# Fate of Chiral Symmetries in the Quark-Gluon Plasma from an Instanton-Based Random Matrix Model of QCD

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We propose a new way of understanding how chiral symmetry is realized in the high temperature phase of QCD. Based on the finding that a simple free instanton gas precisely describes the details of the lowest part of the spectrum of the lattice overlap Dirac operator, we propose an instanton-based random matrix model of QCD with dynamical quarks. Simulations of this model reveal that even for small quark mass the Dirac spectral density has a singularity at the origin, caused by a dilute gas of free instantons. Even though the interaction, mediated by light dynamical quarks, creates small instanton–anti-instanton molecules, those do not influence the singular part of the spectrum, and this singular part is shown to dominate Banks-Casher type sums in the chiral limit. By generalizing the Banks-Casher formula for the singular spectrum, we show that in the chiral limit the chiral condensate vanishes if there are at least two massless flavors. Our model also indicates a possible way of resolving a long-standing debate, as it suggests that for two massless quark flavors the $U(1)_\mathrm{A}$ symmetry is likely to remain broken up to arbitrarily high finite temperatures.

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*Introduction.*—Quantum chromodynamics (QCD), the theory of strong interactions, has been around for 50 years, but we still have not explored all of its consequences. One of the outstanding problems is the lack of a full understanding of the finite temperature transition from hadrons to the quark-gluon plasma, and in particular the restoration of the spontaneously broken chiral symmetry, and the loss of quark confinement.

The lightest quarks in nature, the $u$ and $d$, making up most of the visible matter around us, have much smaller mass than the characteristic scale of QCD. As a result, the $SU(2)_\mathrm{A} \times U(1)_\mathrm{A}$ chiral symmetry that would be exact for massless quarks is a useful approximate symmetry of QCD. While the $U(1)_\mathrm{A}$ part is broken by the anomaly, the $SU(2)_\mathrm{A}$ part is spontaneously broken at low temperatures. The crossover from the hadronic to the quark-gluon plasma state is signaled by a large drop of its approximate order parameter, the chiral condensate. The Banks-Casher relation [1] connects the chiral condensate to the spectral density of the quark Dirac operator at zero as

$$\lim_{m_q \to 0} \langle \bar{\psi}\psi \rangle \propto \rho(0). \tag{1}$$

This suggests that the realization of chiral symmetry is intimately connected to the lowest part of the Dirac spectrum. Indeed, for a long time it was believed that in the high temperature phase $\rho(0)$ vanishes in the chiral limit, signaling the restoration of chiral symmetry. This view, however, was challenged by lattice QCD studies when a more precise exploration of the lowest part of the Dirac spectrum became possible, using chirally symmetric lattice Dirac operators. A study of the overlap Dirac spectrum on quenched gauge backgrounds found that rather than going to zero, the spectral density develops a narrow spike at zero [2]. The statistics of the eigenvalues in the spike was shown to be consistent with mixing instanton–anti-instanton zero modes of a free instanton gas.

This finding was largely ignored for some time, partly because it could be dismissed as a quenched artifact, the result of using an approximation omitting the back reaction of the quarks on the gauge field. Indeed, in the presence of dynamical quarks, each configuration receives in the path integral an additional weight, proportional to the determinant of the Dirac operator, and this is expected to suppress the spike of the spectral density at zero. More recently, detailed studies of the Dirac spectrum with dynamical quarks on finer lattices revealed that the spike is not a quenched artifact [3–6]. Some doubts, however, still remained, as these works used staggered or Wilson sea quarks that, due to their lack of exact chiral symmetry, cannot precisely resolve small Dirac eigenvalues, which could lead to an improper suppression of the spike. Indeed, studies by the JLQCD Collaboration show that for the proper suppression of the peak, the sea quark action should also be chiral [7]. However, it is possible that even though the peak is still there, chiral quarks suppress it so much that larger volumes would be needed to detect it, especially at temperatures farther above the crossover, where the instanton density is already strongly suppressed by the temperature. These calculations with chiral fermions are extremely expensive, and presently cannot be performed on very large volumes. The situation became even more complicated when the spectral spike was claimed to be singular at zero, and based on this the presence of a new, intermediate phase of QCD was suggested [8].

