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Editors' Suggestion

Large-N Principal Chiral Model in Arbitrary External Fields

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We report the explicit solution for the vacuum state of the two-dimensional SU(N) principal chiral model at large N for an arbitrary set of chemical potentials and any interaction strength, a unique result of such kind for an asymptotically free quantum field theory. The solution matches one-loop perturbative calculation at weak coupling, and in the opposite strong-coupling regime exhibits an emergent spacial dimension from the continuum limit of the SU(N) Dynkin diagram.

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Introduction.—The principal chiral model (PCM) is a remarkable representative of a large class of integrable relativistic $(1+1)$-dimensional QFTs [1,2] possessing key features of QCD: asymptotic freedom, mass-gap generation, and a nontrivial topological expansion in 't Hooft's large-N limit. These features attracted considerable attention to the nonperturbative dynamics of PCM that has been studied via the exact S matrix [2–4] and Bethe ansatz [5–10] giving access to thermodynamics of the model [2]. While the S matrix is explicitly known for PCM [5], the linear Bethe equations for the vacuum as a function of chemical potentials or the nonlinear thermodynamic Bethe ansatz equations are difficult to solve at any coupling strength. They can be studied perturbatively [7] or numerically [8–10], but their analytic solution is beyond reach. Fortunately, there is an exception: the large-N limit of the vacuum Bethe equations is exactly solvable [11,12] for a specific choice of external fields (chemical potentials of conserved charges) repeating the mass spectrum profile in their dependence on the SU(N) Dynkin label [13]. In this Letter we lift this restriction and solve the model at large N for any set of chemical potentials. It gives us the first exact analytic solution for an asymptotically free theory at general chemical potentials.

The FKW solution [11,12] obeys the semicircle law in rapidity and can be systematically extended to higher orders in $1/N$; moreover, a double-scaling (DS) limit combining large N with strong coupling resums all orders of the $1/N$ expansion [15]. In this regime the Dynkin labels give rise to a new emergent dimension in addition to the $1+1$ physical dimensions of space-time, as was pointed out already in [11,12]. The DS limit presumably describes a three-dimensional noncritical string theory, by analogy to the $c=1$ matrix quantum mechanics [16] whose DS limit [17–20] is dual to noncritical strings in two dimensions [21–27].

We will report the solution for the vacuum state of large-N PCM with arbitrary chemical potentials. The rapidity dependence happens to be always semicircular with the support $(-B_a, B_a)$ varying along the Dynkin diagram. We derived an integral equation for the limit shape $B(\alpha)$ assuming its continuity. We then match the free energy with the known one-loop result [7] at weak coupling. At strong coupling, we observe the locality along the emergent dimension for the small deviations from the first, Perron-Frobenius (PF) mode of the external field. This may have bearing on the currently unknown effective theory of the dual three-dimensional string.

PCM with general chemical potentials at large N.—The PCM is defined by the Lagrangian,

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$$S = \frac{N}{\lambda_0}\int d^2x \mathrm{tr} D_\mu g^\dagger D^\mu g, \tag{1}$$

where the field $g(x) \in \mathrm{SU}(N)$. The mass spectrum consists of $N-1$ particle species in the bifundamental representations of $\mathrm{SU}(N)\times\mathrm{SU}(N)$. The lightest particle acquires its mass by dimensional transmutation while others can be seen as its bound states:

$$m_a = m\frac{\sin\frac{\pi a}{N}}{\sin\frac{\pi}{N}}, \qquad a = 1\ldots N-1. \tag{2}$$

A finite density of particles is introduced by gauging the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ global symmetry by constant chemical potentials: $D_0 = \partial_0 g - (i/2)(Hg + gH)$, $D_1 = \partial_1$, where $H = \mathrm{diag}(q_1,\ldots,q_N)$ is a traceless diagonal matrix with the eigenvalues organized in the descending order: $q_1 \geq q_2 \geq \cdots \geq q_N$. The chemical potential of the ath species, transforming in the rank-a antisymmetric representation, is then $h_a = \sum_{b=1}^{a} q_b$ (ordering of q_a's is important for this conclusion). At large N we introduce a continuous coordinate $\alpha = \pi a/N \in (0,\pi)$ and define the limit shape functions $q(\alpha) = q_a$ and $h(\alpha) = \pi h_a/N$, smooth at $N\to\infty$ and related by $h'(\alpha) = q(\alpha)$. Trace and ordering conditions on q_a translate into the boundary conditions on $h(\alpha)$:

