Floquet Fermi Liquid

Li-kun Shi,[1,*] Oles Matsyshyn,[2,*] Justin C. W. Song,[2] and Inti Sodemann Villadiego[1,†]
[1]Institut für Theoretische Physik, Universität Leipzig, Brüderstraße 16, 04103, Leipzig, Germany
[2]Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore

(Received 7 October 2023; revised 22 January 2024; accepted 12 March 2024; published 3 April 2024)

We demonstrate the existence of a nonequilibrium "Floquet Fermi liquid" state arising in partially filled Floquet Bloch bands weakly coupled to ideal fermionic baths, which possess a collection of "Floquet Fermi surfaces" enclosed inside each other, resembling matryoshka dolls. We elucidate several properties of these states, including their quantum oscillations under magnetic fields which feature slow beating patterns of their amplitude reflecting the different areas of the Floquet Fermi surfaces, consistent with those observed in microwave induced resistance oscillation experiments. We also investigate their specific heat and thermodynamic density of states and demonstrate how by controlling properties of the drive, such as its frequency, one can tune some of the Floquet Fermi surfaces toward nonequilibrium Van Hove singularities without changing the electron density.

DOI: 10.1103/PhysRevLett.132.146402

Introduction.—Characterizing the landscape of nonequilibrium quantum states of matter is a major open frontier in the study of quantum many-body systems. Recent years have seen substantial progress in our understanding and experimental control of periodically driven quantum systems also known as Floquet systems. In particular, the flexibility afforded by tuning the properties of the drive has led to a variety of interesting proposals to realize nontrivial Floquet topological band structures [1–4]. However, a crucial point that has been emphasized in several pioneering works [5–14] is that Floquet topological bands should not be occupied according to naive equilibrium distributions, such as the Fermi-Dirac distribution, and this is at the heart of their markedly different physical behavior relative to their equilibrium counterparts. Despite all these important efforts in understanding the occupation of Floquet topological insulators, much less attention has been devoted to elucidating the ultimate nature of the analog of the metallic state for Floquet systems.

Our study tries to precisely fill in this gap by systematically investigating the fate of Fermi liquids and their Fermi surfaces when they are driven far away from equilibrium by periodic time-dependent perturbations. To address this we will revisit the more general question of how should Floquet states be occupied by fermions? To answer this question it is important to consider a system in contact with a bath, because periodically driven closed systems that are thermalizing tend to have trivial infinite temperature steady states [15–17], and those that are not thermalizing tend to retain memory of initial conditions [16,18–22], making their steady states not unique. In contrast to a closed system, the coupling to the bath allows the system to release the energy that it gains from the work performed by the periodic drive every cycle, enabling it to reach a nontrivial steady state at late times. We will consider an "all fermion" setting, where the system and the bath are both comprised only of fermions.

Within such a setting, we have a found a remarkable answer to this question: a nonequilibrium steady state with a sizable energy density difference relative to the ground state but which retains its quantum nature, which we call the Floquet Fermi liquid. Unlike its equilibrium counterpart where states are occupied according to the Fermi Dirac distribution, the Floquet Fermi liquid features a staircase-shaped occupation of the Floquet band with multiple jumps that evolve into sharp discontinuities at zero temperature giving rise to a collection of enclosed Floquet Fermi surfaces (see Fig. 1). We will investigate the fingerprints left by Floquet Fermi surfaces in various observables, such as the appearance of a slow beating of the quantum oscillations amplitude, as well as the density of states and the specific heat.

Fermi Dirac staircase periodic Gibbs ensemble.—Consider a model of noninteracting fermions in contact with a fermionic bath, with a single particle Hamiltonian of the system plus bath of the form

$$H(t) = \begin{bmatrix} H_S(t) & H_{SB} \\ H_{BS} & H_B \end{bmatrix}. \tag{1}$$

The system can be viewed as a tight-binding model, where each site can tunnel (via H_{SB}) to a collection of bath sites that are a set of independent energy levels (described by H_B). This is a "grand-canonical" setting where the energy and particle number of the system can fluctuate. We assume the bath to be "featureless," namely with

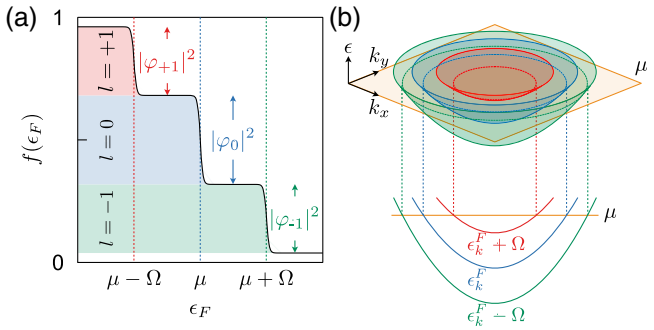

FIG. 1. (a) Fermi Dirac Staircase occupation of Floquet states, from Eq. (3); (b) associated Floquet Fermi surfaces from Eq. (5). We are choosing an extended zone scheme for the Floquet energy, ϵ_k^F (blue line), so that it approaches the undriven band energy in the limit of vanishing amplitude of the drive at fixed frequency Ω. Therefore, the Floquet Fermi surface for $\epsilon_k^F=\mu$ (dotted blue line) approaches the undriven Fermi surface, while additional Floquet Fermi surfaces appear for $\epsilon_k^F=\mu-\Omega$ (red dotted) and $\epsilon_k^F=\mu+\Omega$ (green dotted) [see Eq. (5)].

energy-independent density of states and tunneling amplitudes over a bandwidth that is much larger than the system's, as it is frequently assumed [13,23–32].

