Amplitudes at Strong Coupling as Hyper-Kähler Scalars

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Alday and Maldacena conjectured an equivalence between string amplitudes in $\mathrm{AdS}_5 \times S^5$ and null polygonal Wilson loops in planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. At strong coupling this identifies SYM amplitudes with areas of minimal surfaces in anti–de Sitter space. For minimal surfaces in AdS_3, we find that the nontrivial part of these amplitudes, the *remainder function*, satisfies an integrable system of nonlinear differential equations, and we give its Lax form. The result follows from a new perspective on "Y systems," which defines a new psuedo-hyper-Kähler structure *directly* on the space of kinematic data, via a natural twistor space defined by the Y-system equations. The remainder function is the (pseudo-)Kähler scalar for this geometry. This connection to pseudo-hyper-Kähler geometry and its twistor theory provides a new ingredient for extending recent conjectures for nonperturbative amplitudes using structures arising at strong coupling.

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Introduction.—The spaces of kinematic data $\mathcal{K}_n$ on which $\mathcal{N} = 4$ super Yang-Mills amplitudes are defined has a rich combinatorial that has been a fertile ground for advancing the understanding of scattering amplitudes [1–3]. This Letter discovers new geometric structures that arise from the analysis of the amplitude at strong coupling [4,5] and that complements the combinatorial cluster variety and positivity structures that arise at weak coupling. The successes of the bootstrap of [6,7] hinges on the cluster geometry of kinematic spaces $\mathcal{K}_n$ but more recently features from strong coupling have played an instrumental role in generating the nonperturbative conjectures of [8,9]. Our new integrable geometric structures encode the full structure of the strong coupling amplitude and will provide foundations for further advances in this direction.

Alday-Maldacena [10,11] conjectured a three-way correspondence between planar amplitudes, $\mathcal{A}$: planar null-polygonal Wilson loops, $\langle \mathcal{W}_\gamma \rangle$, both for $\mathcal{N} = 4$ super-Yang-Mills, and type IIB string amplitudes in $\mathrm{AdS}_5 \times S^5$

$$\mathcal{A} = \langle \mathcal{W}_\gamma \rangle = \int_{\partial\Sigma=\gamma} \mathcal{D}[\Sigma \subset \mathrm{AdS}_5 \times S^5] \mathrm{e}^{-\frac{1}{\alpha'} S_{\mathrm{string}}}. \quad (1)$$

Here γ is a null polygon made up from the null momenta in the amplitude. The α' string parameter is related to the 't Hooft coupling, λ, by $R^2_{\mathrm{AdS}}/\alpha' = \sqrt{\lambda}$. The first equality has been proved (The tree-level MHV amplitude is removed in the definition of $\mathcal{A}$.) in perturbation theory [12,13]. The second equality is a conjecture arising from the AdS/CFT correspondence. It has only been systematically investigated at strong coupling as $\lambda \to \infty$ (and $\alpha' \to 0$), where the semi-classical approximation for the string gives

$$\langle \mathcal{W}_\gamma \rangle \sim \mathrm{e}^{-\mathrm{Area}(\Sigma)/\alpha'}, \quad (2)$$

where $\mathrm{Area}(\Sigma)$ is the area of the minimal surface, Σ, bounded by γ. Like the Wilson loop $\langle \mathcal{W}_\gamma \rangle$ the area of the minimal surface Σ is divergent at its cusps where it meets the boundary at infinity. These correspond precisely to the infrared divergences of the amplitude. These divergences can be removed compatibly with all three interpretations leading to a regularized area or remainder function $R(\gamma)$, which is our main object of study.

Alday-Maldacena reformulate minimal surfaces in AdS as a Hitchin system and express the area as the Hamiltonian for a certain circle action on the kinematic data [4]. Hitchin moduli spaces are often hyper-Kähler [14,15] but discrete symmetries are imposed to give minimal surfaces so that standard results (from, e.g., [16,17]) do not apply, and our space is not expected to be hyper-Kähler from these arguments, see Sec. 3.3 of [4]. However, we will show that these smaller moduli spaces are often *pseudo-hyper-Kähler*, i.e., the analog of hyper-Kähler appropriate to metrics of split signature.