$$h(0) = 0 = h(\pi), \qquad h''(\alpha) < 0. \tag{3}$$

The Bethe equations for the ground state of PCM take the form of integral equations for the pseudo-energies [11]:

$$\varepsilon_a(\theta) + \sum_{b=1}^{N-1}\int_{-B_b}^{B_b} d\theta\, R_{ab}(\theta-\theta')\varepsilon_b(\theta) = h_a - m_a\cosh\theta, \tag{4}$$

which hold on the intervals $(-B_a, B_a)$ where the pseudo-energies are positive. The condition that they vanish at the end points, $\varepsilon_a(\pm B_a) = 0$, closes the system. Once the pseudo-energies are known, the vacuum energy is obtained by simple integration:

$$E = -\sum_{a=1}^{N-1} m_a \int_{-B_a}^{B_a}\frac{d\theta}{2\pi}\varepsilon_a(\theta)\cosh\theta. \tag{5}$$

The kernels of the integral equation originate from scattering between particles in the Fermi sea and can be extracted from the exact S matrix of PCM (their Fourier transform is explicitly given in [11]):

$$R_{ab}(\theta) = G\left(\frac{\pi(a+b)}{N} + i\theta\right) - G\left(\frac{\pi|a-b|}{N} + i\theta\right) + \text{c.c.}, \tag{6}$$

where

$$4\pi^2 G(x) = \psi\left(\frac{x}{2\pi}\right) + \psi\left(-\frac{x}{2\pi}\right) - \frac{2\pi}{x}, \tag{7}$$

and $\psi(x)$ is the logarithmic derivative of the gamma function.

Taking the large-N limit is now straightforward: the Dynkin labels are promoted to continuous variables α and β, while summation is replaced by integration. Quite remarkably, the Dynkin coordinate and rapidity combine into complex variables:

$$z = \alpha + i\theta, \qquad w = \beta + i\theta', \tag{8}$$

such that the kernel (6) becomes almost holomorphic:

$$\begin{aligned} R(z,\bar z; w,\bar w) &= G(z+\bar w) - \frac{1}{2}G(z-w) - \frac{1}{2}G(w-z) \\ &\quad + \frac{\mathrm{sign}(\alpha-\beta)}{2\pi(z-w)} + \text{c.c.} \end{aligned} \tag{9}$$

We remark here that for symmetric $\varepsilon(\beta,\theta') = \varepsilon(\beta,-\theta')$ the first term $G(z+\bar w)$ is equivalent to $G(z+w)$ and then the first line becomes holomorphic in both variables, but this is immaterial for our purposes: holomorphy in z alone will suffice. More importantly, the second line breaks holomorphy explicitly, but just locally on the line $\alpha = \beta$. The sign function originates from $|a-b|$ in the kernel (6), before the large-N limit $|\alpha-\beta|$ comes in multiples of π/N, and the sign function should be smeared on the interval $(-\varepsilon,\varepsilon)$, where ε is a proxy for π/N. At the end ε will just define the contour deformation prescription.

Finally we get the Bethe equations in the two-dimensional form:

$$\int_{\mathcal{D}} d^2w\, R(z,\bar z; w,\bar w)\varepsilon(w,\bar w) = h(\alpha) - \frac{m}{2}(\sin z + \sin\bar z), \tag{10}$$

where integration is over a complex domain,

$$\mathcal{D} = \{\alpha + i\theta | 0 < \alpha < \pi, -B(\alpha) < \theta < B(\alpha)\}, \tag{11}$$

and the equation holds for $z \in \mathcal{D}$.

Solution of Bethe equation.—Solution of the large-N integral equation crucially relies on holomorphy of the kernel. Every holomorphic function is harmonic, and thus, applying the Laplacian $\partial^2/\partial z\partial\bar z$ to both sides of (10) picks only the anomalous term which moreover becomes local in the Dynkin variable. At the end we get a singular integral equation for each α:

$$\frac{2}{\pi}\fint_{-B(\alpha)}^{B(\alpha)} d\theta' \, \frac{\varepsilon(\alpha,\theta')}{(\theta-\theta')^2} = h''(\alpha), \tag{12}$$

the solution of which is the famous semicircle law:

$$\varepsilon(\alpha,\theta) = -\frac{h''(\alpha)}{2}\sqrt{B(\alpha)^2-\theta^2}. \tag{13}$$

