

## From Maximum of Inter-Visit Times to Starving Random Walks

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Very recently, a fundamental observable has been introduced and analyzed to quantify the exploration of random walks: the time  $\tau_k$  required for a random walk to find a site that it never visited previously, when the walk has already visited  $k$  distinct sites. Here, we tackle the natural issue of the statistics of  $M_n$ , the longest duration out of  $\tau_0, \dots, \tau_{n-1}$ . This problem belongs to the active field of extreme value statistics, with the difficulty that the random variables  $\tau_k$  are both correlated and nonidentically distributed. Beyond this fundamental aspect, we show that the asymptotic determination of the statistics of  $M_n$  finds explicit applications in foraging theory and allows us to solve the open  $d$ -dimensional starving random walk problem, in which each site of a lattice initially contains one food unit, consumed upon visit by the random walker, which can travel  $\mathcal{S}$  steps without food before starving. Processes of diverse nature, including regular diffusion, anomalous diffusion, and diffusion in disordered media and fractals, share common properties within the same universality classes.

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The territory covered by random walks (RWs) constitutes a fundamental property with significant implications in quantifying the efficiency of diverse stochastic exploration processes, ranging from animal foraging behaviors [1] to the trapping of diffusing molecules [2]. Usually, this explored territory is quantified by the number  $N(t)$  of distinct sites visited at time  $t$  [3]. Its average, variance and, in some cases, full distribution have been determined analytically [4–7]. However, since  $N(t)$  is a cumulative quantity, it does not describe the detailed dynamics of the exploration process. In particular, it does not differentiate between trajectories in which new sites are discovered at an almost regular rate and those in which they are essentially found toward the end of exploration, corresponding to long periods of time with no new sites discovered.

Very recently, as a first step to account for this disparity between random trajectories, another fundamental quantity was introduced [8,9]: the time  $\tau_k$  required for the RW to find a site that it never visited previously when  $k$  distinct sites have already been visited; see Fig. 1. These random variables are indeed very useful because they encompass the full dynamics of the visitation statistics [8–10]. The knowledge of the statistics of a given  $\tau_k$  variable is, however, insufficient to characterize the long periods of time with no new sites discovered, which can deeply impact the exploration process [see Fig. 1(a)].

In this Letter, we provide a quantitative characterization of these long time periods by determining the asymptotic statistics of the maximum  $M_n$  of the  $\tau_k$ ,  $M_n = \max(\tau_0, \dots, \tau_{n-1})$  [11]. This fundamental question belongs to the domain of extreme value statistics, which has attracted considerable attention in recent years due to its connection with the statistics of extreme events [12–15].

Applications are found in fields as diverse as disordered systems [16,17], random matrices [18,19], and search algorithms [20–22]. Here, the technical difficulty is that the territory visited by the RW is incessantly updated. As a result, the random variables  $\tau_k$  are both correlated and nonidentically distributed. Importantly, these characteristics are not given *a priori* but are generated by the RW itself.

