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Weyl Fermions on a Finite Lattice

David B. Kaplan [1,*] and Srimoyee Sen [2,†]

[1] *Institute for Nuclear Theory, Box 351550, Seattle, Washington 98195-1550, USA*

[2] *Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011, USA*

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The phenomenon of unpaired Weyl fermions appearing on the sole $2n$-dimensional boundary of a $(2n+1)$-dimensional manifold with massive Dirac fermions was recently analyzed in D. B. Kaplan [preceding Letter, Chiral gauge theory at the boundary between topological phases, Phys. Rev. Lett. **132**, 141603 (2024).]. In this Letter, we show that similar unpaired Weyl edge states can be seen on a finite lattice. In particular, we consider the discretized Hamiltonian for a Wilson fermion in $(2+1)$ dimensions with a $1+1$ dimensional boundary and continuous time. We demonstrate that the low lying boundary spectrum is indeed Weyl-like: it has a linear dispersion relation and definite chirality and circulates in only one direction around the boundary. We comment on how our results are consistent with Nielsen-Ninomiya theorem. This work removes one potential obstacle facing the program outlined in D. B. Kaplan, preceding Letter, for regulating chiral gauge theories.

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Understanding how to regulate chiral gauge theories on the lattice has been a longstanding challenge, brought into focus by the early work on anomalies by Karsten and Smit [1] and the no-go theorem of Nielsen and Ninomiya [2]. The decades of failed attempts might have been enough to stop research in the field if it were not for the fact that the world is well described by a chiral gauge theory (the standard model) and we need to fully understand how it works nonperturbatively and how it can defined. Furthermore, strongly coupled chiral gauge theories are expected to display interesting phenomena such as massless composite fermions and would be interesting to simulate numerically. Recently it was proposed that Weyl fermions could appear as edge states on a finite five-dimensional manifold with a single boundary, and that a chiral gauge theory could be constructed on the four-dimensional boundary by gauging the exact vectorlike symmetry in the five-dimensional world. In particular, the Dirac equation on a solid torus in odd dimensions was analyzed in the continuum and shown to support massless 2D or 4D Weyl fermions on the boundary. The problem was reduced to considering the Dirac equation on the 2D disk that forms the cross section of the torus, with the coordinate around the edge of the disk to be identified with one of our spatial dimensions [3] (for related work, see Ref. [4]). The proposal [3] exploits the fact that massless fermion edge states appear at the boundary between two different topological phases, which can exist in discretized theories as well as the continuum, and as such are not subject to fine-tuning [5–7]. Furthermore, it is clear that there exist nonlocal interactions in such a theory with a coefficient proportional to the gauge anomaly, so it follows algebraically that an anomalous local 4D chiral gauge theory could never be constructed using this method, an important check on whether the method is sensible. There are several speculative components to the idea, and here we address one of the simplest: whether the free fermion spectrum discussed in the continuum in Ref. [3] can be realized on a lattice. The answer is yes, and we demonstrate that Weyl edge state modes are ubiquitous in extremely simple lattice theories of free fermions.

It is not obvious that the continuum analysis should apply to the lattice. As a technical issue, the chiral nature of the edge states in the continuum follows from discarding solutions to the Dirac eigenvalue equation, which are divergent at the center of the disk. If one considered an annulus with a hole in the center instead of a disk, in place of a single Weyl fermion edge state in the spectrum with level spacing proportional to the inverse disk radius, $1/R$, there would have been two of opposite chirality, the second solution circling the interior boundary of the annulus with level splitting set by the inner radius, $1/R'$. Since a lattice is full of holes, one should expect such mirror states when space is discretized, potentially ruining the mechanism. However, if we consider the lattice to be like the annulus, but with holes at the scale of the lattice spacing $R' \sim a$, it may be reasonable to expect such mirror states to be gapped at the lattice cutoff. This gapping mechanism does not break the exact $U(1)$ symmetry [or $U(N_f)$ for multiple flavors], in which we hope to embed a chiral gauge group, as it does not result from a mass for the unwanted Weyl mode but instead from forcing it to have high angular momentum around the small holes.

