Achieving the Fundamental Quantum Limit of Linear Waveform Estimation

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Sensing a classical signal using a linear quantum device is a pervasive application of quantum-enhanced measurement. The fundamental precision limits of linear waveform estimation, however, are not fully understood. In certain cases, there is an unexplained gap between the known waveform-estimation quantum Cramér-Rao bound and the optimal sensitivity from quadrature measurement of the outgoing mode from the device. We resolve this gap by establishing the fundamental precision limit, the waveform-estimation Holevo Cramér-Rao bound, and how to achieve it using a nonstationary measurement. We apply our results to detuned gravitational-wave interferometry to accelerate the search for postmerger remnants from binary neutron-star mergers. If we have an unequal weighting between estimating the signal's power and phase, then we propose how to further improve the signal-to-noise ratio by a factor of $\sqrt{2}$ using this nonstationary measurement.

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In our efforts to probe fundamental physics, we invariably encounter the quantum limit: the irrevocable statistical nature of our reality [1–3]. This fundamental uncertainty of our measurement devices limits the precision at which we can sense classical signals.

We consider the general problem of estimating a real classical signal $s(t)$ for all times t using a linear quantum device as shown in Fig. 1(a) [4]. The device evolves linearly according to a Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_{\text{int}}(t)$ with the interaction $\hat{H}_{\text{int}}(t) = -s(t)\hat{G}$ [5]. Observables that do not commute with the generator $\hat{G}$ (an internal degree of freedom) respond linearly to $s(t)$. We assume that the device is in a stationary state of $\hat{H}_0$ which is time invariant. In the input-output formalism [6], information about the signal leaks out of the device into the environment imprinted on an outgoing mode of a bosonic field. By measuring this mode, we obtain a classical estimate of the classical signal mediated by the quantum device. The outgoing bosonic mode at each position and time is a harmonic oscillator with canonical quadratures $\hat{x}$ and $\hat{p}$ which obey $[\hat{x}, \hat{p}] = i$ (let $\hbar = 1$ henceforth). Let $\hat{x}_\theta \coloneqq \cos(\theta)\hat{x} + \sin(\theta)\hat{p}$ for real θ such that the outgoing mode is [7,8]

$$\hat{x}_\theta(t) = \hat{x}_\theta^{(0)}(t) + \int_{-\infty}^{\infty} \mathrm{d}t'\, \chi_{x_\theta G}(t-t')s(t') \tag{1}$$

where the superscript (0) denotes the free evolution under $\hat{H}_0$ and the susceptibility is $\chi_{x_\theta G}(t-t') \coloneqq i[\hat{x}_\theta^{(0)}(t), \hat{G}^{(0)}(t')]\Theta(t-t')$ with Θ the Heaviside function. In the Fourier domain, Eq. (1) becomes

$$\hat{x}_\theta(\Omega) = \hat{x}_\theta^{(0)}(\Omega) + \chi_{x_\theta G}(\Omega)\tilde{s}(\Omega) \tag{2}$$

where the mode at each frequency Ω is displaced by the signal's complex Fourier component $\tilde{s}(\Omega) \coloneqq \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\Omega t} s(t)$. Since $s(t)$, $\hat{x}_\theta(t)$, and $\chi_{x_\theta G}(t)$ are real, their Fourier components obey $\tilde{s}(-\Omega) = \tilde{s}^\dagger(\Omega)$ etc. such that it suffices to consider only positive frequencies.

Quantum metrology extends the classical theory of estimating parameters from a probability distribution. In particular, the quantum Cramér-Rao bound (QCRB) [9,10], sets a fundamental precision limit: a lower bound on the variance of unbiased estimation of parameters encoded in a quantum state. This limit only depends on the state itself and not on the measurement scheme. In the real single-parameter case, the QCRB can always be saturated by the optimal measurement if the sample size is large [9].

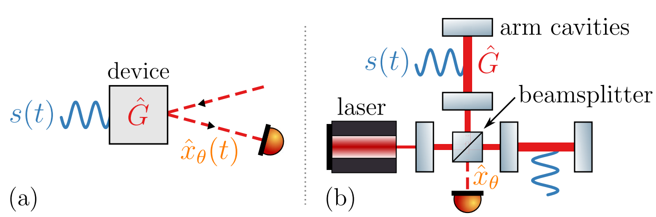

FIG. 1. (a) A linear quantum device coupled to a classical signal. (b) For example, a gravitational-wave interferometer.