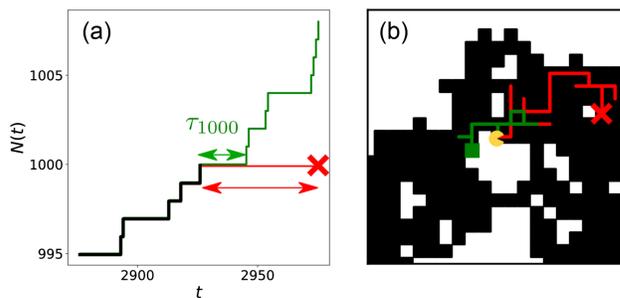


FIG. 1. Inter-visit times. (a) The inter-visit times  $\{\tau_k\}$  (horizontal steps), defined as the time intervals between increments of the number  $N(t)$  of distinct sites visited, control the exploration process. After visiting  $N(t) = 1000$  sites the discovery of the 1001th new site can be either “short” (green arrow) or “long” (red arrow). Here, we determine the statistics of the maximum  $M_n$  of  $\{\tau_k\}_{k < n}$ . The starving RW model. (b) Each site of a lattice initially contains one food unit, consumed upon visit by the RW, which can travel  $\mathcal{S}$  steps without food before starving. Two sample trajectories (green and red) associated with the evolution scenarios of  $N(t)$  in (a) are displayed. A forager (yellow) has eaten  $N(t) = 1000$  food units (the domain depleted of food is black, and  $\mathcal{S} = 50$ ). Following the green trajectory, it finds rapidly a new food unit (green square). On the red trajectory, it fails to find food before starving (at red cross).

Beyond this theoretical aspect, the determination of the statistics of  $M_n$  finds explicit applications in foraging theory, particularly in the context of the starving RW model [23–27], which describes depletion-controlled starvation of a RW forager. In the original version of this model [24], the RW survives only if the time elapsed until a new food-containing site is visited is less than an intrinsic metabolic time  $\mathcal{S}$  [see Fig. 1(b)]. Such a situation is commonly encountered at various scales [28], ranging from microscopic organisms, such as bacteria [29], to larger creatures like insects, foraging mammals [30], and robots [31]. So far, the analytical results on starving RWs have essentially been limited to one dimension (infinite dimension, corresponding to a mean-field solution), where it was found that (i) the mean number  $\langle N_S \rangle$  of units of food collected at starvation is proportional to  $\mathcal{S}^{1/2}$  (exponential in  $\mathcal{S}$ ), (ii) the mean lifetime  $\langle T_S \rangle$  is proportional to  $\mathcal{S}$  (exponential in  $\mathcal{S}$ ), and (iii) the distributions of these two observables admit a single-parameter scaling. Since the original model [24], several lines of extension have been considered, including resource renewal [32], long-range food detection [33,34], penalties on long moves [35], and switching on several modes of motion [36]. The only studies in higher dimensions concern the mean-field approach mentioned above [24,25], numerical simulations [33,34], and a scaling result on the mean lifetime  $\langle T_S \rangle \propto \mathcal{S}^2$  in the particular case of the 2D situation [9].

Here, we provide analytical results for the  $d$ -dimensional starving nearest-neighbor RW model, which constitutes an open problem. Our starting point is the observation that the knowledge of the statistics of the maximum  $M_n$  is a key step to describe the dynamics of a starving RW. Indeed, a starving RW is still alive after  $n$  units of food have been collected if (and only if)  $M_n < \mathcal{S}$ . In this Letter, we derive analytically the long-time asymptotic distribution of, first,  $M_n$ , and, second, key observables pertaining to starving RWs: the lifetime  $T_S$ , the number  $N_S$  of units of food collected when starvation occurs, and the position  $\bar{R}_S$  of the walker when it starves. A wide range of processes, including  $d$ -dimensional regular diffusion, anomalous diffusion, and diffusion in disordered media and fractals, fall into the same universality classes.

*Maximum of the inter-visit times.*—We consider the general situation of a discrete-time symmetric Markovian RW on a lattice of fractal dimension  $d_f$  ( $d_f$  being equal to  $d$  in the particular case of a  $d$ -dimensional Euclidean lattice). The RW dynamics is characterized by the walk dimension  $d_w$  given by the typical displacement  $r(t) \propto t^{1/d_w}$  after  $t$  steps. Recurrent (shown [4,37] to be obtained for  $\mu \equiv d_f/d_w < 1$ ) and marginal ( $\mu = 1$ ) RWs visit any site with probability one, whereas transient ( $\mu > 1$ ) RWs have a finite probability not to visit it. We define by  $\tau_k$  the time elapsed between the visits to the  $k$ th and the  $(k+1)$ st distinct sites [9,24] and by  $M_n$  the maximum of the inter-visit times  $\tau_k$ ,  $M_n = \max(\tau_0, \dots, \tau_{n-1})$ . We report here that