Although the main structures we use are available for full kinematics, in this Letter we work with momenta and the Wilson loop lying in $1 + 1$ dimensions with the spanning minimal surface living in AdS_3. Although this might seem restrictive, in practice the extension to full kinematics is well known to be a straightforward but elaborate extension to larger cluster varieties [18,19] where the extra complexity will obscure the essential ideas. We therefore postpone this discussion. We prove here that the regularized area of this surface is a Kähler scalar for a pseudo-hyper-Kähler structure on $\mathcal{K}_n$, when it has $4k$ dimensions. We do this by using the Y system [5] to define a twistor space for $\mathcal{K}_n$, analogous to the twistor spaces for full Hitchin moduli spaces; we expect this novel connection between Y systems and twistor constructions to be of much wider applicability. Then we derive a system of integrable equations satisfied by the regularized area, which can be used to solve for the area. In Sec. 2 we introduce the kinematic space $\mathcal{K}_n$ and both its cluster and associated Poisson or symplectic structure. In Sec. 3 we recall the Y system [5] and explain how it defines a twistor space for $\mathcal{K}$. In Sec. 4 we find the hyper-Kähler structure explicitly and show the regularized area is a Kähler scalar for a split signature analog of a hyper-Kähler metric that satisfies an integrable system of generalized Plebanski equations. Finally in Sec. 5 we mention a number of checks and further developments including applications to amplitudes at finite coupling.

The spaces of kinematic data in $1+1$ *dimensions.*—Our kinematic space $\mathcal{K}_n$ here will be the moduli space of $2n$-sided null polygonal Wilson loops in $1+1$ dimensions. Such Wilson loops are given by a set of ordered null momenta (the "edges" of the loop) that sum to zero (so that the loop closes). Take null coordinates (X^+, X^-) on Minkowski space with metric

$$\mathrm{d}s^2 = 2\mathrm{d}X^+\mathrm{d}X^-. \tag{3}$$

The edges of a polygonal Wilson-loop alternate between lines of constant X^+ and constant X^-. The kinematic data for a $2n$-sided Wilson loop in AdS_3 is therefore given by two cyclically ordered sets of real numbers $\{X_i^+\}, \{X_i^-\}$, with $i = 0, \ldots, n-1$. Vertices of the polygon are given by the points (X_i^+, X_{i-1}^-), (X_i^+, X_i^-), then (X_{i+1}^+, X_i^-) and so on as illustrated in Fig. 1. Conformal invariance means that our functions of these parameters should be invariant under Möbius transformations on the X_i^+ and X_i^- separately. Thus the space of kinematic data $\mathcal{K}_n$ is

$$\mathcal{K}_n = \mathcal{M}_{0,n}^{\mathbb{R}} \times \mathcal{M}_{0,n}^{\mathbb{R}}, \tag{4}$$

where

$$\mathcal{M}_{0,n}^{\mathbb{R}} = \{X_i^\pm, i = 1, \ldots, n\}/PSL_2 \tag{5}$$

is the moduli space of n points on $\mathbb{RP}^1$ modulo Möbius transformations.

Möbius invariant functions on $\mathcal{K}$ are functions of cross ratios

$$(ij|kl)^\pm = \frac{(X_i^\pm - X_j^\pm)(X_k^\pm - X_l^\pm)}{(X_i^\pm - X_l^\pm)(X_j^\pm - X_k^\pm)}. \tag{6}$$

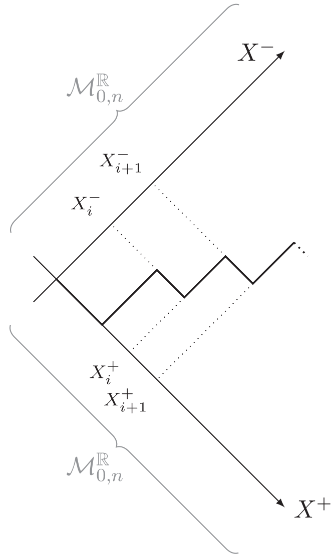

FIG. 1. Null coordinates on $\mathbb{M}^{1,1}$.

Our charts for $\mathcal{K}$ are sets of cross ratios called *clusters* [20]. A cluster is specified by choosing a *triangulation* of the n-gon. For a fixed triangulation, the chords are indexed by $s = 1, \ldots, n-3$. Such a chord connects say vertex i to k making the diagonal $i-k$ of some quadrilateral (i, j, k, l) formed by two triangles. To this chord associate the coordinate

$$\chi_s^\pm = (il|kj)^\pm. \tag{7}$$

The set of these cross ratios, $\{\chi_s^\pm\}$ define a cluster of coordinates on $\mathcal{K}_n$.