Additionally, if the parameter appears as the shift in the mean of a Gaussian state, then the QCRB can be saturated for any sample size. In the multiparameter case, however, reaching the QCRB is not guaranteed. In general, the QCRB can be saturated if and only if the symmetric logarithmic derivatives with respect to the parameters weakly commute [11]. For real parameters s_j encoded as the shift in the mean of a Gaussian state by the unitary transformation $\exp(-i\sum_j s_j \hat{G}_j)$, this condition is equivalent to the generators $\hat{G}_j$ weakly commuting, i.e., $\langle[\hat{G}_j, \hat{G}_k]\rangle = 0,\ \forall\ j, k$.

Here, we want to simultaneously estimate the continuum of independent complex parameters $\tilde{s}(\Omega)$ in Eq. (2), one for each positive frequency Ω.

We assume that the linear device is at the quantum limit such that it is in a pure Gaussian state. The waveform-estimation QCRB S_Q^{wave} [4] sets a lower bound on the error S_{ss} of unbiased estimation of $\tilde{s}(\Omega)$ of

$$\frac{S_{x_\theta x_\theta}(\Omega)}{|\chi_{x_\theta G}(\Omega)|^2} =: S_{ss}(\Omega) \geq S_Q^{\mathrm{wave}}(\Omega) \coloneqq \frac{1}{S_{GG}(\Omega)} \tag{3}$$

where the power spectral density $S_{z_1 z_2}$ for *stationary* random processes $\hat{z}_1$ and $\hat{z}_2$ is defined as

$$2\pi\delta(\Omega - \Omega')S_{z_1 z_2}(\Omega) = \langle\{\hat{z}_1(\Omega), \hat{z}_2^\dagger(\Omega')\}\rangle. \tag{4}$$

The QCRB depends only on the reciprocal of the fluctuations of the generator S_{GG}.

This can be explained by the generalized uncertainty principle for waveform estimation [12]

$$S_{x_\theta x_\theta}(\Omega)S_{GG}(\Omega) \geq |\chi_{x_\theta G}(\Omega)|^2 + |S_{x_\theta G}(\Omega)|^2 + c(\Omega) \tag{5}$$

where, in the quantum limit, this is an equality and $c(\Omega)$ is a term that vanishes [5].

From Eq. (5), Ref. [5] showed that measuring the stationary complex quadrature $\hat{x}_\theta(\Omega)$ saturates the QCRB in Eq. (3) if and only if it is uncorrelated with the generator, i.e., $S_{x_\theta G} = 0$. Furthermore, Ref. [5] showed that when $S_{x_\theta G} \neq 0,\ \forall\ \theta$ the optimal error S_{ss} is still within a factor of 2 of the QCRB.

It is not necessary *a priori*, however, that we measure a stationary complex quadrature $\hat{x}_\theta(\Omega)$. Equation (3) only applies to such stationary measurements. This leaves several important questions unanswered. What is the QCRB in general and when can it be saturated? If it cannot be saturated, then what is the optimal precision? And, what measurement attains this limit? We will answer these questions and demonstrate our results using gravitational-wave interferometry.

Cosine and sine phases.—At the positive frequency Ω, let $\tilde{s}(\Omega) = \pi T(A + iB)$ where A and B are independent real degrees of freedom and T is the finite integration time. In the time domain, this component of the signal is $A\cos(\Omega t) + B\sin(\Omega t)$ at a given time t where A and B are the cosine and sine phases of the signal $s(t)$ at frequency Ω, respectively.

Our goal of measuring the signal s, therefore, is equivalent to simultaneously estimating A and B at each Ω.

Our weighted figure of merit for the precision at Ω is $\Sigma = 2w\mathrm{Var}[\hat{A}] + 2(1-w)\mathrm{Var}[\hat{B}]$ where $w \in (0,1)$ and $\hat{A}$ and $\hat{B}$ are unbiased estimates of A and B, respectively. (Without loss of generality, we can assume that our weight matrix is diagonal.) The weights may be unequal $w \neq 0.5$ for several reasons. For example, if we want to estimate the signal's power $|\tilde{s}|^2 \propto A^2 + B^2$ more than its phase, then the weights would be unequal because the derivatives of $|\tilde{s}|^2$ depend on A and B. We assume a uniform prior on A and B and distinguish that, while unequal weights indicate how much more is wanted to be known *a posteriori* about A than B, a nonuniform prior would indicate how much more is known *a priori* about A than B [13–15].