Crucially, the system Hamiltonian $H_S(t)$, can be time dependent, allowing us to drive it away from thermal equilibrium with the bath. By assuming that the bath is prepared in a thermal ensemble in a distant past with a Fermi Dirac distribution, $f_0(\epsilon)=1/[1+e^{\beta(\epsilon-\mu)}]$ with inverse temperature $\beta=1/k_BT_0$ and chemical potential μ, one can rigorously show [33] that at late times the system approaches a unique steady state, with its exact one-body density-matrix given by

$$\rho_S(t)=\Gamma\int_{-\infty}^{+\infty}\frac{d\epsilon}{\pi}f_0(\epsilon)U_\Gamma(t,\epsilon)U_\Gamma^\dagger(t,\epsilon),$$
$$U_\Gamma(t,\epsilon)=\int_{-\infty}^{t}dt' e^{\Gamma(t'-t)-i\epsilon t'}U_S(t,t'), \tag{2}$$

where $U_S(t,t')$ satisfying $i\partial_t U_S(t,t')=H_S(t)U_S(t,t')$ is the unitary evolution operator of the isolated system and $\Gamma=\lambda^2\nu_0/2$ is the particle tunneling rate into the bath, which parametrizes the strength of system-bath coupling. Equation (2) generalizes Eq. (41) of Ref. [13] to arbitrary off-diagonal time-dependent system Hamiltonians. Now by assuming that the drive is periodic, $H(t)=H(t+T)$, and the coupling to the bath is infinitesimal, $\Gamma\to 0$, so that it would act as an "ideal" bath in equilibrium, then Eq. (2) reduces to [33]:

$$\lim_{\Gamma\to 0}\rho_S(t)=\sum_a f_a|\psi_a^F(t)\rangle\langle\psi_a^F(t)|,$$
$$f_a=\sum_{l=-\infty}^{+\infty}|\varphi_{a,l}|^2 f_0(\epsilon_a^F+l\Omega). \tag{3}$$

Here, $|\psi_a^F(t)\rangle$ are the complete basis of solutions of the single particle time dependent Schrödinger equation, ϵ_a^F are their Floquet energies, $\Omega=2\pi/T$, $|\varphi_{a,l}|^2\equiv\langle\varphi_{a,l}|\varphi_{a,l}\rangle$ with $|\varphi_{a,l}\rangle$ the lth harmonic of the Floquet wave function, related as $|\psi_a^F(t)\rangle=\sum_l e^{-i\epsilon_a^F t-il\Omega t}|\varphi_{a,l}\rangle$ [39]. Notice that while there is a "gauge" freedom to redefine the Floquet energy and wave function as $\epsilon_a'^F=\epsilon_a^F+l_0\Omega$, $|\varphi'_{a,l}\rangle=|\varphi_{a,l+l_0}\rangle$ for any chosen integer l_0, which leaves the physical single-particle time-dependent wave-function invariant, the occupation of each physical time-dependent state obtained from Eq. (3) is invariant under such redefinition and thus unambiguously defined.

Equation (3) is an example of a periodic Gibbs ensemble [16,18–20], but in contrast to the setting of Refs. [16,18–20] we have obtained this ensemble by coupling the system to a bath and not as a result of many-body self-thermalization. In the context of self-thermalization the occupations f_a would not be fixed but determined by initial conditions of the quasiparticles, but in our context the f_a are uniquely fixed by the state of the bath. Notably, the occupations f_a viewed as a function of the Floquet energy ϵ_a^F are not given by the equilibrium Fermi-Dirac function but instead by a Fermi-Dirac staircase (see Fig. 1), generalizing the results of Ref. [13] to off-diagonal Hamiltonians. These staircase occupations have also appeared in Eq. (12) of Ref. [5] and Eq. (1) of Ref. [14], and in discussions of the Tien-Gordon effect [40,41] in driven mesoscopic systems.

The Floquet Fermi liquid.—Let us now specialize to the case of a Floquet Bloch band. For simplicity, we take a system with a single band arising from a tight-biding model with one site per unit cell with dispersion $\epsilon(\mathbf{k})$, and driven by a time-periodic and spatially uniform electric field with vector potential $\mathbf{A}(t)=\mathbf{A}(t+T)$ so that the Hamiltonian remains diagonal in crystal momentum and is given by $\epsilon_{\mathbf{k}}(t)\equiv\epsilon(\mathbf{k}-\mathbf{A}(t))$. In this case, the density matrix is indeed time-independent and the occupations can be obtained from Eq. (3) by replacing $a\to\mathbf{k}$:

$$f_{\mathbf{k}}=\sum_l|\varphi_{\mathbf{k},l}|^2 f_0(\epsilon_{\mathbf{k}}^F+l\Omega), \tag{4}$$

where the Floquet energy and the harmonics of the Floquet wave functions are given by $\epsilon_{\mathbf{k}}^F=\langle\epsilon_{\mathbf{k}}(t)\rangle_T$, $\varphi_{\mathbf{k},l}=\langle e^{-i\int_0^t dt'[\epsilon_{\mathbf{k}}(t')-\epsilon_{\mathbf{k}}^F-l\Omega]}\rangle_T$, and $\langle\cdots\rangle_T=\int_0^T(\cdots)dt/T$ denotes the time average over one period. Notice that here we are choosing an extended zone scheme where the Floquet energy is not restricted to an interval of size Ω, but instead fixing the ambiguity of the Floquet energy $\epsilon_{\mathbf{k}}^F$, so that it approaches time-independent band dispersion $\epsilon_{\mathbf{k}}$ in the limit of vanishing driving electric field amplitude while fixing the frequency, and correspondingly, $f_{\mathbf{k}}\to f_0(\epsilon_{\mathbf{k}})$ with $|\varphi_{\mathbf{k},l}|^2\to\delta_{l,0}$. $f_{\mathbf{k}}$ in Eq. (4) describes the occupation of canonical crystal momentum $\mathbf{k}$, which is related to the physical gauge invariant crystal momentum via