Since the Banks-Casher relation contains the spectral density at the singular point, it is not clear, how the restoration of chiral symmetry occurs. The fate of the flavor singlet axial symmetry $U(1)_A$ is especially interesting, as the question of whether it gets restored at high temperature has important consequences for the phase structure of QCD-like theories [9]. In spite of all the efforts in lattice QCD, there is still an ongoing debate about whether $U(1)_A$ gets restored [4,7].

Here, following Ref. [2], we propose to interpret the accumulation of small Dirac eigenvalues in the spectral spike, in terms of instantons. Indeed, in a high statistics quenched lattice QCD study we showed that the distribution of the topological charge and the number of eigenvalues in the spike are both consistent with an ideal instanton gas [10]. Meanwhile, the noninteracting nature of the instanton gas above $T_c$ was independently confirmed [11–13].

In this Letter we provide further evidence that the spike in the spectral density is due to a free instanton gas, and find that it survives for arbitrarily high temperatures and nonzero quark masses. We show that in the thermodynamic limit the spectral density is singular at the origin, and this singularity dominates the chiral condensate and the $U(1)_A$ breaking susceptibility in the chiral limit. This suggests a whole new picture of the manifestation of chiral symmetry in high temperature QCD, and also solves the problem of $U(1)_A$ restoration.

Our results are based on a simple random matrix model of the Dirac operator in the subspace spanned by instanton zero modes, the so called zero mode zone (ZMZ). Several versions of this model were used in the past [14,15]. However, our proposal is much simpler than any previous version, and more importantly, it precisely captures the details of the spectrum of the lattice overlap Dirac operator on quenched gauge backgrounds. This not only lends further support to the instanton explanation of the spectral spike, but also makes it possible to explore its fate in the chiral limit, a question still impossible to address directly with state of the art lattice simulation techniques.

*The model for quenched QCD.*—We construct a random matrix model, describing the Dirac operator, restricted to the subspace of instanton zero modes. In an ideal gas the number of instantons $n_i$ and anti-instantons $n_a$ are randomly distributed with independent and identical Poisson distributions of mean $\chi_0 V/2$, where $\chi_0 = (1/V)\langle Q^2\rangle = (1/V)\langle (n_i - n_a)^2\rangle$ is the topological susceptibility and $V$ is the four-volume of the system. At high temperatures, the instanton size is limited by the (Euclidean) temporal size of the system, therefore instantons typically occupy all the available space in the temporal direction. Thus their location is only characterized by the three spatial coordinates, which, as the instanton gas is ideal, are chosen randomly in a 3D box of size $L^3$.

We now construct the random matrix, representing the Dirac operator in the ZMZ of such an instanton configuration. The size of the matrix is $(n_i + n_a) \times (n_i + n_a)$, the dimensionality of the ZMZ being equal to the total number of would be zero modes of the topological objects. Since the zero modes of same type objects are protected from mixing by the index theorem, the matrix has two diagonal blocks of zeros of size $n_i \times n_i$ and $n_a \times n_a$. The rest of the matrix elements connect instanton and anti-instanton zero modes that are expected to fall off exponentially with an exponent $\pi T$ [16]. We set the matrix element, connecting instanton $k$ and anti-instanton $l$ to be $w_{kl} = iA \cdot e^{-\pi T r_{kl}}$, where $r_{kl}$ is their distance.

The model has two parameters. The topological susceptibility $\chi_0$ that sets the instanton density and $A$. We use a set of 20 000 quenched $32^3 \times 8$ lattice gauge configurations, produced at $T = 1.1T_c$ (Wilson $\beta = 6.13$) to fix the parameters. The susceptibility is determined by counting the number of zero eigenvalues of the lattice overlap Dirac operator, yielding $\langle Q^2\rangle = 2.58(3)$. We fit $A$ to reproduce the distribution of the lowest eigenvalue of the overlap Dirac operator on the lattice configurations.

