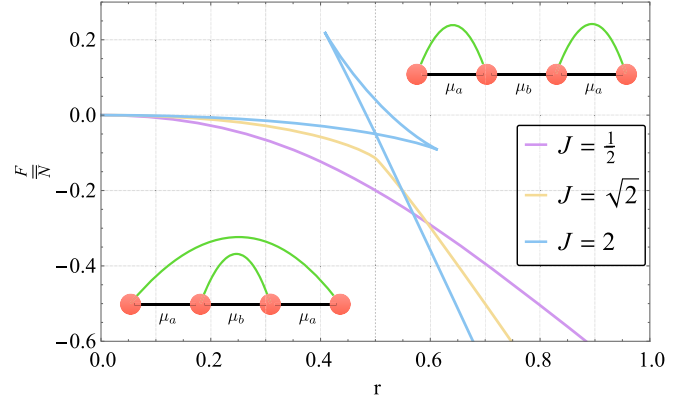


FIG. 1. Free energy of the four-site chain with potential  $h_{x,x+1}(\xi) = \mu_x(1 - e^{J\xi})/(2J)$  for different interaction strengths  $J$ . We observe a nonanalyticity at  $r \equiv \mu_a/\mu_b = 1/2$  for  $J$  above the critical value  $J_c = \sqrt{2}$ , signaling a phase transition. The two phases of the system are characterized by the correlation structure given by (10), whose limiting cases for  $r \rightarrow 0$  and  $r \rightarrow \infty$  are shown in the two embedded diagrams.

parameter. We identify the thermodynamically dominant solutions by minimizing the free energy  $F$  obtained from the effective action (2) [25]. This free energy is shown in Fig. 1 for different values of the interaction strength below, at, and above the critical point. We see that the free energy exhibits nonanalyticity at  $r = 1/2$  for interaction strengths  $J > J_c$ . Thus, the system is characterized two phases for  $J > J_c$  and it undergoes a first-order quantum phase transition [30] across  $r = 1/2$ , the existence of which constitutes the second main result of this work. At the critical point  $J = J_c$ , this transition is of second order. Let us stress that this phase transition is present for a large class of potentials other than the exponential [25]. The two phases differ by the order parameter  $\tan\theta$  (11), which characterizes the correlation structure via (10). Two limiting regimes of this structure when  $r \rightarrow 0$  and  $r \rightarrow \infty$  are shown in the insets in Fig. 1.

We now turn our attention to the study of entanglement in the four-site model and consider a connected two-site subregion  $A$ , which we take to be, e.g., the sites  $x = 1, 2$ . The reduced density matrix  $\rho_A$  of this system can be written as a thermal density matrix of a two-site chain of the form (6), with  $H$  now to be thought of as the one-sided modular Hamiltonian. Its explicit form is not known in our case, so we take as an ansatz the general form given in (1). For this ansatz to describe a proper reduced density matrix,  $\rho_A$  should reproduce the expectation values of local operators in the subregion. In particular, it must reproduce the correlations given by the Green's function (10) restricted to the subregion  $A$ . We must, therefore, impose a constraint for the one-sided modular Hamiltonian at large  $N$ :

$$G_{xy}(0) = \frac{1}{N} \sum_j \text{Tr}(\rho_A \psi_x^j \psi_y^j), \quad x, y \in A, \quad (12)$$

where  $G_{xy}(0)$  is given in (10). Equation (12) uniquely determines the modular Hamiltonian only when  $J = 0$  [22]. Nevertheless, we can still use it to compute the entanglement entropy for  $J \geq 0$ . Indeed, we have shown that, for a density matrix of the form given in (6), the entropy density of the two-site system can be computed for any form of the one-sided modular Hamiltonian ansatz and is given by (9). Because of the constraint (12),  $s(X)$  needs to be evaluated on the SC solution  $X = -2iG_{12}(0) = \sin \theta$  obtained from (11), with  $X \in [0, 1)$  since  $\theta \in [0, \pi/2)$ . In this way, even without knowing the explicit form of  $\rho_A$ , we find that  $s(X) = S_A/N$  is the entanglement entropy of subregion  $A$ . The resulting entanglement entropy density as a function of  $r$  [recall  $X$  depends on  $r$  via (11)] is shown in Fig. 2 for different values of  $J$ . The phase transition reflects itself in the entanglement entropy evaluated on SC solutions as it becomes discontinuous for  $J > J_c$ .