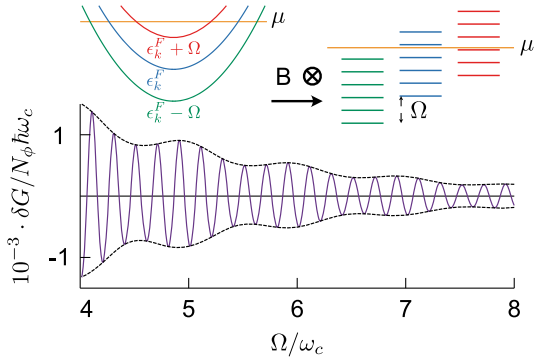

FIG. 2. Illustration of Floquet Landau levels (top right) and free energy oscillations computed from Eq. (7), for $T_0 = 0.05\mu$, $\Omega = 0.2\mu$, $\mathbf{E}/(\Omega\sqrt{m\mu}) = 0.05\cdot\mathbf{n}_\mathbf{E}/\|\mathbf{n}_\mathbf{E}\|$, $\mathbf{n}_\mathbf{E} = (1, i/2)$.

$\mathbf{k}_{\text{phys}} = \mathbf{k} - \mathbf{A}(t)$. Since the occupation of canonical momenta is time independent, the occupation of physical momenta oscillates as $\mathbf{A}(t)$. Therefore, the occupation develops a collection of sharp steps at several surfaces in crystal momentum that are enclosed inside each other and are given by (see Fig. 1)

$$\epsilon^F_\mathbf{k} = \mu - l\Omega, \qquad l \in \mathbb{Z}. \tag{5}$$

We will refer to these surfaces as the Floquet Fermi surfaces (FFS) and the corresponding nonequilibrium steady state as the Floquet Fermi liquid (FFL). Notice that the height of the jump at the lth FFS, given by $|\varphi_{\mathbf{k},l}|^2$, is in general a function of the momentum within a given FFS (see Fig. 1).

Quantum oscillations of the FFL.—As we have seen, a periodically driven system of fermions in contact with a fermionic bath approaches a nontrivial FFL steady state with a collection of enclosed FFS's. We would like to investigate how these FFS's manifest directly through observable properties. We note that a few previous studies have indeed indirectly dealt with the FFL through its manifestation in properties such as Ruderman-Kittel-Kasuya-Yosida interactions [42], susceptibility functions [43], and similar staircase occupations have also been discussed in driven Luttinger liquids [44–46].

Systems with a Fermi surface display characteristic quantum oscillations of many observables in the presence of applied magnetic fields, with a periodicity of the form $\sim\cos(S/B)$, where S is the area of the Fermi surface. As we will show, the FFSs give rise to a sum of quantum oscillations with different frequencies $\sim\cos(S_l/B)$, where S_l is the area of the lth FFS. We will show that in the regime where the cyclotron energy is smaller than the driving frequency this will lead to a slow beating of the amplitude of quantum oscillations, which, remarkably, has the same period measured in two-dimensional electron systems in the regime where microwave induced resistance oscillations (MIRO) coexist with the Shubnikov–de Haas (SdH) oscillations [47,48], suggesting that the Floquet Fermi liquid has indeed already been achieved in these experiments.

To illustrate this, we consider parabolic fermions, coupled simultaneously to a uniform magnetic field, $\mathbf{B} = \nabla\times\mathbf{A}_0(\mathbf{r})$, and a time-dependent electric field, $\mathbf{E}(t) = -\partial_t\mathbf{A}(t) = \mathbf{E}e^{-i\Omega t} + \text{c.c.}$, with Hamiltonian: $H_S(t) = [\mathbf{k} - \mathbf{A}_0(\mathbf{r}) - \mathbf{A}(t)]^2/(2m)$. The solutions of the time dependent Schrödinger equation for this Hamiltonian are time-dependent Landau levels wave functions, $|\psi^F_N(t)\rangle$, with Floquet energy ϵ^F_N and labeled by a principal cyclotron index $N = 0, 1, 2\ldots$, and with a guiding-center degeneracy N_ϕ [33]. By replacing $a \to N$ in the formula for the steady state from Eq. (3), we can compute various observables of the system. Here, we will focus on the oscillations of an effective Floquet free energy defined as follows:

$$\beta G \equiv -N_\phi\sum_{N,l}|\varphi_{N,l}|^2\log\left[1 + e^{-\beta(\epsilon^F_N + l\Omega - \mu)}\right]. \tag{6}$$