In Fig. 1 we show the distribution of the lowest eigenvalue of the Dirac operator on the lattice and in the random matrix model with the best fit parameter $A = 0.40$ (in lattice units). To better resolve these small eigenvalues, we plotted the distribution of the natural log of the eigenvalues. It is already nontrivial that the distribution can be fitted with just this parameter. A further test is provided by comparing the distribution of the lowest eigenvalues on different volumes, illustrated in the same figure. Similar comparisons in different topological sectors, and for the full spectral density, all show good agreement between the lattice and the random matrix model. This gives us confidence that the model captures all the essential features of the lowest part of the lattice overlap Dirac spectrum in the quenched case. Preliminary data also suggest that the parameter $A$ practically does not depend on the temperature in the range 1.05–1.15$T_c$, whereas $\chi_0$ changes by a factor of 3.

*Dynamical quarks.*—On the lattice, including dynamical quarks amounts to supplementing the quenched Boltzmann weight with the determinant of the quark Dirac operator, obtained after integrating out the quarks in the path integral. The determinant can be written as

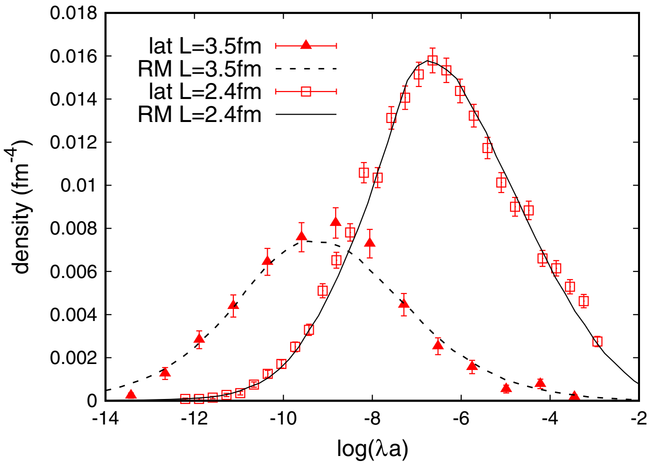


FIG. 1. The distribution of the log of the lowest Dirac eigenvalue in two different volumes on the lattice and in the random matrix model. The smaller volume was used to fit $A$, and for the larger volume the prediction of the model is shown, without any further fitting.

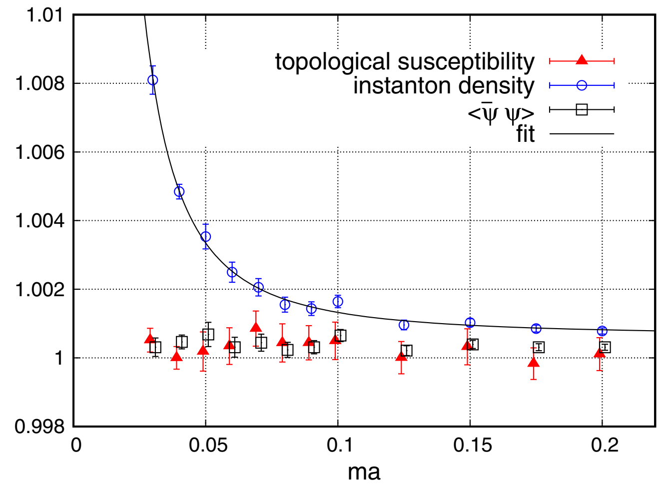


FIG. 2. The topological susceptibility and the instanton density normalized by $m^2\chi_0$, and the chiral condensate normalized by $m\chi_0$ as a function of the quark mass. The fit is of the form $c/m^2 + d$, where $c$ and $d$ are constants.