This picture is simplistic but helps to address which of the four underlying assumptions the Nielsen-Ninomiya theorem is being violated [8]. As seen in the continuum example, the chiral symmetry of the edge states on the disk descends directly from the exact fermion number symmetry of the higher dimension theory, and this is true on the lattice as well. Therefore the chiral symmetry assumption is not being violated, the way it is by Ginsparg-Wilson fermions. With the starting point being Wilson fermions in $d+1$ dimensions there are no doublers, and the correct continuum limit can be achieved at zero wave number, satisfying two more of the Nielsen-Ninomiya assumptions. What is being violated apparently is the assumption that the fermion operator be periodic and analytic in momentum. That assumption forbids the fermion energy levels as a function of momentum to cross zero an odd number of times, ruling out a Weyl fermion, while breaking it makes a single Weyl mode possible. This is plausible in the picture of the annulus: the continuum limit is the large-R limit, keeping R' fixed. Thus the modes on the outer edge develop a continuous spectrum while those of opposite chirality on the inner edge do not.

We show here that in fact such a picture seems to be realized. What we find is that a very standard lattice theory of Wilson fermions can describe a left-handed fermion bound on the outer rim of lattice manifold, which develops a continuous spectrum at low energy as the size of the lattice is taken to be large. There are no long wavelength states one might identify as a mirror Weyl fermion.

In this Letter we examine the spectrum for an extremely simple system: the discretized Hamiltonian for free Wilson fermions in $2+1$ dimensions,

$$H = \gamma_0 \mathcal{D} = \gamma_0 \sum_{\mu=1}^{2} \gamma_\mu \partial_\mu + M + \frac{r}{2}\Delta,$$
$$\gamma_0 = \sigma_3, \qquad \gamma_1 = \sigma_1, \qquad \gamma_2 = \sigma_2. \tag{1}$$

The ∂_μ are symmetric lattice derivatives with $\mu = 1, 2$, and Δ is the 2D lattice Laplacian (see the Supplemental Material [9] for details of our lattice implementation). Since the features of the fermion spectrum we are interested in are topological in nature, they are determined by both the ratio M/r, [5–7] and by the boundary conditions [10]. We take $M = r = 1$, ensuring a nontrivial topological phase in the bulk. For the boundary conditions, as discussed in Ref. [3], a $(d+1)$-dimensional manifold ($d = 2n$) with no boundaries is expected to support no massless states; a $(d+1)$ manifold with two disconnected boundaries is expected to support a light Dirac fermion (two Weyl fermions of opposite chirality with exponentially small interactions), and a $(d+1)$ manifold with a single boundary should have a single massless Weyl fermion with exponentially small nonlocal interactions. These considerations are independent of details of the shape of the lattice or whether the low energy theory possesses $d = 1+1$ Lorentz invariance. Therefore we start by considering the simplest possible lattice, a square lattice, cut into a square shape of dimension $L \times L$, with three possible boundary conditions: (i) periodic boundary conditions in both coordinates, $\psi(x+L, y) = \psi(x, y)$, $\psi(x, y+L) = \psi(x, y)$. This is the usual boundary condition for Wilson fermions, the lattice has the topology of a 2-torus without boundary, and the spectrum is gapped without any continuum low lying modes. (ii) Periodic boundary conditions in one direction and open boundary conditions in the other: $\psi(x+L, y) = \psi(x, y)$ and $\psi(x, 0) = \psi(x, L+1) = 0$. This is the prescription for conventional domain wall fermions proposed by Shamir [10]. In this case the lattice has the topology of an open cylinder with a circle for the boundary at each end. With two disconnected pieces to the boundary, the lattice supports two Weyl fermions with opposite chirality. They have an interaction vanishing exponentially with the length of the cylinder, turning them into a very light Dirac fermion with mass exponentially small in ML, and have no exact chiral symmetry to be naively gauged [11]. (iii) Open boundary conditions in both directions, $\psi(0, y) = \psi(L+1, y) = \psi(x, 0) = \psi(x, L+1) = 0$. This is the case of interest, realizing a single massless Weyl mode in its spectrum (open boundary conditions have been considered in a different context, see Ref. [19]). The expected behavior is confirmed by plotting the eigenvalues of H for these three different boundary conditions, as shown in Fig. 1.