This free energy approaches the equilibrium free energy in the limit of static Hamiltonians [$\mathbf{A}(t) \to 0$]: $\beta G_{\text{eff}} \to -\ln Z(\beta,\mu)$, where $Z(\beta,\mu)$ is the Grand-canonical partition function in the absence of drive. Conceptually, the effective Floquet free energy in Eq. (6), can be viewed as a sum of the effective free energies of each Floquet quasienergy state, $\epsilon^F_N + l\Omega$, weighed by the amplitude of the corresponding harmonic of the Floquet wave function $|\varphi_{N,l}|^2$, and thus it is a mathematically natural extension of the relevant equilibrium free energy in a grand-canonical ensemble to a Floquet system. We have utilized this free energy for conceptual illustration in the main text because it displays much simpler oscillations than other conceptually more natural quantities such as the energy averaged over one period. Nevertheless, the theory of the oscillations of such time averaged energy are presented in Supplemental Material [33] (see also Refs. [13,32,34–38] therein). In the absence of magnetic fields, and to second order in driving electric fields, only the Floquet bands shifted in energy by $\pm\hbar\Omega$ contribute. In a magnetic field to second order in electric fields, we would have two Floquet copies of the Landau level spectrum (see Fig. 2), since higher Floquet harmonics have weights with higher powers of electric field [33]. We will thus compute this free energy to second order in the electric fields. By performing a similar analysis to the equilibrium calculation [49], we have found the oscillating part of the Floquet free energy in the limit where many Landau levels are occupied $\mu \gg \hbar\omega_c$ is [33]

$$\frac{\delta G}{N_\phi\hbar\omega_c} \approx \sum_{l=\pm1,0}\sum_{k=1}^{\infty} G_k R_k\left(\delta_{l,0} + \frac{b_l\mu_l}{\hbar\omega_c}\right)\cos\left(\frac{kS_l}{B}\right), \tag{7}$$

where $G_k = (-1)^k/(2k^2\pi^2)$, $R_k = \lambda k/\sinh(\lambda k)$ is the Lifshitz-Kosevich factor, $\lambda = 2\pi^2/(\beta\hbar\omega_c)$, $S_l \approx 2m\pi(\mu - l\Omega)$ is the area of lth FFS, $b_{\pm1} = -b_0/2 = R_+ = |z_+|^2/(\omega_c - \Omega)^2 + |z_-|^2/(\omega_c + \Omega)^2$, and $|z_\pm|^2 = \omega_c(|\mathbf{E}|^2 \pm i[\mathbf{E}\times\mathbf{E}^*]_z)/(2m\Omega^2)$. We therefore see that the FFSs give

rise to additional frequencies of the quantum oscillations controlled by their effective areas, resembling a multiband system in equilibrium, as illustrated in Fig. 2.

The above oscillations resemble closely those observed in two-dimensional electron systems in the regime where MIRO and SdH oscillations coexist [47], where a rich variety of nonequilibrium phenomena have been observed [50–54], that also have been realized for electrons in the surface of helium [55] and more recently in graphene [56]. To make a more direct connection with these, let us compute also the oscillations of an effective nonequilibrium thermodynamic density of states (DOS), defined as

$$\nu \equiv \left(\frac{\partial n}{\partial \mu}\right)_{T_0}, \qquad n = \frac{1}{2\pi l_B^2}\sum_N f_N, \tag{8}$$

where $l_B^2 = \hbar c/eB$ denotes the magnetic length, and μ, T_0 are the chemical potential and temperature of the bath. This nonequilibrium DOS reduces to the equilibrium DOS in the absence of drive, and the oscillations of DOS tend to resemble those of resistivity in equilibrium [49,57], making them a more relevant observable to contrast with MIRO photoconductivity measurements. The oscillatory part of the DOS [33], can be shown to be

$$\delta\nu \approx \frac{2}{h\omega_c l_B^2}\sum_{k=1}^{\infty}(-1)^k R_k F_E \cos\left(k\frac{S_0}{B}\right), \tag{9}$$

where $S_0 \approx 2m\pi\mu$ is the area of the FS in equilibrium and the factor $F_E = 1\text{–}4R_+(\mu/\hbar\omega_c)\sin^2(\pi\Omega/\omega_c)$ describes the oscillations of the envelope of the fast oscillations (see Fig. 2), imprinted by the ac drive. The frequency and phase of these envelope oscillations agrees exactly with that of photoresistivity theories from Refs. [53,54], which agrees with the point of view that these are dominated by density of states oscillations. The frequency of oscillations of the envelope also agrees with those of the photoresistivity in MIRO experiments but not with their phase [47], for which there is no current detailed understanding, although it is expected to depend on the intensity of radiation [54], and on details of the scattering mechanisms [53]. Therefore, the FFL and its collection of FFSs, provide a simple overarching conceptual framework that positions MIRO as a natural nonequilibrium counterpart to conventional equilibrium quantum oscillations. We hope this picture can contribute to clarify and guide experiments in the future. Our fermionic bath is by no means a realistic approximation to the relevant relaxation mechanisms in typical 2DEGs where MIRO is observed. While some properties of the quantum oscillations of FFLs, such as the frequency of the slow beating of their amplitude, are expected to be robust to the details of relaxations, other properties might depend on the nature of the bath and relaxation mechanisms. This is an avenue we hope to further investigate more in future studies.

DOS and nonequilibrium Van Hove singularities of the FFL.—The thermodynamic DOS plays a central role in equilibrium and is directly measurable via capacitive measurements of compressibility [58–63]. Notably, a noninteracting system with its chemical potential tuned at a Van Hove singularity, for which the DOS diverges, would generically become unstable toward broken symmetry states for weak interactions (see, e.g., Ref. [64]). Here, we would like to demonstrate that FFLs possess a greater degree of tunability relative to their equilibrium counterparts, because the parameters controlling the radiation, such as the frequency, can be used to tune it toward a Van Hove singularity of its nonequilibrium DOS, without the need to change the electron density. To demonstrate this we consider a single band model. Using Eq. (4), the nonequilibrium DOS can be expressed as a sum of an effective DOS of each FFS:

$$\nu(\mu) = \lim_{T_0\to 0}(\partial n/\partial\mu)_{T_0} = \sum_l \nu_l(\mu),$$
$$\nu_l(\mu) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}|\varphi_{\mathbf{k},l}|^2\delta(\mu - l\Omega - \epsilon^F_{\mathbf{k}}). \tag{10}$$