Different choices of clusters of coordinates are related by *mutation relations*. Flipping a chord s inside a quadrilateral that it is a diagonal of, gives a new chord, s', and the new cross ratios are related to the old ones by

$$\mu(\chi_s) = \chi_s^{-1}, \qquad \mu(\chi_t) = \chi_t(1+\chi_s^{\epsilon_{st}})^{\epsilon_{st}}. \tag{8}$$

The only cross ratios that change are those sharing a triangle with s in the triangulation.

Finally, there is a 2-form on $\mathcal{K}_n$ that is symplectic when $\mathcal{K}_n$ is even dimensional (n odd). Fixing a triangulation of the n-gon as above, define an antisymmetric matrix, $\epsilon_{ss'}$ where for chords s and s', write $\epsilon_{ss'} = 0$ if the two chords do not share a triangle. If they share a triangle, write $\epsilon_{ss'} = 1$ if s' is clockwise of s, or write $\epsilon_{s's} = -1$ is s' is counterclockwise of s. On each copy of $\mathcal{M}_{0,n}^{\mathbb{R}}$ define

$$\omega^\pm = \sum_{i,j} \epsilon_{ij} \mathrm{d}\log\chi_i^\pm \wedge \mathrm{d}\log\chi_j^\pm. \tag{9}$$

We fix, for example, the zigzag triangulation of Fig. 2, as in [5]. This gives cross ratios

$$\chi_s^\pm = \begin{cases} (s-1, s|-s-1, -s)^\pm & s\,\text{odd}, \\ (s-1, s|-s, -s+1)^\pm & s\,\text{even}, \end{cases} \qquad s = 1, \ldots, n-3, \tag{10}$$

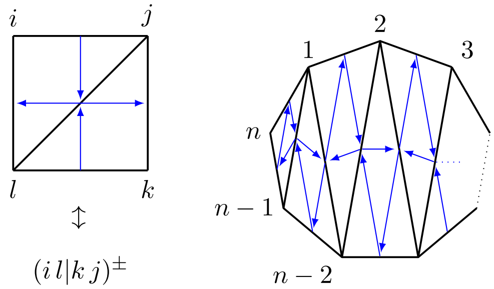

FIG. 2. Left: The correspondence between chords of a triangulation and cross ratios. Right: The zigzag triangulation of the polygon.

where we specify vertices of the polygon mod n. The matrix $\epsilon_{ss'}$ is given by $\epsilon_{s,s+1} = 1$, s odd, and $\epsilon_{s,s+1} = -1$ for s even giving

$$\omega^{\pm} = \sum \mathrm{d}\log\chi^{\pm}_{2i} \wedge (\mathrm{d}\log\chi^{\pm}_{2i-1} - \mathrm{d}\log\chi^{\pm}_{2i+1}). \quad (11)$$

The symplectic structures is invariant under mutations [20]. After one mutation using (8), the symplectic 2-form becomes

$$\omega^{\pm} = \mu(\omega^{\pm}) \coloneqq \sum_{i,j} \tilde{\epsilon}_{ij}\mathrm{d}\log\mu(\chi^{\pm}_i) \wedge \mathrm{d}\log\mu(\chi^{\pm}_j), \quad (12)$$

where $\mu(\chi^{\pm}_i)$ are the new cross ratios, and $\tilde{\epsilon}_{ij}$ is the matrix of the mutated triangulation. Thus, a series of mutations preserves $\omega^{\pm}$ so $\omega^{\pm}$ is independent of the choice of cluster.