For a realistic simulation of a $1+1$-dimensional theory of a Weyl fermion, one would want to equate spatial translations in the continuum theory with rotations of the two-dimensional lattice Hamiltonian. Even though short distance modes will always be sensitive to the symmetry of the underlying square lattice, to study the appropriate long wavelength edge modes one will need to consider a square lattice cut in the shape of a disk. To construct the lattice Hamiltonian on such a disk we start with an $L \times L$ square lattice with open boundary conditions, and then act on both sides of the Hamiltonian with projectors setting to zero the field on all sites at radius $r \geq L/2 - 1$, discarding eigenvectors corresponding to those sites. Alternatively, one can use the square lattice with a radially dependent mass with $M = 1$ for $r < L/2$ and $M \ll -1$ for $r \geq L/2$, discarding eigenvectors with $|\omega_n| \gg 1$. Both methods give similar results. As shown in Ref. [3], linear momentum of the edge state is then given by J/R, where J is the angular momentum operator $J = (\vec{r} \times \vec{p} + S)$, where $S = i/4[\gamma_x, \gamma_y]$.

There are several properties we expect to find belong to the Weyl modes in the gap $-1 \leq \omega_n \leq 1$. (i) Their energy eigenvalues should obey the dispersion relation appropriate

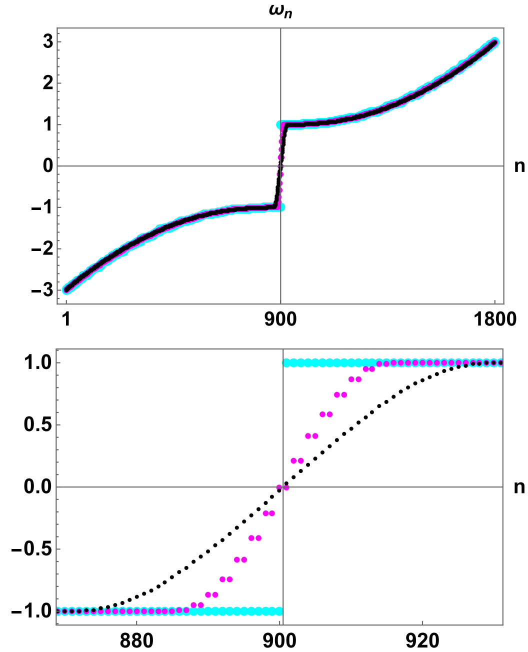

FIG. 1. The 1800 ordered eigenvalues ω_n on the y axis versus n on the x axis for the free Wilson fermion Hamiltonian with $a = M = r = 1$ on a 30×30 lattice, with a magnified view of the crossing point in the lower panel. Cyan points are for purely periodic boundary conditions, corresponding to a spatial manifold with no boundary, and exhibit a gap. Magenta indicates mixed periodic and open boundary conditions, as specified by Shamir [10] for domain wall fermions, supporting a Weyl fermion on each of its two S^1 surfaces with opposite chiralities, exact zero modes, and evident degeneracy. The nondegenerate black points are for purely open boundary conditions, representing the situation described in Ref. [3]: a manifold with a single boundary (a square, in this case) supporting a single Weyl fermion without an exact zero mode.

for a massless chiral fermion, $\omega = -\langle J\rangle/R$ (with $O(1/MR)$ corrections), where $\langle J\rangle$ is the total angular momentum quantum number. (ii) These Weyl states should be localized on the boundary. (iii) The edge states should be chiral, only rotating clockwise around the boundary in the simple model we consider. States of opposite chirality can be obtained by reversing the signs of both M and r in H.

The expected dispersion relation characteristic of a massless Weyl fermion is evident in Fig. 2, where we have plotted the energy of each energy eigenstate versus the expectation value of the total angular momentum operator J for a disk-shaped lattice with radius $R = 34$ sites. (For this calculation we used improved derivative operators for the orbital part of J, canceling lattice artifacts to $O(a^7)$, thereby gaining a modest improvement in the range of linear behavior in ω versus $-\langle J\rangle/R$.) The Weyl modes have an increasingly dense spectrum as one increase R, but rather then being periodic in J, the spectrum turns back at large energy and dissolves into a sea of lattice artifact states. This