$$\det\,(D(A)+m)^{N_f} = \prod_{ZMZ}(\lambda_i+m)^{N_f}\times\prod_{\text{bulk}}(\lambda_i+m)^{N_f}, \qquad (2)$$

where $D(A)$ is the gauge field dependent covariant Dirac operator, and for simplicity we assume $N_f$ degenerate flavors of quarks with mass $m$. The product of eigenvalues is split into the lowest eigenvalues (ZMZ) and the rest (bulk). At high temperatures, the ZMZ and the bulk are well separated in the spectrum, and are not expected to be correlated. Therefore, if the quark mass is small, the suppression of the eigenvalues in the ZMZ depends only on the eigenvalues in the ZMZ. This part of the spectrum is well-described by our random matrix model, so including the determinant of the random matrices in the weight will give a self-consistent description of the suppression of the spectral spike by dynamical quarks.

*Simulation of the model and results.*—The instanton-based random matrix model can be easily simulated numerically, even with the additional determinant factor. For that we used the previously described parameters and two degenerate quark flavors. (After including dynamical quarks, the effective temperature in units of $T_c$ will be higher than the original quenched $1.1T_c$, since $T_c$ with dynamical quarks is smaller than in the quenched case.) In Fig. 2 we show the topological susceptibility, the instanton density and the chiral condensate, obtained from these simulations, as a function of the quark mass. For better visibility, the first two were divided by $m^2\chi_0$, while the chiral condensate was divided by $m\chi_0$. We can conclude that to a very good approximation

$$\chi(m)\approx m^2\chi_0 \quad \text{and} \quad \langle\bar\psi\psi\rangle\approx m\chi_0. \qquad (3)$$

A fit to the instanton density reveals that its behavior in the chiral limit is well approximated by $m^2\chi_0 + c$.

For the interpretation of the results it is instructive to inspect the determinant in more detail. On a configuration with $n_{\rm i}$ instantons and $n_{\rm a}$ anti-instantons the random matrix has $|n_{\rm i}-n_{\rm a}|$ exact zero eigenvalues, and the rest of the eigenvalues come in complex conjugate imaginary pairs. So the determinant can be written as

$$\det\,(D+m)^{N_f} = m^{N_f(n_{\rm i}+n_{\rm a})}\prod_{\text{pairs } i}\left(1+\frac{\lambda_i^2}{m^2}\right)^{N_f}. \qquad (4)$$

If the eigenvalues $\lambda_i$ are much smaller than the quark mass, each (anti)instanton contributes with a factor of $m^{N_f}$ to the determinant, and the distribution of (anti)instanton numbers are still Poissonian, but with a density suppressed by a factor $m^{N_f}$. This explains the result for the susceptibility in Eq. (3). One could think that in the $m\to 0$ limit, the condition $|\lambda_i|\ll m$ fails to be satisfied. However, as our simulations reveal, this is not the case, because as the instanton density drops in the chiral limit, and the typical distance between instantons grows, the exponentially small matrix elements and, as a result, the eigenvalues of $D$ become small, and always remain much smaller than $m$.

To demonstrate that even in the chiral limit, a free instanton gas is responsible for the spectral spike, in Fig. 3 we compare the spectral density of the matrix model with light dynamical quarks with that of the quenched theory with the same topological susceptibility. The quenched and the dynamical spectrum fully agree, except for the very largest eigenvalues, where the dynamical spectrum has an excess of eigenvalues.

These eigenvalues are connected to the component of the instanton density that is constant in the chiral limit

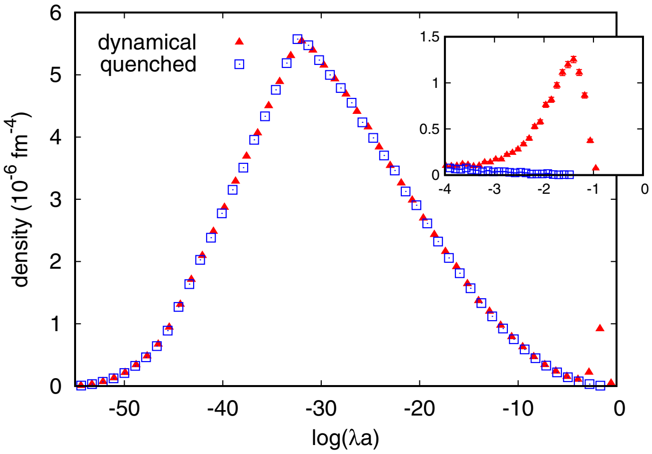


FIG. 3. The logarithmic spectral density of the matrix model with two flavors of quarks of mass $ma = 0.05$ compared to the spectral density of the quenched matrix model with the same topological susceptibility. The inset is an enlargement of the largest eigenvalues where the two spectra significantly differ.