*von Neumann algebras.*—Motivated by recent results in holography [12,13,15–18], we study the classification of operator algebras associated to subsystems of our four-site model. Operator algebras can generally be classified into three types, denoted as type I, II, and III [1,5,8]. Based on the standard trace  $\text{Tr}$ , a type I algebra encapsulates a finite entanglement entropy. Using  $\text{Tr}$ , entanglement entropy is infinite in both type II and type III algebras. To further distinguish the algebras, a key ingredient is the *trace functional*, denoted by lowercase “tr” (to differentiate it from uppercase  $\text{Tr}$ ), which is defined to be a positive, linear, and cyclic functional on the algebra [7,8]. In particular, type II algebras allow for the definition of such a trace functional, while type III algebras do not. For the technical construction of operators in the algebra, we closely parallel [7]. In our system of consideration and

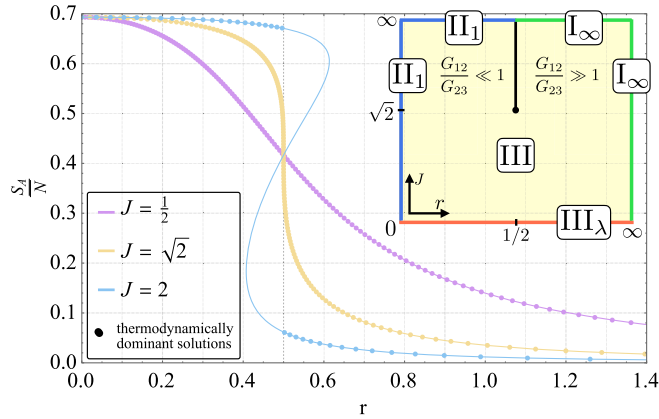


FIG. 2. Entanglement entropy (9) of subregion  $A$  on solutions to (11) as a function of  $r \equiv \mu_a/\mu_b$  for different couplings  $J$ . Solid lines denote all SC solutions, while a sample of physical solutions minimizing  $F$  in Fig. 1 is marked with dots. The phase transition is signaled by the discontinuity for  $J > J_c$ . Inset: von Neumann type of the algebra  $\mathcal{A}_A$  in different regimes of the phase diagram. Each type is denoted by a different color, and the black dot (line) represents a phase transition of second (first) order.

in the  $N \rightarrow \infty$  limit, operators in  $\mathcal{A}_A$  consist of products of finitely many Majoranas located in subregion  $A$ . In particular, these will act trivially on countably infinitely many indices  $j$  of the Majorana color space.

Based on the results for the ground state entanglement of our model, together with the operators in  $\mathcal{A}_A$  defined above, we identify the operator algebras associated to subregion  $A$  in different regimes of the correlation measure  $X = -2iG_{12}(0)$ . This classification is shown in the phase diagram inset in Fig. 2. When  $X \rightarrow 1$ , the entropy  $S_A \rightarrow 0$  [25]. This is consistent with our physical intuition, since we expect the subsystems to completely factorize in this limit. Thus, we find that  $\mathcal{A}_A$  is a type  $\text{I}_\infty$  algebra when  $X \rightarrow 1$  [which implies  $r \rightarrow \infty$  by (11)]. The index in  $\text{I}_\infty$  alludes to the infinite dimensionality of the local Hilbert space.

When  $X < 1$ , the ground state is no longer factorized, and the entropy (9) is infinite, therefore ruling out  $\mathcal{A}_A$  being of type I. To specify the type, we resort to the definition of a trace functional  $\text{tr}$  on  $\mathcal{A}_A$ . When the maximally entangled state  $|\Psi\rangle$  is in the Hilbert space generated by the algebra  $\mathcal{A}_A$ , a well-defined trace functional is given by  $\text{tr}(\mathbf{a}) \equiv \langle \Psi | \mathbf{a} | \Psi \rangle$ , with  $\mathbf{a} \in \mathcal{A}_A$  [7]. Importantly, when  $X = 0$ , we find that the entanglement entropy in our ground state is infinite *and* maximal up to subleading corrections in  $1/N$  [25]. This implies that our ground state can be mapped to  $|\Psi\rangle$  by applying finitely many Majorana operators. Therefore, we conclude that the functional  $\text{tr}$  defines a proper trace when  $X = 0$  [ $r = 0$  by (11)], thus unveiling that  $\mathcal{A}_A$  is of type  $\text{II}_1$  only at this point; cf. Fig. 2.