Similarly to the signal, we can split the complex quadratures of the outgoing light $\hat{x}_\theta(\Omega)$ into their real and imaginary parts in the frequency domain. Or, equivalently, into their cosine and sine phases in the time domain, e.g., see Refs. [16,17]. These parts are Hermitian but *nonstationary*. In this manner, Eq. (2) becomes

$$\vec{\hat{q}} \coloneqq \frac{1}{\sqrt{\pi T}}\begin{bmatrix}\mathrm{Re}[\hat{x}(\Omega)]\\ \mathrm{Re}[\hat{p}(\Omega)]\\ \mathrm{Im}[\hat{x}(\Omega)]\\ \mathrm{Im}[\hat{p}(\Omega)]\end{bmatrix} = \vec{\hat{q}}^{(0)} + A\vec{d}_A + B\vec{d}_B. \tag{6}$$

Let $\vec{\chi} = [\chi_{xG}(\Omega), \chi_{pG}(\Omega)]^{\mathrm{T}}$, then the real signal displacements are

$$\vec{d}_A \coloneqq \sqrt{\pi T}\begin{bmatrix}\mathrm{Re}[\vec{\chi}]\\ \mathrm{Im}[\vec{\chi}]\end{bmatrix}, \qquad \vec{d}_B \coloneqq \sqrt{\pi T}\begin{bmatrix}-\mathrm{Im}[\vec{\chi}]\\ \mathrm{Re}[\vec{\chi}]\end{bmatrix}. \tag{7}$$

These are orthogonal and have the same Euclidean norm

$$l = \sqrt{\pi T\left(|\chi_{xG}(\Omega)|^2 + |\chi_{pG}(\Omega)|^2\right)}. \tag{8}$$

Since $[\hat{x}(\Omega), \hat{p}(\Omega')] = i2\pi\delta(\Omega - \Omega')$, by using $\mathrm{Re}[z] = \frac{1}{2}(z + z^*)$ and $\mathrm{Im}[z] = (1/2i)(z - z^*)$ we have that $[\vec{\hat{q}}_1, \vec{\hat{q}}_2] = [\vec{\hat{q}}_3, \vec{\hat{q}}_4] = i$ with all other commutators zero ($\vec{\hat{q}}_j$ is the jth element of $\vec{\hat{q}}$). The system at each frequency, therefore, comprises two real displaced harmonic oscillators.

We assume that the noise is stationary such that the complex quadratures $\vec{\hat{x}} = [\hat{x}(\Omega), \hat{p}(\Omega)]^{\mathrm{T}}$ have the 2-by-2 covariance matrix $\frac{1}{2}\langle\{\vec{\hat{x}}_j^{(0)}, \vec{\hat{x}}_k^{(0)}\}\rangle = (V_2)_{jk}$ and the parts $\vec{\hat{q}}$ have the 4-by-4 covariance matrix $\frac{1}{2}\langle\{\vec{\hat{q}}_j^{(0)}, \vec{\hat{q}}_k^{(0)}\}\rangle = (V_2 \oplus V_2)_{jk}$. (Without loss of generality, we assume that $\langle\vec{\hat{x}}^{(0)}\rangle = 0$.) Since the device is linear, distinct frequencies are uncorrelated. For the moment, we assume that the pure state at each frequency is vacuum, i.e., $V_2 = \mathrm{diag}(\frac{1}{2}, \frac{1}{2})$, and will generalize later.

To measure A and B from the output light in Eq. (6), the naïve optimal unbiased estimates are $\hat{A}_{\text{naïve}} = l^{-2}\vec{d}_A \cdot \vec{q}$ and $\hat{B}_{\text{naïve}} = l^{-2}\vec{d}_B \cdot \vec{q}$. The variance of each estimate is $\frac{1}{2}l^{-2}$ such that the naïve figure of merit is $\Sigma_{\text{naïve}} = l^{-2}$. [Note that $\Sigma_{\text{naïve}} \propto T^{-1}$ by Eq. (8) such that integrating for longer times reduces the error as expected].