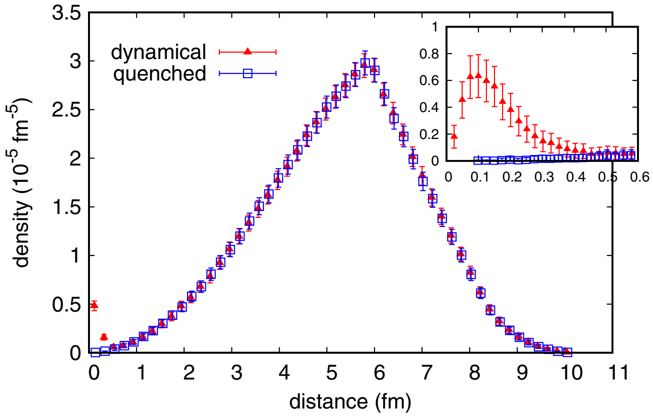


FIG. 4. The distribution of the distance of the closest opposite charged object on the dynamical and quenched matrix model configurations used for Fig. 3.

(see Fig. 2), and appears in addition to the free instanton gas component, causing the spectral spike. Further examination reveals that this component of the instanton gas consists of closely bound instanton–anti-instanton "molecules." To show this, in Fig. 4 we plot the distribution of the distance of the closest opposite charged objects on the dynamical and quenched ensemble. The two distributions exactly match, except for the shortest distance of $d \lesssim 1/T$, indicating that the instantons in excess of the free gas, occur in closely bound molecules.

We emphasize that our random matrix model contains both the gauge action and the fermion action in a self-consistent way. The gauge action, which, as the lattice data show, does not produce any interaction among topological objects, is taken into account by selecting free-instanton configurations with the parameters given by the quenched lattice data. The fermion action is included as an additional weight, proportional to the quark determinant for each configuration. It is only the latter that generates interactions, binding the molecules.

*Singular spectral density.*—Since a singularity in the spectral density, suggested in Ref. [8] can appear only in the thermodynamic limit, a precise determination of its power is only possible by using large spatial volumes. In Fig. 5 we show the density of the log of the eigenvalues for three different system sizes in the quenched matrix model, up to volumes currently not accessible with lattice simulations. If the spectral density is of the form $\rho(\lambda) \propto \lambda^{\alpha}$, then the density of $\tilde{\lambda} = \log(\lambda)$ goes as $\tilde{\rho}(\tilde{\lambda}) \propto e^{(1+\alpha)\tilde{\lambda}}$. Fitting this exponential form to the common envelope of the curves for the different volumes in the figure yields $\alpha = -0.769(5)$. A systematic study of the strength of the singularity as a function of the instanton density is out of the scope of the present work, but performing simulations with a few different densities shows that the singularity becomes stronger with decreasing instanton density. Both the presence of the singularity in the thermodynamic limit, and its strengthening as the instanton gas gets more dilute, can be qualitatively understood by noting that both large volumes and diluteness result in smaller matrix elements.

*The chiral limit.*—Since, as we have already seen, the lowest singular part of the spectrum is due to the free-instanton component of the gas, which persists for arbitrarily small quark masses, the singularity will also remain in the chiral limit. In fact, as $m \to 0$ and the instanton density decreases, the singularity gets stronger. However, the total weight of the singularity (its integral) is equal to the density of instantons in the free component of the gas, and decreases as the topological susceptibility, $\chi(m) \propto m^{N_f}$. Thus the singularity should remain integrable in the chiral limit and might eventually become a Dirac delta at zero, as discussed recently in Ref. [17].