From the Y system to the twistor space.—The Y system of [5] is based on the zigzag cluster coordinates (χ^+_s, χ^-_s) on $\mathcal{K}_n$ [Eq. (2.8)]. The Y system associated with this cluster consists of a family of functions

$$\mathcal{Y}_s = \mathcal{Y}_s(\chi^+_r, \chi^-_r, \zeta) : \mathcal{K}_n \times \mathbb{CP}^1 \longrightarrow \mathbb{C}, \quad s = 1, \ldots, n-3 \quad (13)$$

that are complex analytic in the *spectral parameter* $\zeta \in \mathbb{CP}^1$ fixed by the following four conditions. First, at $\zeta = 1, i$ we have

$$\mathcal{Y}_s(1) = \chi^+_s, \qquad \mathcal{Y}_s(i) = \chi^-_s. \quad (14)$$

Second, the $\mathcal{Y}_s$, are holomorphic except for branching singularities at $\zeta = 0$ and $\zeta = \infty$, with a branch cut along $\mathbb{R}^-$. Third, we require exponential asymptotics at the singularities:

$$\log\mathcal{Y}_s \simeq Z_s\zeta^{-1} + \ldots, \quad \text{as } \zeta \to 0,$$
$$\log\mathcal{Y}_s \simeq \bar{Z}_s\zeta + \ldots, \quad \text{as } \zeta \to \infty \quad (15)$$

for some $Z_s(\chi^+_r, \chi^-_r)$. Finally, we define the analytic continuation $\mathcal{Y}_s$ across $\mathbb{R}^-$. Writing $\mathcal{Y}^{++}_s(\zeta) = \mathcal{Y}(\mathrm{e}^{i\pi}\zeta)$, the analytic continuation for the zigzag cluster is given by

$$\mathcal{Y}^{++}_{2k+1}\mathcal{Y}_{2k+1} = (1 + \mathcal{Y}_{2k+2})(1 + \mathcal{Y}_{2k}),$$
$$\mathcal{Y}^{++}_{2k}\mathcal{Y}_{2k} = (1 + \mathcal{Y}^{++}_{2k+1})(1 + \mathcal{Y}^{++}_{2k-1}). \quad (16)$$

These relations are defined so that analytic continuation implements a series of mutation relations on the $\chi^{\pm}_s$, (8), that *rotate* the initial triangulation by $2\pi/n$. In other words, $\mathcal{Y}^{++}_s(1)$ and $\mathcal{Y}^{++}_s(i)$ are the cross ratios obtained by performing this series of mutations.

The $\mathcal{Y}_s$ are uniquely determined by these conditions as can be seen by iteration of the equivalent integral equations of the thermodynamic Bethe ansatz (TBA) described in [5,21].

We now define the *twistor space* to be (We suppress the proper description of the real slice $|\zeta| = 1$ as treated in the analogous case of [22].) $\mathcal{T}_n = \mathcal{K}_n \times \mathbb{CP}^1$ as a smooth manifold. The $\mathcal{Y}_s$ functions define holomorphic coordinates on $\mathcal{T}_n$, making it a complex manifold.

Proposition 3.1.—$\mathcal{T}_n$ is a complex $n-2$ manifold with local holomorphic coordinates $(\mathcal{Y}_s, \zeta)$ and holomorphic projection: $p\colon \mathcal{T}_n \to \mathbb{CP}^1$. There is a family of symplectic 2-forms $\Sigma(\zeta)$ on the fibers of p. For odd n, $\Sigma(\zeta)$ is nondegenerate. Moreover, $[\mathcal{T}_n, \Sigma(\zeta)]$ is invariant under the holomorphic circle action

$$(\mathcal{Y}_s, \zeta) \longrightarrow (\mathcal{Y}_s, \mathrm{e}^{i\theta}\zeta). \quad (17)$$

Finally, there is an antiholomorphic involution on $\mathcal{T}_n$,

$$(\mathcal{Y}_s, \zeta) \longrightarrow (\bar{\mathcal{Y}}_s, 1/\bar{\zeta}), \quad (18)$$

so that the $\mathcal{Y}_s$ are real on the unit circle $|\zeta| = 1$.

Proof.—Construct $\mathcal{T}_n$ by gluing together holomorphic coordinate patches

$$U = \{-\pi < \arg\zeta < \pi\}, \text{ and } U^{++} = \{0 < \arg\zeta < 2\pi\}, \quad (19)$$

glued by $\zeta \mapsto e^{i\pi}\zeta$. The $\mathcal{Y}_s$ functions are holomorphic coordinates on U and the $\mathcal{Y}^{++}_s$ on U^{++}. These two sets of holomorphic coordinates are glued together on $U \cap U^{++}$ by the Y-system equations (3.4).