Therefore, the frequency can be used to shift the effective chemical potential of lth FFS as $\mu - l\Omega$. As an example, consider a 2D square lattice with nearest neighbor hopping amplitude t, so that in equilibrium it would have dispersion $\epsilon_{\mathbf{k}} = -2t\cos(k_x) - 2t\cos(k_y)$, with a Van Hove singularity at $\mu = 0$ originating from the states near the two special momenta $(\pi, 0), (0, \pi)$ [see Fig. 3(a)]. In the driven case, $\mathbf{A}(t) = [A_x\sin(\Omega t + \phi_x), A_y\sin(\Omega t + \phi_y)]$, the Floquet band energy is [33]:

$$\epsilon^F_{\mathbf{k}} = -2t[\cos(k_x)J_0(A_x) + \cos(k_y)J_0(A_y)], \tag{11}$$

where J_0 is the Bessel function of first kind. This Floquet problem retains a $\mathbf{k} \to -\mathbf{k}$ symmetry which pins the origin of Van Hove singularities of the higher order FFSs to the same two special momenta $(\pi, 0), (0, \pi)$. This symmetry also leads to a vanishing of the odd Floquet wave functions at these momenta, namely $\varphi_{(0,\pi),l} = \varphi_{(\pi,0),l} = 0$ for odd l [33]. However, for l even, the Floquet amplitudes remain finite near these points and as a result such FFSs display additional Van Hove singularities in the nonequilibrium DOS, at the following chemical potentials [see Fig. 3(a)]:

$$\mu = l\Omega \pm [J_0(A_x) - J_0(A_y)], \qquad l \text{ even}. \tag{12}$$

This model illustrates the tantalizing potential of engineering the properties of the ac drive to tune some FFSs into Van Hove singularities, even if at equilibrium there is no DOS singularity at the chemical potential.

Nonequilibrium specific heat of the FFL.—In equilibrium, the low temperature specific heat C_V in a Landau-Fermi liquid is related to the thermodynamic DOS via [65]:

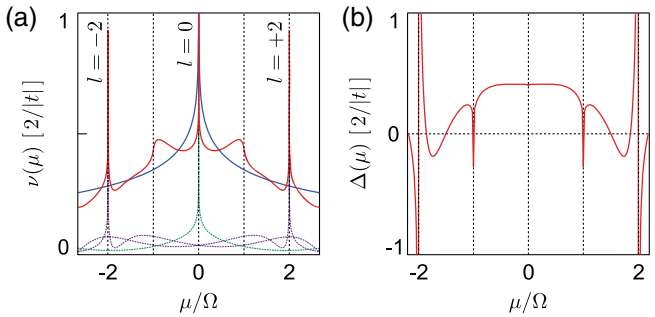

FIG. 3. (a) DOS for the nondriven (blue) and driven (red) square lattice model from Eq. (10). Dashed lines are the additional Van Hove singularities from Eq. (12). Green and purple lines are contributions making the red line coming from $l = 0, \pm 2$ harmonics, respectively. (b) Correction $\Delta(\mu)$ to the equilibrium specific heat coefficient from Eq. (14). Parameters: $\Omega = 1/4$, $A_x = A_y = 4/3$, $\phi_x = 0$, $\phi_y = \pi/2$.

$$\lim_{T_0\to 0} \frac{C_V}{\frac{\pi^2}{3} k_B T_0} = (1 + F_0^s) \lim_{T_0\to 0} \left(\frac{\partial n}{\partial \mu}\right)_{T_0}, \tag{13}$$

where C_V is the specific heat at constant volume, and F_0^s is the spin symmetric Landau parameter, which is nonzero only in the presence of interactions. Therefore for noninteracting fermions, the linear in T_0 coefficient of the specific heat also measures the DOS. However, interestingly, the FFLs violate the above relation between specific heat and thermodynamic DOS even for noninteracting fermions. To show this, we begin by defining the nonequilibrium the specific heat, C_V, as $C_V \equiv (\partial \bar{E}/\partial T_0)_n$, where $\bar{E}$ is the system energy time-averaged over one period, and the derivative is computed at fixed particle density. For our model with single Floquet band, we obtain the following relation [33]:

$$\lim_{T_0\to 0} \frac{C_V}{\frac{\pi^2}{3} k_B T_0} = \lim_{T_0\to 0} \left(\frac{\partial n}{\partial \mu}\right)_{T_0} + \Delta(\mu), \tag{14}$$

where $\Delta(\mu) = \Omega \sum_{l_1 l_2} (l_2 - l_1) \nu_{l_1}(\mu) \nu'_{l_2}(\mu) / [\sum_l \nu_l(\mu)]$. The additional Van Hove singularities in the nonequilibrium DOS also manifest themselves as singularities in the nonequilibrium specific heat as illustrated in Figs. 3(a) and 3(b) for the same square lattice tight-binding model of the previous section.