the rescaled random variable,

$$x_n = \begin{cases} M_n/\langle M_n \rangle & \mu \leq 1 \\ (M_n - \langle M_n \rangle)/\sqrt{\text{Var}(M_n)} & \mu > 1, \end{cases} \quad (1)$$

where the scalings with  $n$  of the averages and standard deviations of  $M_n$  are given by

$$\langle M_n \rangle \propto \begin{cases} n^{1/\mu} & \mu < 1 \\ \sqrt{n} & \mu = 1 \\ (\ln n)^{1/\mu+1} & \mu > 1, \end{cases} \quad (2)$$

and

$$\sqrt{\text{Var}(M_n)} \propto \begin{cases} n^{1/\mu} & \mu < 1 \\ \sqrt{n} & \mu = 1 \\ (\ln n)^{1/\mu} & \mu > 1 \end{cases} \quad (3)$$

is asymptotically ( $n \rightarrow \infty$ ) distributed according to a cumulative distribution function (CDF)  $\Xi_\mu(x)$ . Even if process dependent,  $\Xi_\mu(x)$  displays the following universal asymptotic behaviors, depending on the nature of exploration, recurrent, marginal, or transient.

For recurrent RWs ( $\mu < 1$ ),

$$-\ln \Xi_\mu(x) \propto \begin{cases} x^{-\mu} & \text{for } x \ll 1 \\ E_1(Ax) \propto e^{-Ax}/x & \text{for } x \gg 1, \end{cases} \quad (4)$$

where  $E_1$  is the exponential integral function and  $A$  is a process-dependent constant.

For marginal RWs ( $\mu = 1$ ), the distribution obeys (up to log corrections)

$$-\ln \Xi_1(x) \propto \begin{cases} x^{-2} & \text{for } x \ll 1 \\ e^{-Bx^{1/2}} & \text{for } x \gg 1, \end{cases} \quad (5)$$

where  $B$  is a process-dependent constant.

For transient RWs,  $\Xi_\mu(x)$  does not depend on  $\mu$  and is given by the celebrated Gumbel distribution [14],

$$-\ln \Xi_\infty(x) = \exp[-\pi x/\sqrt{6} - \gamma_E], \quad (6)$$

where  $\gamma_E$  is the Euler constant.

Striking qualitative differences between recurrent and transient RWs emerge: while for recurrent and marginal RWs the standard deviation of  $M_n$  is always comparable to its mean value, this is not the case for transient RWs for which the standard deviation of  $M_n$  is negligible in comparison to its mean. As a consequence,  $M_n$  becomes asymptotically deterministic in the latter case. Besides the average and variance, the asymptotic distribution of the

rescaled maximum in the recurrent and marginal cases is very different from the usual Gumbel distribution for random variables with stretched exponential tails distribution, as can be seen by comparing Eqs. (4) and (5) to Eq. (6). As shown below, this is the signature of strong aging effects for  $\mu \leq 1$ .

We now sketch the main steps involved in obtaining these results [see Supplemental Material (SM) for detailed derivations [38]]. We emphasize that the treatment of the recurrent, marginal, and transient cases has to be differentiated due to the disparities in the visitation process, as described in [9]. The calculations are based on the hypothesis that the events  $\{\tau_k < \mathcal{S}\}$  (with  $k \leq n-1$ ) are asymptotically ( $n \rightarrow \infty$ ) effectively independent. This key hypothesis is extensively checked numerically in SM, and self-consistently analytically checked below. The effective independence of  $\tau_k$  allows us to express the CDF of  $M_n$  via the CDF of  $\tau_k$ :

$$\begin{aligned} \mathbb{P}(M_n \leq T) &\approx \prod_{k=0}^{n-1} \mathbb{P}(\tau_k \leq T) \\ &\approx \exp \left[ - \sum_{k=0}^{n-1} \int_T^\infty F_k(\tau) d\tau \right], \end{aligned} \quad (7)$$

where  $F_k(\tau)$  is the probability distribution function of  $\tau_k$ , whose asymptotics were determined recently in [9] (see also SM for refined characterization in the marginal case).