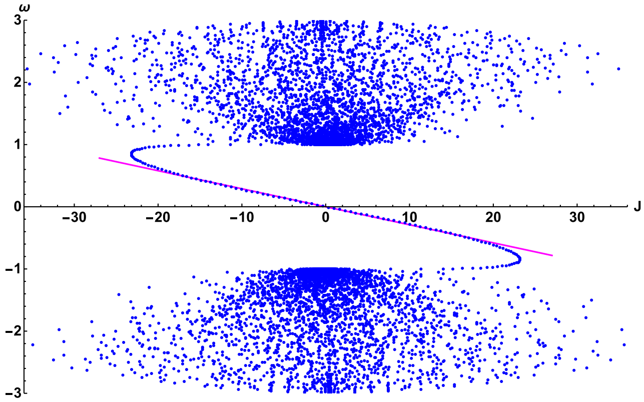

FIG. 2. The energy for every mode plotted versus the expectation value of the angular momentum J on a radius $R = 34$ disk of square lattice with open boundary conditions. The straight line is a plot of the function $\omega = -\langle J\rangle/R$, where $\langle J\rangle/R$ can be identified as the edge state momentum along the boundary.

result contradicts the conventional wisdom derived from the Nielsen-Ninomiya theorem that a Weyl-like dispersion relation cannot arise from a physically sensible microscopic lattice theory of free fermions.

In Fig. 3 we show the density plots for three positive energy eigenstates of j_0 and j_θ, where j_μ is the lattice construction of the continuum fermion number current operator $\bar{\psi}\gamma_\mu\psi$, and $j_\theta = (-\sin\theta j_1 + \cos\theta j_2)$—one with $\omega \ll 1$, one in the middle of the $|\omega| < 1$ "Weyl gap", and one just above the edge of the gap. The plots confirm expectations: the first two modes are well localized at the boundary and exhibit counterclockwise current flow, the latter very much a bulk state with no definite chirality.

We have contrasted the three simplest possible boundary conditions on a two-dimensional square lattice of Wilson fermions in one of their nontrivial topological phases: ({periodic, periodic}, {periodic, open}, and {open, open}). We have demonstrated that they lead to quite different physics: a gapped spectrum of no interest for a continuum theory in the {periodic, periodic} case, the well studied case of conventional domain wall fermions with an almost massless Dirac mode on the surface in the {periodic, open} case, and a new example with a massless Weyl fermion on the surface for {open, open}. When the latter is reformulated as a lattice cut in the approximate shape of a disk with an open boundary condition, the Weyl states display approximate rotational invariance as required if one hopes to attain Lorentz invariance in the IR. Generalizations to higher dimension lattices are expected to lead to even richer phase structure. It is gratifying that insights from the continuum study of a manifold with a single boundary supporting a Weyl fermion carry over directly to the lattice, and that the theory has no problem sidestepping restrictions implied by the Nielsen-Ninomiya theorem. There is still