Discussion.—We have demonstrated the existence of a nonequilibrium FFL steady state in Floquet bands that features a collection of FFSs enclosed inside each other. To realize these states in experiments essentially two criteria should be met: first the driving frequency should exceed thermal broadening $\hbar\Omega \gg k_B T_0$ [66], so that the multiple Floquet quasienergy bands can be resolved. In addition, the size of the additional jumps of the Fermi Dirac staircase occupation, which are the dimensionless numbers $|\varphi_{a,l}|^2$ in Eq. (3), should be sizable. The first nontrivial jump scales as $|\varphi_{l=\pm 1}|^2 \sim (e v_F |\mathbf{E}|/\hbar\Omega^2)^2$, at small field amplitudes, therefore the second criterion is that the light intensity, $I = c\epsilon_0 |\mathbf{E}|^2/2$, is comparable to the intensity scale $I_0 = \hbar\Omega^4/(8\pi\alpha v_F^2)$, with $\alpha \approx 1/137$ the fine structure constant. We believe that these criteria can be comfortably met in a variety of platforms, and, in fact, are likely met in several of those in which MIRO and SdH oscillations are seen to coexist [47,48]. For example, for MIRO experiments [48] with a frequencies of $\Omega/2\pi = 10$ GHz and $v_F = 2 \times 10^5$ m/s, the intensity scale is just $I_0 \approx 0.2$ W/m^2, illustrating that low frequencies greatly help in reducing the required power. However, we believe there can be completely different conditions and material platforms for accessing the FFL regime. For example, for the experiments of Ref. [67] that studied Floquet-Bloch states in the surface of topological insulators with mid-infrared pulses of frequency $\hbar\Omega = 120$ meV, it is estimated that $|\varphi_{l=\pm 1}|^2 \sim (e v_F |\mathbf{E}|/\hbar\Omega^2)^2 \sim 0.25$, and therefore meets the criteria. For the experiments of Ref. [68] realizing the light-induced anomalous Hall effect in graphene ($v_F = 10^6$ m/s) with a similar midinfrared frequency, the intensity scale is $I_0 \approx 4 \times 10^{12}$ W/m^2, which is the same as their typical pulse peak intensity. Therefore, these type of experiments are well posed to prepare the FFL with pump pulses and investigate its subsequent decay. Finally, to comment on the potential impact of electron-electron interactions, we note that they will lead to an increase of collisions among particles and therefore to higher temperature of the electrons in the steady state with respect to the bath, which would tend to smear the sharpness of the staircase distribution. One strategy to mitigate this in experiments is to consider two-dimensional electron systems separated by some thin dielectric from a 3D metallic gate that can screen the interactions and also act as a bath via electron tunneling. But more broadly speaking, understanding the impact of interactions in the steady state as well as its modifications for other kinds of baths such as bosons, are important components of the problem that we hope to address in future studies.

We are thankful to Qianhui Shi, Michael Zudov, and Ivan Dmitriev for valuable discussions. We acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) through research Grant Project No. 518372354 (I. S.) and Ministry of Education Singapore under its Academic Research Fund Tier 2 Grant No. MOE-T2EP50222-0011 (J. C. W. S) and Academic Research Fund Tier 3 Grant No. MOE2018-T3-1-002 (J. C. W. S).

*These authors contributed equally to this letter.
†sodemann@uni-leipzig.de

[1] Jérôme Cayssol, Balázs Dóra, Ferenc Simon, and Roderich Moessner, Floquet topological insulators, Phys. Status Solidi RRL 7, 101 (2013).