Defining a function,

$$F(z) = \int_{\mathcal{D}} d^2w \left[2G(z+\bar{w}) - G(z-w) - G(w-z) + \frac{\operatorname{sign}(\alpha-\beta)}{\pi(z-w)}\right]\varepsilon(w,\bar{w}) - ih'(\alpha)\theta - h(\alpha), \tag{14}$$

we see that it will be holomorphic in $\mathcal{D}$ as soon as $\varepsilon(w,\bar{w})$ satisfies the integral equation (12). This can be checked by computing $\partial F/\partial\bar{z}$. On the other hand, by definition,

$$\int_{\mathcal{D}} d^2w \, R(z,\bar{z};w,\bar{w})\varepsilon(w,\bar{w}) = \frac{F(z)}{2} + \frac{F(\bar{z})}{2} + h(\alpha). \tag{15}$$

In virtue of this equation, (10) factorizes into holomorphic and antiholomorphic parts and boils down to

$$F(z) = -m\sin z. \tag{16}$$

This functional condition can be used for fixing the profile of the Fermi surface $B(\alpha)$. We thus call it the *shape equation*.

A useful form of the shape equation is obtained by turning the branch cuts of the square root inside out, in other words, replacing the short cut $[-B(\alpha), B(\alpha)]$ passing through zero by a long cut passing through infinity. The resulting expression is manifestly analytic in $\mathcal{D}$:

$$F(z) = \int_0^{\pi}\frac{d\beta}{2\pi i}h''(\beta)\sin\beta\int_{B(\beta)}^{\infty}\frac{d\xi\sqrt{\xi^2-B(\beta)^2}}{\cosh(\xi-iz)+\cos\beta} - (z\to -z). \tag{17}$$

We mention in passing that transformation from short to long cuts is well familiar in the theory of integrable systems; for example, it is a crucial step in deriving the quantum spectral curve for AdS/CFT [28,29]. Apart from making holomorphy manifest, this representation reveals the symmetries of the problem: written that way, $F(z)$ is manifestly 2π periodic and antisymmetric. A convenient basis in the space of such functions is $\{\sin pz | p\in\mathbb{N}\}$. Expanding (16) and (17) in this basis and matching the coefficients gives:

$$\int_0^{\pi} d\alpha \, h''(\alpha)B(\alpha)K_1(pB(\alpha))\sin p\alpha = -\frac{\pi m}{2}\delta_{p1}, \tag{18}$$

where K_1 is the modified Bessel function. For the energy, we get accordingly:

$$E = \frac{N^2m}{4\pi^2}\int_0^{\pi} d\alpha \, h''(\alpha)B(\alpha)I_1(B(\alpha))\sin\alpha. \tag{19}$$

These two equations completely characterize the ground state of PCM at arbitrary chemical potentials.

If the chemical potentials repeat the mass spectrum, with only the lowest PF mode present: $h(\alpha) = \mathbb{h}\sin\alpha$, the solution is constant: $B(\alpha) = \mathbb{B}$. Indeed, the shape equations (18) with $p\geq 2$ then follow from orthogonality of $\sin k\alpha$ with different k, and the shape equation for $p=1$ determines $\mathbb{B}$:

$$\mathbb{B}K_1(\mathbb{B}) = \frac{m}{\mathbb{h}}, \qquad E = -\frac{N^2m\mathbb{h}}{8\pi}\mathbb{B}I_1(\mathbb{B}), \tag{20}$$

recovering the results of [11,12]. At large chemical potential the theory is weakly coupled, and the Fermi rapidity grows logarithmically:

$$\mathbb{B} \overset{\mathbb{h}\gg m}{\simeq} \ln\frac{\mathbb{h}}{m} + \frac{1}{2}\ln\ln\frac{\mathbb{h}}{m} + \frac{1}{2}\ln\frac{\pi}{2} + \cdots. \tag{21}$$

It can be shown to satisfy the two-loop renormalization group equation and can thus be identified with the inverse of the running coupling in a particular renormalization scheme [11]:

$$\frac{4\pi}{\lambda(\mathbb{h})} = \mathbb{B}. \tag{22}$$

The Fermi rapidity thus directly controls the interaction strength. In the opposite, strong-coupling regime (small $\mathbb{B}$), we have [11]

$$\mathbb{B} \overset{\mathbb{h}\to m}{\simeq} \sqrt{\frac{4\Delta}{|\ln\Delta|}}, \qquad \Delta = \frac{\mathbb{h}}{m} - 1. \tag{23}$$

In general, the shape equations are an infinite set of nonlinear integral conditions for a real function $B(\alpha)$. These equations can be solved numerically, as illustrated in Fig. 1, however, deriving an analytic solution for a general profile proves challenging. Therefore, our analysis is confined to small perturbations around the constant solution.