As for the regime  $0 < X < 1$ , corresponding to  $0 < r < \infty$  by virtue of (11), we find that the entanglement entropy (9) is infinite but not maximal at leading order as  $N \rightarrow \infty$ . Therefore, our ground state cannot be mapped to the maximally entangled state  $|\Psi\rangle$  by finitely many local operators, and, therefore, the algebra  $\mathcal{A}_A$  does not admit the definition of a trace [7]. This implies that  $\mathcal{A}_A$  is of type III. Recall that the first and second equalities in (11) impose SC for generic potentials, and, therefore, the analysis above is valid in the general interacting case.

In the free case  $J = 0$ , where the entanglement Hamiltonian ought to be quadratic in the fermions, the classification of the algebras can be attained by studying the spectrum of the *modular operator*  $\Delta = \lim_{N \rightarrow \infty} \rho_A \otimes \rho_A^{-1}$  [5,31–33]. Given this setup, we are able to compute the large- $N$  spectrum of  $\Delta$  [25], finding  $\text{Spec}(\Delta) = \{\lambda^n\}_{n \in \mathbb{Z}}$ . Here, the parameter  $\lambda$  is related to the correlations within the subsystem as  $\lambda = [(1 - X)/(1 + X)]$ . When the modular operator has precisely this form, the associated operator algebras are said to be of  $\text{I}_\infty$  when  $\lambda = 0$ , type  $\text{II}_1$  for  $\lambda = 1$ , and type  $\text{III}_\lambda$  for  $\lambda \in (0, 1)$ . Such type  $\text{III}_\lambda$  algebras are known to arise for free fermions on a lattice [34].

At finite  $J$ , these considerations lead to transitions between operator algebras of type  $\text{II}_1 \leftrightarrow \text{III} \leftrightarrow \text{I}_\infty$  in the phase diagram; cf. Fig. 2. Given that the limit of the entropy for  $r \rightarrow 0$  and  $r \rightarrow \infty$  is the same for other types of

potentials, we expect similar phase diagrams to hold for in those cases when  $J$  is finite. Our setup's analytical tractability enables us to study the limit  $J \rightarrow \infty$ , where the solution to the SC equation (11) for the class of interaction potentials with exponential behavior is  $X = \Theta(r - 1/2)$ , with  $\Theta$  the Heaviside step function. In this limit, the entanglement entropy is  $S_A = \Theta(1/2 - r)N \log 2 + \mathcal{O}(1)$  [25]. This results in a direct transition of algebras  $\text{II}_1 \leftrightarrow \text{I}_\infty$  at  $r = 1/2$ , which coincides with the phase transition undergone by the system. The transitions between different types of local operator algebras across the phase diagram provide the third result of our work.

*Closed periodic chains.*—To showcase the generality of our methods, we now consider closed periodic chains. In particular, we focus on a closed chain consisting of  $L$  sites with a Hamiltonian of the form (1) with staggered interaction

$$h_{xy}(\xi) = \delta_{x+1,y}[h_b(\xi)\delta_{\text{mod}_2x,0} + h_a(\xi)\delta_{\text{mod}_2x,1}] + \delta_{x-1,y}[h_a(-\xi)\delta_{\text{mod}_2x,0} + h_b(-\xi)\delta_{\text{mod}_2x,1}], \quad (13)$$

where  $h_a$  and  $h_b$  are generic functions and we have the periodic identification  $L + x \sim x$ . Notice that, by defining cells consisting of adjacent sites interacting by  $h_b$ , we can leverage translational invariance with respect to these cells to solve the model in momentum space [25]. We find that the Green's function is determined by an implicit dependence on its own entries. In particular,  $G$  is an implicit function of only the correlations within a given cell  $G_{2x,2x+1}(0)$  and those connecting adjacent cells  $G_{2x-1,2x}(0)$ . This dependence manifests itself via the parametrization