For fixed ζ, define

$$\Sigma(\zeta) \coloneqq \sum \epsilon_{ij}\mathrm{d}\log\mathcal{Y}_i \wedge d\log\mathcal{Y}_j. \quad (20)$$

We claim that this closed 2-form is preserved by mutations. (A direct proof is to define a generating function for (3.4). A conceptually interesting proof is to apply a series of "mutations on sinks," as in the proof of Zamolodchikov's periodicity conjecture given, for example, by Theorem 8.8 of [23]; see [24] for a review.) In particular, $\Sigma^{++}(\zeta) \equiv \Sigma(e^{i\pi}\zeta) = \Sigma(\zeta)$. So $\Sigma(\zeta)$ is defined for all ζ, except for $\zeta = 0$ and $\zeta = \infty$. Moreover, $\mathcal{Y}_s$ are invariant under the circle symmetry, so Σ is likewise circle invariant.

Finally, the functions $\bar{\mathcal{Y}}_s(1/\bar{\zeta})$ have the same analytic properties and special values as the functions $\mathcal{Y}_s(\zeta)$, and satisfy the same Y-system equations as $\mathcal{Y}_s(\zeta)$. But the $\mathcal{Y}_s$ functions are unique, so that $\bar{\mathcal{Y}}_s(1/\bar{\zeta}) = \mathcal{Y}_s(\zeta)$. ■

Integrable system for the remainder function.—The *remainder function*, $R(\chi^+_r, \chi^-_s)$, is the nontrivial part of the regularized area of the minimal surface in AdS_3.

Here we define R to be the Hamiltonian for the circle action, (3.5) (following Sec. 3 of [4]). Our main result is that R satisfies an integrable system on kinematic space, $\mathcal{K}_n$. In fact, R defines a pseudo-hyper-Kähler structure on $\mathcal{K}_n$.

A pseudo-hyper-Kähler structure is the analog of a hyper-Kähler structure [25,26], but with a split-signature metric, as in [27]. This structure consists of a split-signature metric, together with three symplectic 2-forms, $\omega^\pm$ and Ω. The 2-forms are compatible with the metric if they satisfy the *pseudo-quaternion* relations. Using the metric to raise the indices of the 2-forms to obtain tensors, $J^\pm$ and $\Omega^\sharp$, the pseudo-quaternion relations are: $(J^+)^2 = (\Omega^\sharp)^2 = 1$, $(J^-)^2 = -1$ and $J^-J^+ = -J^+J^- = \Omega^\sharp$, $\{J^\pm, \Omega^\sharp\} = 0$.

Proposition 4.1.—For n odd, $\mathcal{K}_n$ is pseudo-hyper-Kähler, with split-signature metric

$$ds^2 \coloneqq R^{rs} \mathrm{d}x_r^+ \mathrm{d}x_s^-, \tag{21}$$

and the three symplectic 2-forms

$$\begin{aligned}\omega^\pm &= \epsilon^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^+ \pm \epsilon^{rs} \mathrm{d}x_r^- \wedge \mathrm{d}x_s^-, \\ \Omega &= R^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^-.\end{aligned} \tag{22}$$

Proposition 4.3 can be shown as a consequence of the following result, which shows that R satisfies the analog of the *first Plebanski equation*.

Proposition 4.2.—For n odd, the remainder function satisfies

$$R^{pq} R^{rs} \epsilon_{pr} = \epsilon^{qs}, \tag{23}$$

together with circle invariance [Eq. (4.22)]. Here $\epsilon_{pq}\epsilon^{qr} = \delta_p^r$ and R^{pq} is the Hessian:

$$R^{rs} = \frac{\partial^2 R}{\partial x_r^+ \partial x_s^-}, \tag{24}$$

with $x_p^\pm \equiv \log \chi_p^\pm$. This is an integrable system with Lax system $\{\mathcal{L}_r, \tilde{V}\}$, where

$$\begin{aligned}\mathcal{L}_r &\coloneqq (\zeta^2 - 1)\frac{\partial}{\partial x_r^+} + (\zeta^2 + 1) i R^{rs} \frac{\partial}{\partial x_s^-}, \\ \tilde{V} &\coloneqq \epsilon_{rs}\left(\frac{\partial R}{\partial x_s^+}\frac{\partial}{\partial x_r^+} + \frac{\partial R}{\partial x_s^-}\frac{\partial}{\partial x_r^-}\right) + i\zeta \frac{\partial}{\partial \zeta}.\end{aligned} \tag{25}$$

A pseudo-hyper-Kähler structure is the analog of a hyper-Kähler structure, but with a split-signature metric.