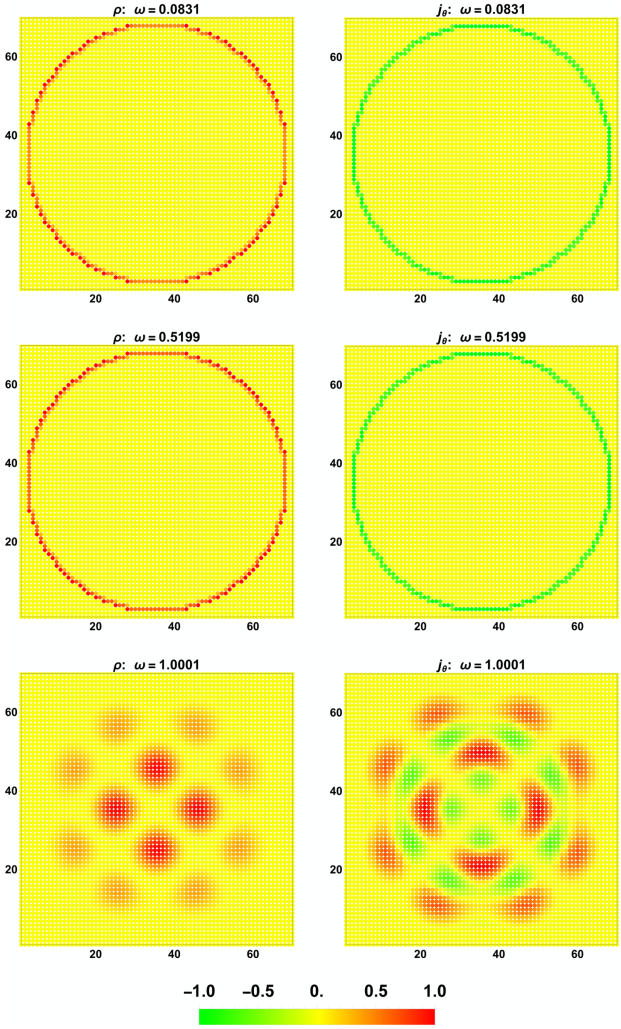

FIG. 3. The fermion number charge densities ρ (left) and angular fermion number current density j_θ (right) on a radius $R = 34$ disk of square lattice with open boundary conditions for energy eigenstates with energies $\omega = 0.0831$, 0.5199, 1.0001 in lattice units (top to bottom). Both the ρ and j_θ plots display sharp edge localization for the two modes in the Weyl gap $|\omega| < 1$, and delocalization for the mode slightly above $\omega = 1$. The Weyl modes exhibit strictly negative j_θ corresponding to clockwise rotation around the lattice boundary, while the bulk mode does not display any net chirality. The data has been rescaled by the maximum magnitude of the density value.

much that can be learned about the lattice discussed here, such as the degree of nonlocality in the propagation of the Weyl surface mode, and whether there is some version of the overlap operator to describe its behavior without reference to the higher dimension world. Ultimately, a realistic computation will involve discretizing time as well as space and including gauge fields as link variables, integrating over their boundary values as discussed in Ref. [3], the latter entailing additional questions and challenges not addressed here.

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*Corresponding author: dbkaplan@uw.edu
†Corresponding author: srimoyee08@gmail.com

[1] L. H. Karsten and J. Smit, Lattice fermions: Species doubling, chiral invariance and the triangle anomaly, Nucl. Phys. **B183**, 103 (1981).
[2] H. B. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice. 1. Proof by homotopy theory, Nucl. Phys. **B185**, 20 (1981); Nucl. Phys. **B195**, 541(E) (1982).
[3] D. B. Kaplan, preceding Letter, Chiral gauge theory at the boundary between topological phases, Phys. Rev. Lett. **132**, 141603 (2024).
[4] S. Aoki and H. Fukaya, Curved domain-wall fermion and its anomaly inflow, Prog. Theor. Exp. Phys. **2023**, 033B05 (2023).
[5] D. B. Kaplan, A method for simulating chiral fermions on the lattice, Phys. Lett. B **288**, 342 (1992).
[6] K. Jansen and M. Schmaltz, Critical momenta of lattice chiral fermions, Phys. Lett. B **296**, 374 (1992).
[7] M. F. L. Golterman, K. Jansen, and D. B. Kaplan, Chern-simons currents and chiral fermions on the lattice, Phys. Lett. B **301**, 219 (1993).
[8] The assumptions being that in the infinite volume limit the fermion operator be continuous and periodic in momentum space, that it be free of doublers (multiple zeros), that it have the correct continuum limit at zero wave number, and that it anticommute with γ_5. The first assumption is required for locality in position space.
[9] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.141604 for details of our lattice implementation.
[10] Y. Shamir, Chiral fermions from lattice boundaries, Nucl. Phys. **B406**, 90 (1993).
[11] This system has been fully described in the literature. In the large volume limit the massless Dirac mode is described by the overlap operator [12–16]. In turn, the overlap operator solves the Ginsparg-Wilson equation [17], which clarifies exactly how the mode violates chiral symmetry, even in the infinite volume limit, which can be thought of as being due to the nondecoupling of massive modes in the bulk that generate a Chern-Simons form, which accounts for the anomaly [18].
[12] H. Neuberger, Exactly massless quarks on the lattice, Phys. Lett. B **417**, 141 (1998).
[13] H. Neuberger, Vectorlike gauge theories with almost massless fermions on the lattice, Phys. Rev. D **57**, 5417 (1998).
[14] R. Narayanan and H. Neuberger, Chiral determinant as an overlap of two vacua, Nucl. Phys. **B412**, 574 (1994).

[15] R. Narayanan and H. Neuberger, Infinitely many regulator fields for chiral fermions, Phys. Lett. B **302**, 62 (1993).

[16] R. Narayanan and H. Neuberger, Chiral fermions on the lattice, Phys. Rev. Lett. **71**, 3251 (1993).

[17] P. H. Ginsparg and K. G. Wilson, A remnant of chiral symmetry on the lattice, Phys. Rev. D **25**, 2649 (1982).

[18] C. G. Callan, Jr. and J. A. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. **B250**, 427 (1985).

[19] M. Lüscher and S. Schaefer, Lattice qcd with open boundary conditions and twisted-mass reweighting, Comput. Phys. Commun. **184**, 519 (2013).