Small fluctuations around PF mode.—Consider the chemical potential of the form

$$h(\alpha) = \mathbb{h}\sin\alpha + \sum_{p=2}^{\infty} h_p\sin p\alpha, \tag{24}$$

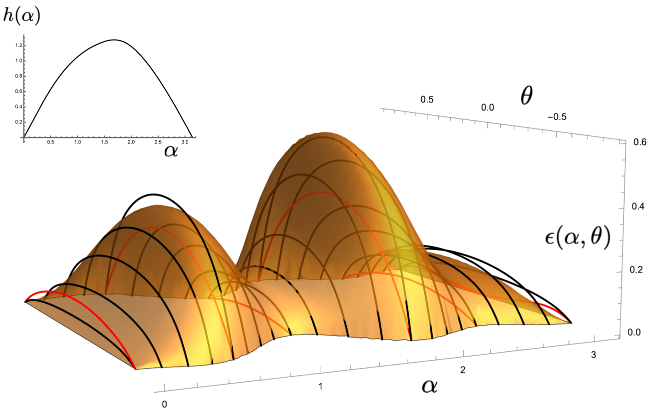

FIG. 1. The semitransparent surface represents the large-N pseudo-energy (13) with the profile $B(\alpha)$ computed from the shape equation (18). For comparison we plot the numerical solution of the exact integral equations (4) for $N = 7$ (red lines) and $N = 22$ (black lines). The inset shows the chemical potential $h(\alpha)$.

with small $h_p \ll \mathbb{h}$. The Fermi surface will deviate little from a constant: $B(\alpha) = \mathbb{B} + b(\alpha)$, and we may expand the shape equations in b. To streamline this expansion we introduce convenient notations.

For two functions on the interval $(0, \pi)$ we define a pairing:

$$\langle f|g\rangle = -\frac{2}{\pi}\int_0^{\pi} d\alpha h''(\alpha) f(\alpha)^* g(\alpha). \tag{25}$$

We also introduce two sets of functions:

$$\langle p| = \sin p\alpha, \qquad |p\rangle = U_{p-1}(\cos\alpha), \tag{26}$$

where U_n are Chebyshev polynomials. Those are orthogonal for the pure PF profile, while for a more general profile (24)

$$\langle p|p'\rangle = \mathbb{h}\delta_{pp'} + \mathcal{O}(h_k). \tag{27}$$

The $\mathcal{O}(h_k)$ terms can be derived explicitly, but we will only need these three:

$$\langle 1|p\rangle = \langle p|1\rangle = p^2 h_p, \qquad \langle 1|1\rangle = \mathbb{h}. \tag{28}$$

The shape equations and the energy are concisely written as

$$\langle p|\mathcal{K}(pB)\rangle = m\delta_{p1}, \qquad E = -\frac{N^2 m}{8\pi}\langle 1|\mathcal{I}(B)\rangle, \tag{29}$$

where $\mathcal{K}(x) = xK_1(x)$ and $\mathcal{I}(x) = xI_1(x)$.

It is now straightforward to expand the shape equations around the constant solution:

$$\mathcal{K}\langle p|1\rangle + p\mathcal{K}'\langle p|b\rangle + \frac{1}{2}p^2\mathcal{K}''\langle p|b^2\rangle + \cdots = m\delta_{p1}. \tag{30}$$

If $p \geq 2$, we may keep only the linear order:

$$\langle p|b\rangle = -\frac{p\mathcal{K}}{\mathcal{K}'}h_p, \tag{31}$$

from which

$$b(\alpha) = \sum_{p=2}^{\infty}\frac{pK_1(p\mathbb{B})}{\mathbb{h}K_0(p\mathbb{B})}h_p U_{p-1}(\cos\alpha). \tag{32}$$

The constant ($p = 1$) mode in $b(\alpha)$ appears at the quadratic order and can be found from the $p = 1$ equation because the linear term there cancels identically and so it is valid at $\mathcal{O}(b^2)$:

$$\langle 1|b\rangle = -\frac{\mathcal{K}''}{2\mathcal{K}'}\langle 1|b^2\rangle. \tag{33}$$

For b appearing on the right-hand side, we can use the linear approximation above.