Proposition 4.3.—For n odd, $\mathcal{K}_n$ is pseudo-hyper-Kähler, with split-signature metric

$$ds^2 \coloneqq R^{rs} \mathrm{d}x_r^+ \mathrm{d}x_s^-, \tag{26}$$

and the three symplectic 2-forms

$$\begin{aligned}\omega^\pm &= \epsilon^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^+ \pm \epsilon^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^+, \\ \Omega &= R^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^-.\end{aligned} \tag{27}$$

Proposition 4.3 follows from the proof of Proposition 4.2.

Proof.—Consider a Laurent series expansion of $\Sigma(\zeta)$ in ζ. Equation (3.2) implies that $\Sigma(\zeta)$ has special values

$$\Sigma(1) = \sum \epsilon_{ij} \mathrm{d}x_i^+ \wedge \mathrm{d}x_j^+, \text{ and } \Sigma(i) = \sum \epsilon_{ij} \mathrm{d}x_i^- \wedge \mathrm{d}x_j^-. \tag{28}$$

Moreover, writing $y_s \equiv \log \mathcal{Y}_s$, $y_s \sim 1/\zeta$ at 0 and $y_s \sim \zeta$ at ∞. So $\Sigma(\zeta)$ has *double poles* at $\zeta = 0$ and ∞. By Proposition 3.1,

$$\Sigma(-\zeta) = \Sigma(\zeta), \tag{29}$$

so that the Laurent series only contains even powers of ζ. Equations (4.8) and (4.9) imply that the Laurent expansion is

$$\Sigma(\zeta) = \frac{(\zeta^2+1)^2}{4\zeta^2}\Sigma(1) - \frac{(\zeta^2-1)^2}{4\zeta^2}\Sigma(i) + \frac{(\zeta^4-1)}{4\zeta^2} i\Omega. \tag{30}$$

for some ζ-independent closed 2-form Ω.

Since $\Sigma(\zeta)$ is nondegenerate with rank $n-3$,

$$[\Sigma(\zeta)]^{(n-1)/2} = 0. \tag{31}$$

Taking the derivative with respect to ζ at $\zeta = 1, i$ gives, respectively,

$$[\Sigma(1)]^{(n-3)/2} \wedge \Omega = 0, \qquad [\Sigma(i)]^{(n-3)/2} \wedge \Omega = 0. \tag{32}$$

Since $[\Sigma(1)]^{(n-3)/2} \neq 0$ is of top degree in the dx_r^+ alone, the first implies that Ω is at least linear in the $\mathrm{d}x_r^+$ and the second that it is at least linear in $\mathrm{d}x_s^-$, so

$$\Omega = \frac{1}{4} J^{rs} \mathrm{d}x_r^+ \wedge \mathrm{d}x_s^- \tag{33}$$

for some $J^{rs}(x^+, x^-)$. But Ω is closed, so

$$J^{rs} = \frac{\partial^2 J}{\partial x_r^+ \partial x_s^-} \tag{34}$$

for some scalar $J(x^+, x^-)$. Again using the vanishing of the leading terms in the expansion of (31), and extracting components with exactly two $\mathrm{d}x^+$'s (or two dx^-'s) gives

$$\epsilon_{rr'} J^{rs} J^{r's'} = \epsilon^{ss'}. \tag{35}$$

There is a Lax system for (4.15). Since $\Sigma(\zeta)$ has rank $n-3$, it has $n-3$ null directions. It can be checked using (4.10) and (4.13) that these null directions are spanned by

$$\mathcal{L}_r \equiv (\zeta^2 - 1)\frac{\partial}{\partial x_r^+} + (\zeta^2 + 1) i J^{rs} \frac{\partial}{\partial x_s^-}, \tag{36}$$

for $r = 1, \ldots, n-3$. Since $\mathcal{L}_r \lrcorner \Sigma(\zeta) = 0$, it follows that the $\mathcal{L}_r$ are involutive. In fact, using (4.15) and (4.16), these vector fields commute: $[\mathcal{L}_r, \mathcal{L}_s] = 0$. Finally, since ϵ_{ij} is nondegenerate, and since the $\mathcal{L}_r$ span the kernel of $\Sigma(\zeta)$, it follows that these vector fields annihilate the Y functions: $\mathcal{L}_r \mathcal{Y}_s(\zeta) = 0$. This can be used to solve for the Y functions. Near $\zeta = 1, i$, we find

$$\log \mathcal{Y}_r(\zeta) = x_r^\pm + (\zeta^2 \mp 1)\epsilon_{rs} i \frac{\partial J}{\partial x_s^\pm} + O[(\zeta^2 \mp 1)^2], \quad (37)$$

and the Lax system further determines all higher order terms.