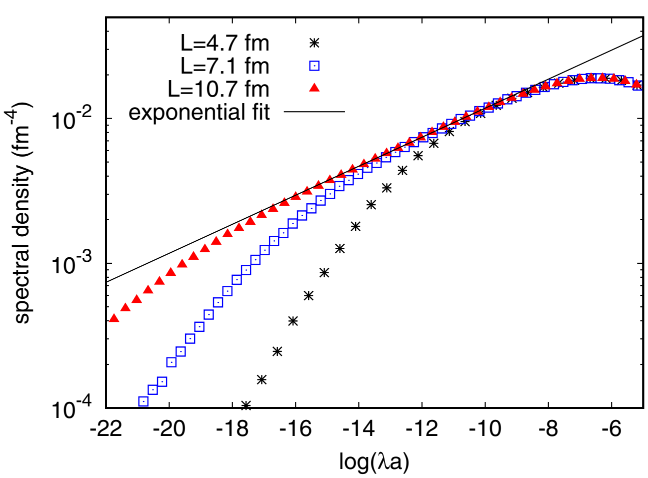


FIG. 5. The spectral density (density of $\log \lambda$) for different system sizes in the quenched matrix model (parameters given in the text).

Let us now discuss the question of chiral symmetry restoration at high temperature in the chiral limit. With a spectral density singular at the origin, it is not immediately clear how to apply the Banks-Casher formula. However, recalling its derivation, we can write the trace of the quark propagator in the chiral limit as

$$\langle\bar{\psi}\psi\rangle \propto \left\langle \sum_i \frac{m}{m^2+\lambda_i^2} \right\rangle = \underbrace{(\text{avg. number of instantons in free gas})}_{m^{N_f}\chi_0 V} \frac{1}{m} = m^{N_f-1}\chi_0 V. \tag{5}$$

Here, we used the fact that eigenvalues, produced by the free instanton component, satisfy $|\lambda| \ll m$, and each of them contributes a term $1/m$ to the sum, whereas the eigenvalues corresponding to the molecules have eventually $|\lambda| \gg m$, and their contribution vanishes as $m/\lambda^2$ in the chiral limit. This explains our simulation result in Eq. (3) for the chiral condensate. It follows that if $N_f \geq 2$, the order parameter of chiral symmetry breaking vanishes, and, as expected, the spontaneously broken $SU(N_f)_A$ is restored, albeit in a nontrivial way. For $N_f = 1$, $\langle\bar{\psi}\psi\rangle$ has a nonvanishing limit, but in this case it is not an order parameter, as there is no corresponding symmetry. In the same fashion, we can also compute the behavior of the $U(1)_A$ breaking susceptibility as

$$\chi_\pi - \chi_\delta \propto \left\langle \sum_i \frac{m^2}{(m^2+\lambda_i^2)^2} \right\rangle \propto m^{N_f}\chi_0 V \frac{1}{m^2} = m^{N_f-2}\chi_0 V, \tag{6}$$

vanishing in the chiral limit only if $N_f \geq 3$, so in the physically interesting $N_f = 2$ case it is nonzero, indicating that the anomalous part of the chiral symmetry is not restored.

*Discussion and conclusions.*—We emphasize that the temperature enters our discussion only through the value of the quenched topological susceptibility, $\chi_0$, and in particular, we only assume that $\chi_0 \neq 0$, which is expected to be true at any finite temperature [18]. This implies that our reasoning applies to any large enough, but finite temperature, and for $N_f = 2$ the $U(1)_A$ part of the symmetry is not restored at any finite temperature. These results are consistent with the quasi-instanton picture of Kanazawa and Yamamoto [19], and also with their chiral expansion of the QCD free energy density [20].