Corrections to the energy are also quadratic:

$$E = -\frac{N^2 m}{8\pi}\left(\mathcal{I}\langle 1|1\rangle + \mathcal{I}'\langle 1|b\rangle + \frac{1}{2}\mathcal{I}''\langle 1|b^2\rangle + \cdots\right). \tag{34}$$

Using (33) and plugging in the solution for b, we get

$$E = -\frac{N^2 m}{8\pi\mathbb{h}}\left(\mathbb{B}I_1(\mathbb{B})\mathbb{h}^2 + \sum_{p=2}^{\infty}\frac{p^2 K_1(p\mathbb{B})^2}{2K_0(\mathbb{B})K_0(p\mathbb{B})^2}h_p^2\right). \tag{35}$$

This result is valid both at weak and strong coupling; the only assumption used in its derivation is smallness of deviations from the PF mode in $h(\alpha)$.

Weak coupling.—At weak coupling (large $\mathbb{B}$), we have

$$E \overset{\mathbb{B}\gg 1}{\simeq} -\frac{N^2}{16\pi}\left[\mathbb{B}\mathbb{h}^2 + \mathbb{B}\sum_p h_p^2 + \frac{1}{2}\sum_p p(p+2)h_p^2\right]. \tag{36}$$

This can be compared with the perturbative expression known to the one-loop order [7] for generic chemical potentials [30]:

$$E^{\text{1-loop}} = -\frac{N^2}{2\pi\lambda(\mathbb{h})}\int d\alpha\, q(\alpha)^2 - \frac{N^2}{16\pi^3}\int d\alpha d\beta[q(\alpha) - q(\beta)]^2 \ln\frac{2|q(\alpha) - q(\beta)|}{e\mathbb{h}}, \tag{37}$$

where $q(\alpha) = h'(\alpha)$ and the running coupling is defined by (22). Plugging in $h(\alpha)$ from (24) and expanding in h_p to the quadratic order, we get the complete agreement with our integrability-based result.

Strong coupling.—In the strong-coupling regime $\mathbb{h}$ approaches m and $\mathbb{B}$ goes to zero, then

$$E \overset{\mathbb{B}\ll 1}{\simeq} -\frac{N^2}{16\pi}\left(\mathbb{B}^2\mathbb{h}^2 + \frac{1}{\mathbb{B}^2|\ln\mathbb{B}|^3}\sum_p h_p^2\right). \qquad (38)$$

Quite remarkably, the energy becomes local along the Dynkin dimension. Using (23), and denoting $\varphi(\alpha) = (h(\alpha) - \mathbb{h}\sin\alpha)/(4\Delta)$, we get

$$E \overset{\Delta\to 0}{\simeq} -\frac{N^2\Delta}{4\pi|\ln\Delta|}\left(m^2 + \frac{1}{|\ln\Delta|}\int d\alpha\,\varphi(\alpha)^2\right). \qquad (39)$$

Deviations from the PF mode are naturally of order Δ, thus φ can be viewed as an order-one variable, albeit we had linearized in fluctuations and thus neglected contributions of higher than quadratic order in φ; moreover, one can show that the very leading term will be the same for any small profile $B(\alpha)$. The fluctuations beyond the PF mode appear to be logarithmically suppressed at strong coupling. Perhaps the PF contribution should be regarded as a subtraction constant and the remainder could possibly be interpreted in terms of an effective theory for fluctuations, of which we calculated just the quadratic term.

Discussion.—We extended the exact solution of large-N PCM [11,12] to the case of arbitrary chemical potentials. The pseudo-energies of physical excitations still follow the semicircular law but now the Fermi surface becomes a new functional degree of freedom that has a nontrivial profile along the Dynkin diagram. This plays the role of a new continuous dimension emerging in the large-N limit. With such new functional degrees of freedom at hand one can model various weak-coupling regimes which may reveal new types of nonperturbative trans-series expansions [31]. At strong coupling, intriguing signs of locality arise, perhaps pointing toward string theory interpretation. It would be interesting to split the Dynkin diagram into two intervals and calculate entanglement entropy as a diagnostic for locality [32].

Similarly to our previous work [15], one can identify the double-scaling limit in the case of general chemical potentials; we will return to this point in the forthcoming publication [33].

Another interesting direction would be to analyze the large-N limit of PCM compactified on the cylinder with twisted boundary conditions; the corresponding analysis at finite N was already initiated in [10]. This corresponds to the putative dual string at finite temperature. Also, it would be interesting to go beyond the vacuum state and analyze the excited states.

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