Let V be the vector field on $\mathcal{K}_n$ generating the circle action that rotates the phases of the Z_r of (3.3). The remainder function, R, was shown in [4] to be the Hamiltonian for V with respect to the symplectic form $\Sigma(1) + \Sigma(i)$ obtained as the coefficient of ζ^0 in $\Sigma(\zeta)$. The Hamiltonian equation is $dR = V \lrcorner [\Sigma(1) + \Sigma(i)]$. This implies that

$$V = \epsilon_{rs} \left(\frac{\partial R}{\partial x_s^+} \frac{\partial}{\partial x_r^+} + \frac{\partial R}{\partial x_s^-} \frac{\partial}{\partial x_r^-} \right). \quad (38)$$

We can relate this to J as follows. The lift $\tilde{V}$ of V to $\mathcal{T}_n$ acts on ζ by $\tilde{V}(\zeta) = i\zeta$ and so is given by $\tilde{V} = V + i\zeta\partial_\zeta$. On $\mathcal{T}_n$, $\tilde{V}$ annihilates $\Sigma(\zeta)$ because $\mathcal{Y}_r$ and hence $\Sigma(\zeta)$ are circle invariant. Thus the coefficients $A^{(\pm 2)}$ of $\zeta^{\pm 2}$ in $\Sigma(\zeta)$ have weights ∓ 2 under V:

$$\pounds_V A^{(\pm 2)} = \mp 2iA^{(\pm 2)}, \quad (39)$$

where $\pounds_V A^{(\pm 2)} = \mathrm{d}(V \lrcorner A^{(\pm 2)})$. Adding both signs of (4.19) gives

$$\frac{\partial^2 R}{\partial x_r^+ \partial x_s^-} = \frac{\partial^2 J}{\partial x_r^+ \partial x_s^-}. \quad (40)$$

Since J is defined only up to a sum of a function of x_r^+ and another of x_r^-, we can fix this freedom by identifying $J \equiv R$. Then the difference between the $\pm$ parts of (4.19) gives

$$0 = \partial_{x_r^+}(J^{ts} V_t^+) + \partial_{x_s^-}(J^{rt} V_t^-), \quad \epsilon_{rs} = \partial_{x_s^\pm}(J^{rt} V_t^\mp). \quad (41)$$

With $J = R$, the first of these equations reads

$$0 = \partial_{x_r^+} \left(\frac{\partial^2 R}{\partial x_t^+ \partial x_s^-} \frac{\partial R}{\partial x_u^+} \epsilon^{tu} \right) + \partial_{x_s^-} \left(\frac{\partial^2 R}{\partial x_r^+ \partial x_t^-} \frac{\partial R}{\partial x_u^-} \epsilon^{tu} \right). \quad (42)$$

Likewise, with $J = R$, the remaining equations simplify, and are solved by (35), which is now

$$0 = \epsilon_{rs} R^{rr'} R^{ss'} + \epsilon^{r's'}. \quad (43)$$

Together, (4.22) and (4.23) are an integrable system for R. In Lax form, the system is $\{\mathcal{L}_r, V + i\zeta\partial_\zeta\}$, where $V + i\zeta\partial_\zeta$ is the circle symmetry generator.

To complete the proof, we show that (4.22) is linearly independent of (4.23). Write $\partial^r = \partial/\partial x_r^+$ and $\partial^{r'} = \partial/\partial x_{r'}^-$. Then (4.23) can be written as

$$\partial^{[r}(R^{s]s'} R^{r'} \epsilon_{r's'} - \epsilon^{s]q} x_q^+) = 0, \quad \text{or,}$$
$$\partial^{[r'}(R^{s']s} R^{r} \epsilon_{rs} - \epsilon^{s']q'} x_{q'}^-) = 0. \quad (44)$$

These are integrability conditions for the existence of functions S and S' satisfying

$$R^{ss'} R^{r'} \epsilon_{r's'} - \epsilon^{sq} x_q^+ = \partial^s S, \quad R^{s's} R^r \epsilon_{rs} - \epsilon^{s'q'} x_{q'}^- = \partial^{s'} S', \quad (45)$$

where S is defined up to functions of $x_{r'}^-$, and S' is defined up to functions of x_r^+. Equation (4.22) becomes $\partial^r \partial^{r'} S' + \partial^{r'} \partial^r S = 0$, which imposes one additional constraint on the system: $S + S' = 0$. ∎

Note that (4.25) together with $S + S' = 0$, provides an alternative form of the integrable system, with one fewer derivatives, at the price of introducing the additional function S.