These measurements, however, may not commute since $[\hat{A}_{\text{naïve}}, \hat{B}_{\text{naïve}}] = i\mu l^{-2}$ where

$$\mu = 2\pi T l^{-2}(\mathrm{Re}[\chi_{pG}]\mathrm{Im}[\chi_{xG}] - \mathrm{Re}[\chi_{xG}]\mathrm{Im}[\chi_{pG}]) \quad (9)$$

such that $0 \le |\mu| \le 1$. (Without loss of generality, we assume that $\mu \ge 0$.) This means that A and B cannot be simultaneously estimated to attain the naïve figure of merit if $\mu \ne 0$. The displacements in Eq. (7) are generated by their conjugate quadratures $\hat{G}_A$ and $\hat{G}_B$ which obey the same commutation relation such that $\mu = 0$ is equivalent to the weak commutativity condition $\langle[\hat{G}_A, \hat{G}_B]\rangle = 0$. The QCRB for simultaneous estimation of A and B, therefore, can be saturated if and only if $\mu = 0$ which is equivalent to $\exists\, \theta \in \mathbb{R}$ such that $\chi_{x_\theta G} = 0$. In fact, the QCRB is precisely the na-ve figure of merit above such that $\Sigma \ge \Sigma_Q = \Sigma_{\text{naïve}}$. This can be shown from the result that the QCRB with respect to s_j given the unitary transformation $\exp(-is_j\hat{G}_j)$ is $(4\mathrm{Var}[G_j])^{-1}$ [9].

Fundamental precision limit.—If $\mu \ne 0$ such that the QCRB cannot be saturated, then the optimal attainable precision is instead the Holevo Cramér-Rao bound (HCRB) Σ_H which accounts for the commutator of the estimates such that $\Sigma \ge \Sigma_H > \Sigma_Q$ [11,15,18,19]. Since the real parameters A and B appear as the shift in the mean of a pure Gaussian state, the HCRB is saturated by the optimal commuting linear combinations of $\vec{q}$ [11]. (This is equivalent to finding the optimal quantum mechanics–free subspace [20].) We calculate the HCRB using the method from Ref. [18] in the Supplemental Material [21].

We show that the ratio of the HCRB Σ_H to the QCRB $\Sigma_Q = l^{-2}$ reduces to single-parameter optimization

$$\frac{\Sigma_H}{\Sigma_Q} = \min_{\phi \in (0,\pi]} \left(\frac{w}{\cos(\phi)^2} + \frac{1-w}{\cos[\phi + \arcsin(\mu)]^2} \right) \quad (10)$$

and we find analytic solutions in certain limits

$$\frac{\Sigma_H}{\Sigma_Q} \overset{\mu=1}{\to} 1 + 2\sqrt{w(1-w)}, \qquad \frac{\Sigma_H}{\Sigma_Q} \overset{w=\frac{1}{2}}{\to} \frac{2}{1+\sqrt{1-\mu^2}}. \quad (11)$$

Figure 2(a) shows that the ratio of the HCRB to the QCRB is at most two which agrees with Ref. [5]. The HCRB increases monotonically with μ and decreases as the weights become less equal as shown in Fig. 2(b). The HCRB reduces to the QCRB for single-parameter estimation at $w = 0$, 1.

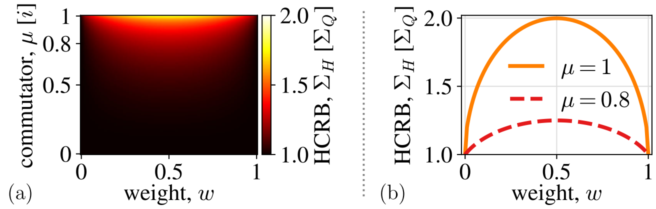

FIG. 2. (a) HCRB versus the commutator and weight. (b) HCRB versus the weight for different commutators.