The main assumption underlying our instanton-based random matrix model is that the bulk of the spectrum is not correlated with the ZMZ, and the suppression of the spectral spike is dominated by the eigenvalues in the ZMZ. This is certainly true at high enough temperatures, as in this limit the bulk and the ZMZ get more separated, since the location of the lowest part of the bulk is controlled by the Matsubara frequency $\pi T$, whereas the spike becomes narrower in a more dilute instanton gas at higher temperature.

In the explanation of the simulation results we used the property of the ZMZ that the eigenvalues satisfy $|\lambda| \ll m$, even in the chiral limit. At higher temperatures the eigenvalues in the ZMZ will become even smaller, and this condition holds even more precisely. In contrast, for lower temperatures, toward and beyond $T_c$, the free-instanton picture eventually breaks down, and our model is not expected to be applicable. Current large-scale overlap simulations around $T_c$ might be able to tell how exactly this happens [21].

We saw that for light quarks, the only deviation of the system from a free-instanton gas is the appearance of a component of instanton–anti-instanton molecules of the size $\lesssim 1/T$. The density of these molecules depends on the details of the instanton–anti-instanton interaction, which we do not have access to. However, as the quenched spectrum shows, the gauge interaction is extremely short-ranged, and so is the fermionic interaction, as our simulations reveal (molecules are smaller than $1/T$). Therefore, whatever the details of the gauge interaction, it does not significantly affect the size of the molecules, so the corresponding Dirac eigenvalues will remain at a fixed scale in the chiral limit, and their contribution to Banks-Casher type sums of the form Eqs. (5) and (6) vanishes with some power of the quark mass. It is only the singular part of the spectrum, produced by the free-instanton gas, that controls whether these quantities are nonvanishing in the chiral limit.

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[1] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[2] R. G. Edwards, U. M. Heller, J. E. Kiskis, and R. Narayanan, Phys. Rev. D **61**, 074504 (2000).
[3] A. Alexandru and I. Horváth, Phys. Rev. D **92**, 045038 (2015).
[4] O. Kaczmarek, L. Mazur, and S. Sharma, Phys. Rev. D **104**, 094518 (2021).
[5] H. T. Ding, S. T. Li, S. Mukherjee, A. Tomiya, X. D. Wang, and Y. Zhang, Phys. Rev. Lett. **126**, 082001 (2021).
[6] A. Alexandru, I. Horváth, and N. Bhattacharyya, Phys. Rev. D **109**, 014501 (2024).

[7] S. Aoki, Y. Aoki, G. Cossu, H. Fukaya, S. Hashimoto, T. Kaneko, C. Rohrhofer, and K. Suzuki (JLQCD Collaboration), Phys. Rev. D **103**, 074506 (2021).

[8] A. Alexandru and I. Horváth, Phys. Rev. D **100**, 094507 (2019).

[9] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).

[10] R. A. Vig and T. G. Kovacs, Phys. Rev. D **103**, 114510 (2021).

[11] C. Bonati, M. D'Elia, H. Panagopoulos, and E. Vicari, Phys. Rev. Lett. **110**, 252003 (2013).

[12] S. Borsanyi, R. Kara, R. Vig, Z. Fodor, D. A. Godzieba, P. Parotto, and D. Sexty, Phys. Rev. D **107**, 054514 (2023).

[13] S. Borsanyi and D. Sexty, Phys. Lett. B **815**, 136148 (2021).

[14] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. **A560**, 306 (1993).

[15] T. Schäfer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).

[16] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).

[17] V. Azcoiti, Phys. Rev. D **107**, 114516 (2023).

[18] S. Borsanyi, M. Dierigl, Z. Fodor, S. D. Katz, S. W. Mages, D. Nogradi, J. Redondo, A. Ringwald, and K. K. Szabo, Phys. Lett. B **752**, 175 (2016).

[19] T. Kanazawa and N. Yamamoto, Phys. Rev. D **91**, 105015 (2015).

[20] T. Kanazawa and N. Yamamoto, J. High Energy Phys. 01 (2016) 141.

[21] Z. Fodor, A. Y. Kotov, and K. Szabo, Proc. Sci., LATTICE2023 (**2024**) 179.