$$\frac{1-v}{1+v} = \frac{h'_a(-2iG_{2x-1,2x}(0))}{h'_b(-2iG_{2x,2x+1}(0))}, \quad (14)$$

where  $v \in [-1, 1]$ . By the aforementioned translational invariance, the correlations entering (14) are independent of  $x$ . Explicit expressions for  $G_{2x-1,2x}(0)$  and  $G_{2x,2x+1}(0)$  at zero temperature and in the limit  $L \rightarrow \infty$  can be found in terms of  $v$  itself and read

$$G_{2x-1,2x}(0) = \frac{i}{2}g(-v), \quad G_{2x,2x+1}(0) = \frac{i}{2}g(v), \quad (15)$$

$$g(v) = \frac{2\text{sgn}(v)}{\pi} \left[ \frac{E(1-1/v^2)}{1+1/v} + \frac{K(1-1/v^2)}{1+v} \right], \quad (16)$$

with  $K(\xi)$  and  $E(\xi)$  the complete elliptic integrals of the first and second kind, respectively. In the spirit of our methods, we can impose self-consistency inserting (15) into (14) and solving for  $v$ . The model is, thus, completely solved once this value of  $v$  has been found. Notice that  $v \rightarrow -v$  exchanges  $G_{2x-1,2x}(0)$  and  $G_{2x,2x+1}(0)$  by virtue of (15), and this amounts to exchanging  $h_a$  and  $h_b$ . Since  $g(1) = 1$ , the limits  $v \rightarrow 1$  and  $v \rightarrow -1$  correspond to maximal correlations between nearest-neighbor sites within and across cells, respectively. For periodic chains with  $L = 4$ , we can parallel

the previous discussions on the entanglement structure and, consequently, on the typification of the operator algebras across the phase diagram.

*Conclusions and future work.*—We determine the phase structure and entanglement for a large class of Majorana models with  $O(N)$  symmetry in the large- $N$  limit. Despite that we mostly focus on models defined on few sites, these are enough to exhibit a rich phase diagram and entanglement structure yet also sufficiently tractable in the large- $N$  limit such as to explicitly compute key quantities like free energy and entanglement entropy. The von Neumann algebras underlying this entanglement structure are summarized in Fig. 2. Remarkably, all three types of algebra are featured throughout the phase diagram of our model. Therefore, our class of exactly solvable models gives rise to nontrivial operator algebra transitions in a highly controllable way. This allows us to track the parameter regimes in which the correlations signal the factorization of the ground state into a product state, as shown in Fig. 1. While here we use correlations for characterizing factorization, a state-based approach consists of using entanglement orbits [18]. In spite of the different approaches used, we see a similar relation between factorization and the value of entanglement entropy. Analyzing these similarities is a promising line of future investigations.

Intriguingly, the algebra transition we find in the strong coupling limit  $J \rightarrow \infty$  coincides with a phase transition that connects a factorized and a nonfactorized state, similarly to the holographic Hawking-Page phase transition [12,13]. Differently from our case (i.e.,  $\text{II}_1 \leftrightarrow \text{I}_\infty$ ), this is a transition between algebras of type  $\text{I}_\infty$  and  $\text{III}_1$  as a function of the temperature. Remarkably, we observe an analogous algebra transition, although our model differs from previous works in the context of holography [27–29,35–38], which consider on-site random interactions leading to a nonzero entropy at zero temperature.

Further relations to holography can be obtained by including random disorder into our model by attaching an SYK model to each lattice site. This enlarges the phase diagram, and we expect the competition between spatially inhomogeneous hoppings and the locally random disorder to change the renormalization group properties of critical points. Additionally, it is promising to investigate the mentioned connections with Brownian SYK [26,39–41].

Since our setup allows for general spatially disordered interactions, it is also relevant for further infinite disordered chains [42,43], where the hopping parameters are distributed according to a binary aperiodic sequence. These so-called aperiodic spin chains have recently been considered [23,44–46] as a step toward establishing a holographic duality on discrete spaces. Finally, in the free case, our model can be interpreted as an instance of a Kitaev chain [47], which has physical realizations in terms of superconducting quantum wires. It would be intriguing to investigate the changes to this physical picture in the interacting case.

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