These results generalize to squeezed states. Let $\hat{S}$ be the conjugate squeezing operator such that $V_2 \mapsto \mathrm{diag}(\frac{1}{2}, \frac{1}{2})$. This unitary transformation does not affect μ or the bounds but does map the signal displacements as $\vec{d} \mapsto (S \oplus S)\vec{d}$ such that

$$l \mapsto l' \coloneqq \sqrt{\pi T(\|S\mathrm{Re}[\vec{\chi}]\|^2 + \|S\mathrm{Im}[\vec{\chi}]\|^2)}. \quad (12)$$

The general stationary pure Gaussian state case, therefore, is equivalent to the vacuum case with the same μ and w but with $\Sigma_Q = (l')^{-2}$. We emphasize that we only apply $\hat{S}$ mathematically to derive the bounds; it is not required experimentally.

Optimal measurement scheme.—There exists a unique symplectic transformation of the two harmonic oscillators $\vec{q} \mapsto \vec{X} = (\hat{X}_1, \hat{P}_1, \hat{X}_2, \hat{P}_2)^{\mathrm{T}}$ that maps the normalized displacements as $l^{-1}\hat{d}_A \mapsto \hat{X}_1$ and $l^{-1}\hat{d}_B \mapsto \mu\hat{P}_1 + \sqrt{1-\mu^2}\hat{X}_2$ such that their commutator remains $i\mu$. In this basis, the optimal commuting unbiased estimates are [21]

$$\hat{A} = [l\cos(\bar{\phi})]^{-1}[\cos(\bar{\phi})\hat{X}_1 - \sin(\bar{\phi})\hat{P}_2]$$
$$\hat{B} = \{l\cos[\bar{\phi} + \arcsin(\mu)]\}^{-1}[\cos(\bar{\phi})\hat{X}_2 - \sin(\bar{\phi})\hat{P}_1] \quad (13)$$

where $\bar{\phi}$ is the optimal angle in Eq. (10).

These estimates are two nonstationary quadratures: arbitrary real linear combinations of $\vec{q}$. Compare this to the stationary complex quadrature $\hat{x}_\theta(\Omega)$ with the real part $\cos(\theta)\vec{q}_1 + \sin(\theta)\vec{q}_2$ and imaginary part $\cos(\theta)\vec{q}_3 + \sin(\theta)\vec{q}_4$. For squeezed states, similarly, the optimal measurement consists of two nonstationary quadratures (mathematically, first apply $\hat{S}$ and then the symplectic transformation).

We propose how to experimentally realize these nonstationary measurements at a given Ω. We expand $\vec{q}$ into the time domain using $\mathrm{Re}[\hat{x}_\theta(\Omega)] = \int_{-\infty}^{\infty} \mathrm{d}t \cos(\Omega t)\hat{x}_\theta(t)$ and $\mathrm{Im}[\hat{x}_\theta(\Omega)] = \int_{-\infty}^{\infty} \mathrm{d}t \sin(\Omega t)\hat{x}_\theta(t)$. Since the measurements are linear combinations of $\vec{q}$, therefore, they are $\hat{A} = \int_{-\infty}^{\infty} \mathrm{d}t c_A(t)\hat{x}_{\theta_A(t)}(t)$ and $\hat{B} = \int_{-\infty}^{\infty} \mathrm{d}t c_B(t)\hat{x}_{\theta_B(t)}(t)$ for some

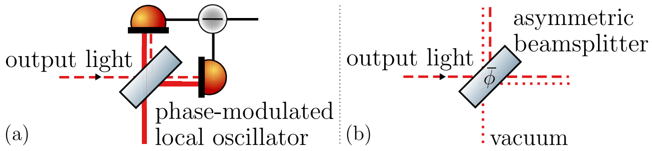

FIG. 3. (a) Phase-modulated balanced homodyne readout. (b) Asymmetric beam splitter with power reflectivity $\cos(\bar{\phi})^2$.

real amplitudes $c_A(t)$ and $c_B(t)$ and phases $\theta_A(t)$ and $\theta_B(t)$ [21]. For example, if $w = 1$ such that we only want to estimate A, then, as shown in Fig. 3(a), we use homodyne readout with a phase-modulated local oscillator with phase $\theta_A(t)$ to obtain the time series $\hat{x}_{\theta_A(t)}(t)$. By integrating this time series multiplied by $c_A(t)$ in postprocessing, we can achieve $\hat{A}$.