[2] Takashi Oka and Sota Kitamura, Floquet engineering of quantum materials, Annu. Rev. Condens. Matter Phys. **10**, 387 (2019).
[3] Fenner Harper, Rahul Roy, Mark S. Rudner, and S. L. Sondhi, Topology and broken symmetry in Floquet systems, Annu. Rev. Condens. Matter Phys. **11**, 345 (2020).
[4] Mark S. Rudner and Netanel H. Lindner, Band structure engineering and nonequilibrium dynamics in Floquet topological insulators, Nat. Rev. Phys. **2**, 229 (2020).
[5] Karthik I. Seetharam, Charles-Edouard Bardyn, Netanel H. Lindner, Mark S. Rudner, and Gil Refael, Controlled population of Floquet-Bloch states via coupling to Bose and Fermi baths, Phys. Rev. X **5**, 041050 (2015).
[6] Hossein Dehghani, Takashi Oka, and Aditi Mitra, Out-of-equilibrium electrons and the Hall conductance of a Floquet topological insulator, Phys. Rev. B **91**, 155422 (2015).
[7] Thomas Iadecola, Titus Neupert, and Claudio Chamon, Occupation of topological Floquet bands in open systems, Phys. Rev. B **91**, 235133 (2015).
[8] Thomas Bilitewski and Nigel R. Cooper, Scattering theory for Floquet-Bloch states, Phys. Rev. A **91**, 033601 (2015).
[9] Maximilian Genske and Achim Rosch, Floquet-Boltzmann equation for periodically driven Fermi systems, Phys. Rev. A **92**, 062108 (2015).
[10] Iliya Esin, Mark S. Rudner, Gil Refael, and Netanel H. Lindner, Quantized transport and steady states of Floquet topological insulators, Phys. Rev. B **97**, 245401 (2018).
[11] Götz S. Uhrig, Mona H. Kalthoff, and James K. Freericks, Positivity of the spectral densities of retarded Floquet green functions, Phys. Rev. Lett. **122**, 130604 (2019).
[12] Sein Park, Wonjun Lee, Seong Jang, Yong-Bin Choi, Jinho Park, Woochan Jung, Kenji Watanabe, Takashi Taniguchi, Gil Young Cho, and Gil-Ho Lee, Steady Floquet-Andreev states in graphene Josephson junctions, Nature (London) **603**, 421 (2022).
[13] Oles Matsyshyn, Justin C. W. Song, Inti Sodemann Villadiego, and Li-kun Shi, Fermi-Dirac staircase occupation of Floquet bands and current rectification inside the optical gap of metals: An exact approach, Phys. Rev. B **107**, 195135 (2023).
[14] Rekha Kumari, Babak Seradjeh, and Arijit Kundu, Josephson-current signatures of unpaired Floquet Majorana bound states, arXiv:2301.07707.
[15] Luca D'Alessio and Marcos Rigol, Long-time behavior of isolated periodically driven interacting lattice systems, Phys. Rev. X **4**, 041048 (2014).
[16] Achilleas Lazarides, Arnab Das, and Roderich Moessner, Equilibrium states of generic quantum systems subject to periodic driving, Phys. Rev. E **90**, 012110 (2014).
[17] Pedro Ponte, Anushya Chandran, Z Papić, and Dmitry A. Abanin, Periodically driven ergodic and many-body localized quantum systems, Ann. Phys. (Amsterdam) **353**, 196 (2015).
[18] Achilleas Lazarides, Arnab Das, and Roderich Moessner, Periodic thermodynamics of isolated quantum systems, Phys. Rev. Lett. **112**, 150401 (2014).
[19] Achilleas Lazarides, Arnab Das, and Roderich Moessner, Fate of many-body localization under periodic driving, Phys. Rev. Lett. **115**, 030402 (2015).
[20] Vedika Khemani, Achilleas Lazarides, Roderich Moessner, and Shivaji L. Sondhi, Phase structure of driven quantum systems, Phys. Rev. Lett. **116**, 250401 (2016).
[21] Dominic V. Else, Bela Bauer, and Chetan Nayak, Floquet time crystals, Phys. Rev. Lett. **117**, 090402 (2016).
[22] Dominic V. Else, Bela Bauer, and Chetan Nayak, Prethermal phases of matter protected by time-translation symmetry, Phys. Rev. X **7**, 011026 (2017).
[23] Antti-Pekka Jauho, Ned S. Wingreen, and Yigal Meir, Time-dependent transport in interacting and noninteracting resonant-tunneling systems, Phys. Rev. B **50**, 5528 (1994).
[24] Kristinn Johnsen and Antti-Pekka Jauho, Quasienergy spectroscopy of excitons, Phys. Rev. Lett. **83**, 1207 (1999).
[25] Sigmund Kohler, Jörg Lehmann, and Peter Hänggi, Driven quantum transport on the nanoscale, Phys. Rep. **406**, 379 (2005).
[26] Takashi Oka and Hideo Aoki, Photovoltaic Hall effect in graphene, Phys. Rev. B **79**, 081406(R) (2009).
[27] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, England, 2011).
[28] Takuya Kitagawa, Takashi Oka, Arne Brataas, Liang Fu, and Eugene Demler, Transport properties of nonequilibrium systems under the application of light: Photoinduced quantum Hall insulators without Landau levels, Phys. Rev. B **84**, 235108 (2011).
[29] Takahiro Morimoto and Naoto Nagaosa, Topological nature of nonlinear optical effects in solids, Sci. Adv. **2**, e1501524 (2016).
[30] Oles Matsyshyn, Francesco Piazza, Roderich Moessner, and Inti Sodemann, Rabi regime of current rectification in solids, Phys. Rev. Lett. **127**, 126604 (2021).
[31] Jinhong Park, Omri Golan, Yuval Vinkler-Aviv, and Achim Rosch, Thermal Hall response: Violation of gravitational analogs and Einstein relations, Phys. Rev. B **105**, 205419 (2022).
[32] Li-kun Shi, Oles Matsyshyn, Justin C. W. Song, and Inti Sodemann Villadiego, Berry-dipole photovoltaic demon and the thermodynamics of photocurrent generation within the optical gap of metals, Phys. Rev. B **107**, 125151 (2023).
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.146402, which contains Refs. [13,32,34–38]. Supplemental Material contains (1) density matrix of a general system Hamiltonian coupled to the featureless fermionic bath; (2) density matrix of a Floquet system Hamiltonian coupled to the ideal fermionic bath; (3) Floquet perturbation theory; (4) magnetic oscillations; (5) nonequilibrium specific heat at fixed particle number; (6) Floquet amplitudes and Van Hove singularities in square lattice tight-binding models; (7) pair correlation function of the Floquet Fermi liquid; (8) unequal time correlations; (9) relation to the Keldysh formalism and two-time Green functions.
[34] Gabriele Giuliani and Giovanni Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press, Cambridge, England, 2008).
[35] Alexander L. Fetter and John Dirk Walecka, *Quantum Theory of Many-Particle Systems* (Courier Corporation, New York, 2012).
[36] A. P. Jauho, Introduction to the Keldysh nonequilibrium Green function technique, Lect. Notes **17** (2006).

[37] J. K. Freericks, H. R. Krishnamurthy, and Th Pruschke, Theoretical description of time-resolved photoemission spectroscopy: Application to pump-probe experiments, Phys. Rev. Lett. **102**, 136401 (2009).

[38] Fahad Mahmood, Ching-Kit Chan, Zhanybek Alpichshev, Dillon Gardner, Young Lee, Patrick A. Lee, and Nuh Gedik, Selective scattering between Floquet-Bloch and Volkov states in a topological insulator, Nat. Phys. **12**, 306 (2016).

[39] These jumps satisfy the normalization condition $\sum_l |\varphi_{a,l}|^2 = 1$, and are therefore constrained to satisfy $|\varphi_{a,l}|^2 \leq 1$.

[40] Yuli V. Nazarov and Yaroslav M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, Cambridge, England, 2009).

[41] P. K. Tien and J. P. Gordon, Multiphoton process observed in the interaction of microwave fields with the tunneling between superconductor films, Phys. Rev. **129**, 647 (1963).

[42] Mahmoud M. Asmar and Wang-Kong Tse, Floquet control of indirect exchange interaction in periodically driven two-dimensional electron systems, New J. Phys. **23**, 123031 (2021).

[43] Atsushi Ono and Sumio Ishihara, Nonequilibrium susceptibility in photoinduced Floquet states, Phys. Rev. B **100**, 075127 (2019).