For recurrent walks ( $\mu < 1$ ), the probability distribution function of  $\tau_k$  presents a scaling form,  $F_k(\tau) = k^{-1-1/\mu} \psi(\tau/k^{1/\mu})$ , where  $\psi(u)$  is algebraic at small  $u$  and exponential at large  $u$  [9]. This implies that the CDF of  $M_n$  also has a scaling form, since

$$\begin{aligned} -\ln \mathbb{P}(M_n \leq T) &\approx \sum_{k=0}^{n-1} \int_T^\infty k^{-1-1/\mu} \psi(\tau/k^{1/\mu}) d\tau \\ &\approx \int_0^{n/T^\mu} \frac{dv}{v} \int_{v^{-1/\mu}}^\infty \psi(u) du \\ &= -\ln \Xi_\mu(T/n^{1/\mu}). \end{aligned} \quad (8)$$

This leads to the asymptotics of Eq. (4).

For marginal walks ( $\mu = 1$ ), the CDF of  $M_n$  is shown to be dominated by the behavior of  $F_k(\tau)$  in the regime  $\sqrt{k} \sim \tau$  corresponding to the typical time needed to exit the largest fully visited spherical domain, determined in [9,49]. Extending this approach to the determination of the scaling of the exit time of the next largest fully visited domains, we show in SM that  $F_k(\tau) = k^{-3/2} \psi(\tau/\sqrt{k})$  with  $\psi(u) \propto u^{-3}$  at small  $u$  and  $-\ln \psi(u) \propto \sqrt{u}$  at large  $u$  (up to log corrections). In turn, this scaling form allows one to adapt the steps of Eq. (8) to the marginal case. We obtain that the CDF of  $M_n/\sqrt{n}$  has asymptotically a single scaling

parameter [ $\langle M_n \rangle \sim \sqrt{\text{Var}(M_n)} \sim \sqrt{n}$  up to log prefactors] and converges to the cumulative distribution  $\Xi_1$  of Eq. (5).

For transient walks ( $\mu > 1$ ), for times  $\tau \ll k^{1+1/\mu}$ , the probability distribution function of  $\tau_k$  is independent of  $k$  and stretched exponentially distributed of exponent  $[\mu/(1+\mu)]$  [9]. By showing that the CDF of  $M_n$  is controlled by this early time regime of  $F_k(\tau)$ , we obtain that the limit distribution ( $n \rightarrow \infty$ ) is the Gumbel law displayed in Eq. (6).

Finally, we provide a self-consistent analytical check of the effective independence of the  $\{\tau_k\}$  used in Eq. (7). This constitutes an extension of the argument of Ref. [50], originally given for Gaussian correlated but identically distributed random variables. The idea is that one can neglect the effect of the correlations on the statistics of the maximum if these correlations are typically much smaller than the maximum's fluctuations induced by the random variables without these correlations.

To make this criteria quantitative, we consider the typical correlation in the  $n$  random variables  $\{\tau_k\}_{k < n}$ ,  $\sqrt{\text{Cov}(\tau_{n/4}, \tau_{3n/4})}$ , and compare them to the maximum's standard deviation,  $\sqrt{\text{Var}(M_n)}$ , supposing that these  $\tau_k$  are independent, which are given by (3). An upper bound of the inter-visit times correlation is given by the Cauchy-Schwarz inequality,  $\text{Cov}(\tau_{n/4}, \tau_{3n/4}) \leq \sqrt{\text{Var}(\tau_{n/4})\text{Var}(\tau_{3n/4})} \propto \text{Var}(\tau_n)$ , which is known from [9] for any type of Markovian RW. For recurrent RWs, this leads to

$$\text{Cov}(\tau_{n/4}, \tau_{3n/4}) \leq n^{2/\mu-1} \ll n^{2/\mu} \sim \text{Var}(M_n), \quad (9)$$

and for marginal RWs, to

$$\text{Cov}(\tau_{n/4}, \tau_{3n/4}) \leq \sqrt{n} \ll n \sim \text{Var}(M_n). \quad (10)$$