Suppose that instead we want to measure both $\hat{A}$ and $\hat{B}$. Although $\hat{A}$ and $\hat{B}$ commute, their integrands $c_A(t)\hat{x}_{\theta_A(t)}(t)$ and $c_B(t)\hat{x}_{\theta_B(t)}(t)$ above may not commute at a given time. This prevents directly performing simultaneous modulated homodyne measurements. If $\mu = 1$ such that the normalized displacements are $\hat{X}_1$ and $\hat{P}_1$, however, then this can be overcome by using an asymmetric beam splitter with reflectivity $\cos(\bar{\phi})^2$ to mix in an ancillary mode (i.e., uncorrelated vacuum) as shown in Fig. 3(b). Then, measuring $\vec{d}_A$ and $\vec{d}_B$ on the two output modes from the beam splitter using modulated homodyne readouts commutes at each time and saturates the HCRB for any w [21]. The added noise from the ancilla is responsible for the gap from the QCRB.

For the general case of any μ and w, we propose a joint homodyne-heterodyne readout scheme. To obtain the individual estimate of $\hat{A}$ above, we integrated the time series $c_A(t)\hat{x}_{\theta_A(t)}(t)$, but information is available at other frequencies too. In particular, the 2Ω Fourier component beats with the time series which oscillates at Ω to produce a linear combination of the quadratures at Ω and 3Ω. This can realize a heterodyne measurement of $\hat{B}$ at Ω [21,22]. The added heterodyne noise at 3Ω can be suppressed by squeezing the output mode using two cascaded, detuned, and narrow band filter cavities—one for each of the upper and lower sidebands at 3Ω—without affecting the estimates or the fundamental limits at Ω. The HCRB, therefore, can be saturated in the narrow band around Ω using a homodyne measurement of $\hat{A}$ and a simultaneous heterodyne measurement of $\hat{B}$.

Gravitational-wave interferometry.—We demonstrate our results for a gravitational-wave interferometer like the Laser Interferometric Gravitational-wave Observatory (LIGO) [23,24] operated in a hypothetical detuned configuration. For simplicity, we model LIGO as a power-recycled Fabry-Pérot Michelson interferometer as shown in Fig. 1(b) with vacuum input into the "dark port" of the beam splitter [25,26]. In our detuned configuration, the 4-km arm cavities with 750 kW of circulating power are detuned away from the input carrier laser frequency of 282 THz by $\Delta = 2\pi \times 3$ kHz [27–33]. We are interested in detecting 1–4 kHz gravitational-wave signals, e.g., from the postmerger remnant of binary neutron-star mergers, to test our theories of extreme matter [34–42]. (Since we focus on the kilohertz response, we ignore quantum radiation pressure noise [3].) Detuning the interferometer makes it resonant at Δ which improves the peak sensitivity without increasing the circulating power [43–45]. We emphasize that operating LIGO in a detuned configuration presents many technical challenges [33] and here we only want to establish the fundamental limit of achievable sensitivity at a given frequency to better evaluate this configuration.

The differential optical mode of the interferometer can be approximated as a single mode in a detuned cavity linearly coupled to the gravitational-wave strain $s(t)$ by $\hat{H}_{\text{int}} = gs(t)\hat{x}_{\text{cav}}$. Here, g is the effective coupling rate (mediated by free masses in the transverse-traceless gauge) and $\hat{x}_{\text{cav}}$ is the amplitude quadrature of the intracavity mode such that $\hat{G} = -g\hat{x}_{\text{cav}}$ [46]. The resulting susceptibility is [21]

$$\vec{\chi} = \frac{\sqrt{2\gamma}g}{\Delta^2 + (\gamma - i\Omega)^2}\begin{bmatrix} \Delta \\ -\gamma + i\Omega \end{bmatrix} \tag{14}$$

where $\gamma = 2\pi \times 42$ Hz is the half-width at half-maximum readout rate of the arm cavities. By Eq. (9), $\mu = [2\Delta\Omega/(\gamma^2 + \Delta^2 + \Omega^2)]$ such that the QCRB cannot be saturated for $\Delta \neq 0$ which agrees with Ref. [5].

In Fig. 4, we compare the HCRB versus frequency to the sensitivity using the optimal stationary quadrature (also known as "variational readout") and nonstationary quadrature measurements. For equal weights, the stationary measurement saturates the HCRB such that the gap to the QCRB is insurmountable. For unequal weights, however, our nonstationary measurement is required to saturate the HCRB.