[44] Christian D. Graf, Guillaume Weick, and Eros Mariani, Parametric resonance and spin-charge separation in 1D fermionic systems, Europhys. Lett. **89**, 40005 (2010).

[45] Yu Kagan and L. A. Manakova, Parametric generation of density and spin modes with formation of stationary condensed states in ultracold two-component quasi-one-dimensional Fermi gases, Phys. Rev. A **80**, 023625 (2009).

[46] Marin Bukov and Markus Heyl, Parametric instability in periodically driven Luttinger liquids, Phys. Rev. B **86**, 054304 (2012).

[47] Q Shi, P. D. Martin, A. T. Hatke, M. A. Zudov, J. D. Watson, G. C. Gardner, M. J. Manfra, L. N. Pfeiffer, and K. W. West, Shubnikov–de Haas oscillations in a two-dimensional electron gas under subterahertz radiation, Phys. Rev. B **92**, 081405(R) (2015).

[48] I. A. Dmitriev, A. D. Mirlin, D. G. Polyakov, and M. A. Zudov, Nonequilibrium phenomena in high Landau levels, Rev. Mod. Phys. **84**, 1709 (2012).

[49] David Shoenberg, *Magnetic Oscillations in Metals* (Cambridge University Press, Cambridge, England, 2009).

[50] M. A. Zudov, R. R. Du, J. A. Simmons, and J. L. Reno, Microwave photoresistance measurements of magneto-excitations near a 2D Fermi surface, arXiv:cond-mat/9711149.

[51] M. A. Zudov, R. R. Du, J. A. Simmons, and J. L. Reno, Shubnikov–de Haas-like oscillations in millimeterwave photoconductivity in a high-mobility two-dimensional electron gas, Phys. Rev. B **64**, 201311(R) (2001).

[52] P. D. Ye, L. W. Engel, D. C. Tsui, J. A. Simmons, J. R. Wendt, G. A. Vawter, and J. L. Reno, Giant microwave photoresistance of two-dimensional electron gas, Appl. Phys. Lett. **79**, 2193 (2001).

[53] X. L. Lei and S. Y. Liu, Temperature dependence of microwave-induced magnetoresistance oscillation in two-dimensional electron systems, Appl. Phys. Lett. **94** (2009).

[54] Ivan A. Dmitriev, Nonequilibrium magnetooscillations in spatially nonuniform quantum Hall systems, in J. Phys. Conf. Ser. **334**, 012015 (2011).

[55] Denis Konstantinov and Kimitoshi Kono, Novel radiation-induced magnetoresistance oscillations in a nondegenerate two-dimensional electron system on liquid helium, Phys. Rev. Lett. **103**, 266808 (2009).

[56] Erwin Mönch, Denis A. Bandurin, Ivan A. Dmitriev, Isabelle Y. Phinney, Ivan Yahniuk, Takashi Taniguchi, Kenji Watanabe, Pablo Jarillo-Herrero, and Sergey D. Ganichev, Observation of terahertz-induced magnetooscillations in graphene, Nano Lett. **20**, 5943 (2020).

[57] Alexei Alekseevich Abrikosov, *Fundamentals of the Theory of Metals* (Courier Dover Publications, New York, 2017).

[58] T. P. Smith III, W. I. Wang, and P. J. Stiles, Two-dimensional density of states in the extreme quantum limit, Phys. Rev. B **34**, 2995 (1986).

[59] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Negative compressibility of interacting two-dimensional electron and quasiparticle gases, Phys. Rev. Lett. **68**, 674 (1992).

[60] Jens Martin, Nitzan Akerman, G. Ulbricht, T. Lohmann, J. H. v Smet, K. Von Klitzing, and Amir Yacoby, Observation of electron-hole puddles in graphene using a scanning single-electron transistor, Nat. Phys. **4**, 144 (2008).

[61] Benjamin E. Feldman, Benjamin Krauss, Jurgen H. Smet, and Amir Yacoby, Unconventional sequence of fractional quantum Hall states in suspended graphene, Science **337**, 1196 (2012).

[62] Fangyuan Yang, Alexander A. Zibrov, Ruiheng Bai, Takashi Taniguchi, Kenji Watanabe, Michael P. Zaletel, and Andrea F. Young, Experimental determination of the energy per particle in partially filled Landau levels, Phys. Rev. Lett. **126**, 156802 (2021).

[63] There are also optical measurement techniques [69].

[64] Rahul Nandkishore, Leonid S. Levitov, and Andrey V. Chubukov, Chiral superconductivity from repulsive interactions in doped graphene, Nat. Phys. **8**, 158 (2012).

[65] David Pines, *Theory of Quantum Liquids: Normal Fermi Liquids* (CRC Press, Boca Raton, 2018).

[66] And an analogous criteria should be met for other quasiparticle broadening effects such as disorder and electron lifetimes.

[67] Y. H. Wang, Hadar Steinberg, Pablo Jarillo-Herrero, and Nuh Gedik, Observation of Floquet-Bloch states on the surface of a topological insulator, Science **342**, 453 (2013).

[68] James W. McIver, Benedikt Schulte, F.-U. Stein, Toru Matsuyama, Gregor Jotzu, Guido Meier, and Andrea Cavalleri, Light-induced anomalous Hall effect in graphene, Nat. Phys. **16**, 38 (2020).

[69] Zhengchao Xia, Yihang Zeng, Bowen Shen, Roei Dery, Kenji Watanabe, Takashi Taniguchi, Jie Shan, and Kin Fai Mak, Optical readout of the chemical potential of two-dimensional electrons, arXiv:2304.09514.