Discussion.—The remainder function R is the key observable for SYM amplitudes. We find that at strong coupling R satisfies an integrable system (for n odd), analogous to the first Plebanski equation for 4d self-dual gravity. Our result follows from a new perspective on the Y systems of [18]: they define twistor spaces for the kinematic space $\mathcal{K}_n$ of null polygonal Wilson loops with $2n$ sides. Moreover, we find that $\mathcal{K}_n$ carries a pseudo-hyper-Kähler structure, for which R is the pseudo-Kahler scalar. The Poisson structures for the n-even cases are necessarily degenerate, and the Y-system can be directly solved for the combinations of Y-functions in the kernel of the Poisson structure; our methods then apply to the symplectic leaves on which these functions are held constant. These results establish a new geometry underpinning the structure of Wilson loops at strong coupling whose study, following the strategy of the conjectures of [8,9], should give insights into the amplitudes to all orders. Similar ideas apply to other SYM operators, see, for example, [28,29] for examples where the imprint from strong coupling can be seen in the full nonperturbative correlator.

We stress that the new pseudo-hyper-Kähler spaces studied here are *not* related to other hyper-Kähler structures that arise in studies of Hitchin systems. Hitchin showed that moduli spaces of regular Hitchin systems admit hyper-Kähler structures, [15] and this result has been partially extended to the irregular case in [16] and more analogously by Gaiotto, Moore, and Neitzke in [17,30]. However, neither of these apply directly to amplitudes at strong coupling, because $\mathcal{K}_n$ parametrizes an *invariant subspace* of the Hitchin moduli space under an involution [4,5,21]. We have verified that the standard hyper-Kähler structures do not restrict to this subspace to yield our results, even in simple examples.

We comment on implications of our results. First, these methods apply to other cluster varieties or Y systems, such as the ADE-type cases [31], and the affine and surface-type cluster algebras. Physically, type D corresponds to form factors at strong coupling for restricted kinematics. Beyond these cases, Grassmannian cluster algebras appear when computing strong coupling amplitudes and form factors for full $\mathcal{N} = 4$ SYM kinematics. The Y systems associated to these cluster algebras are known, [5] as are generalizations incorporating form factors [19,32]. Our strategy developed here will lead to integrable systems for the amplitudes in all of these cases, albeit with novelties arising beyond restricted kinematics.

It is well known that soft and multicollinear limits provide a system of boundary conditions on the remainder function, see [4,32], and in particular equation (2.10) of [33], which gives the boundary condition

$$R_n \to R_{n-m} + R_{m+4}, \tag{46}$$

see also [7] for a recent summary at weak coupling. Coupled with the differential equations we have found here, this will lead to a unique determination of the remainder function starting with the smallest nontrivial boundary conditions provided by the octagon, which is treated in full detail in [4]: this clearly shows that the full solution is highly nontrivial with a nontrivial infinite series beyond the trivial quadratic solution $R = \epsilon_{rs} x_+^r x_-^s$.

Finally, our results suggest avenues beyond the strong coupling limit. The work of [8,9] identifies lines in the full kinematic space where quadratic log solutions are valid at strong coupling, and explains how formulas for all values of the coupling may be obtained from the solutions to the Y system there may be found. Thus these structures are likely to be important for extensions to these conjectures. Our differential equations at strong coupling, and associated structures might therefore be deformable to some that hold beyond strong coupling. In this direction, there are several other connections to explore. Our integrable system can be recovered from a twistor sigma model action [34,35]; quantizing analogous models might allow computations beyond the strong coupling limit. Related structures arise for the anomalous dimension spectrum at finite coupling in the form of the Y system of the quantum spectral curve [36], with the coupling constant incorporated via the "Joukowski correspondence," [37–39].

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