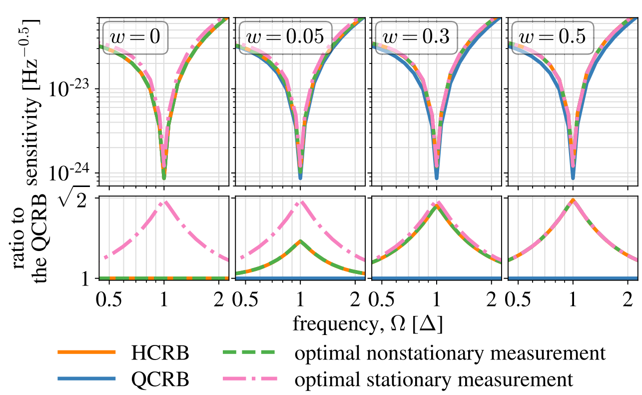

FIG. 4. Strain sensitivity for the detuned LIGO-like interferometer versus frequency for different weights in (top row) effective amplitude spectral density units and (bottom row) ratio to the QCRB.

This unequal weight regime is relevant because, e.g., astrophysically we care more about knowing some parameters of the neutron-star equation of state than others. This can be reduced to having an unequal weighting between the signal's power and phase and, therefore, between A and B. For example, we particularly want to estimate the primary peak of the kilohertz power spectrum to inform our understanding of the equation of state [47]. In the future, with more precise numerical models of the postmerger signal, we may be able to confidently determine the phase of the postmerger signal from a strong enough detection of the inspiral phase. We then only need to estimate the postmerger signal's power which is equivalent to having weights $w = 0$ or 1. In this limiting single-parameter case, we have shown that our nonstationary measurement scheme improves the signal-to-noise ratio by up to a factor of $\sqrt{2}$ at the detuning frequency, an improvement which cannot be surpassed using a different measurement scheme. This corresponds to up to a factor of 2.83 improvement in the volume of the Universe searched for kilohertz signals at the peak frequency, in addition to the gain provided from detuning the interferometer. This could be a significant boost to LIGO's search for kilohertz gravitational waves should the challenges with detuned interferometry be overcome. More realistically, we may instead have partial prior knowledge of the postmerger signal's phase and perform weighted simultaneous estimation of the signal's power and phase. We hypothesize that the sensitivity can still be similarly improved in this regime and defer a detailed study of this application to future work.

Losses limit the possible quantum enhancement of LIGO (where Fig. 4 shows the lossless sensitivity). If we assume optical losses of 100 ppm ($\gamma_l = 2\pi \times 0.3$ Hz) in the arm cavities and $\eta = 0.1$ in the output, then $l \mapsto \sqrt{1-\eta}l$ and $\mu \mapsto \{2\Delta\Omega/[(\gamma+\gamma_l)^2 + \Delta^2 + \Omega^2]\}$ [21]. Since $\gamma_l \ll \gamma \ll \Delta$, μ is unchanged. This implies that the gap from the HCRB to the relevant QCRB, $\Sigma_Q \approx (1-\eta)^{-1} l^{-2}$, is also unchanged and our nonstationary measurement can still achieve up to a factor of $\sqrt{2}$ improvement with losses.

Conclusions.—We have shown how to achieve the fundamental precision limit for the estimation of a classical signal using a linear quantum device. Previous work on linear waveform estimation found an unexplained gap of up to a factor of $\sqrt{2}$ in the signal-to-noise ratio between the optimal stationary quadrature measurement and the QCRB. We showed that this gap stems from the noncommutativity of the na-ve estimates of the cosine and sine phases of the signal at each frequency. This allowed us to establish the fundamental limit of attainable precision and propose how to experimentally realize the optimal nonstationary measurement scheme. We applied these results to the search for postmerger gravitational-wave signals from binary neutron-star mergers using a detuned LIGO-like interferometer. We showed that this nonstationary measurement scheme could significantly increase the volume of the Universe probed for such signals at a given frequency in the unequal weight regime.

Future work could determine the broadband optimal measurement scheme and apply our results to a dual-recycled LIGO-like interferometer with injected squeezed states [48–50] and extend them to other systems, e.g., $\mathcal{PT}$-symmetric interferometers [51–56], axion detectors [57–60], and displacement noise-free interferometers [17,61,62].

Our code is available online [63] and was written using *Mathematica* [64] and PYTHON [65–69].

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