For transient RWs the variance of the inter-visit time is constant so that

$$\text{Cov}(\tau_{n/4}, \tau_{3n/4}) \leq \text{const.} \ll (\ln n)^{2/\mu} \sim \text{Var}(M_n). \quad (11)$$

We conclude that, in all cases, the typical fluctuations dominate the typical cross-correlations for all RW classes so that  $\{\tau_k\}$  are effectively independent and hence Eq. (7) is self-consistently checked. Note that, contrary to the central limit theorem where long-range correlations can deeply impact the asymptotic law of the sum of  $n$  random variables, the maximum is less sensitive to cross-correlations as its fluctuations are relatively (compared to the mean) larger. As an example, for  $n$  i.i.d. random variables with finite variance, while the relative fluctuations of the sum decays as  $1/\sqrt{n}$ , the relative fluctuations of the maximum decay logarithmically as  $1/\ln n$  [14].

Further validation of our results is provided in Fig. 2, in which we numerically test Eqs. (2)–(6) on representative

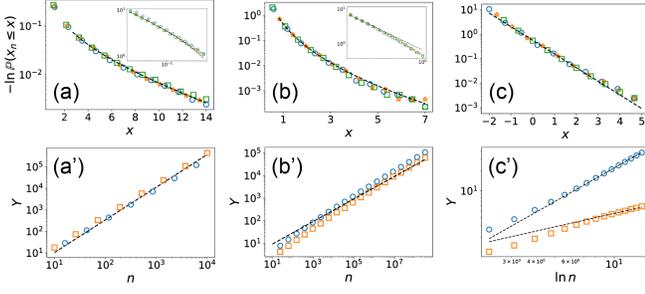


FIG. 2. Maximum of inter-visit times. (a)–(c)  $M_n$  CDF as a function of the rescaled variable  $x_n$  defined in Eq. (1) (insets show them at small  $x_n$  values) and (a')–(c') the corresponding averages (blue circles) and standard deviations (orange squares) of  $M_n$ . The black dashed lines correspond to the best fit of Eqs. (2)–(6). Different universality classes are represented by (a) RWs on a percolation cluster,  $\mu \approx 0.659$  (recurrent),  $n = 1389, 3727$ , and  $10^4$ ; (b) nearest-neighbor 2D RWs,  $\mu = 1$  (marginal),  $n \approx 2 \times 10^7, 10^8$ , and  $3 \times 10^8$  [51]; (c) nearest-neighbor 3D RWs,  $\mu = 3/2$  (transient),  $n \approx 4 \times 10^4, 2 \times 10^5$ , and  $10^6$  [52]. Increasing values of  $n$  are represented by blue circles, orange stars, and green squares.

recurrent, marginal, and transient RW models. We see a very good agreement between our analytical predictions and numerical simulations. The diversity of these examples demonstrates the broad applicability of our theoretical approach.

*Starving random walks.*—We now show that the knowledge of the CDF of  $M_n$  is an essential tool to quantify the interplay between the amount of the resource consumed and the lifetime of a starving RW, as introduced above. We first consider the number of sites visited at starvation  $N_S$ , which is a key observable to quantify the exploration efficiency of starving RWs [24]. At starvation, at least  $n$  sites have been visited if and only if all the first  $n$  times between two visits are strictly less than the metabolic time  $S$ ,  $\tau_0 < S, \dots, \tau_{n-1} < S$ . In other words,

$$\mathbb{P}(N_S \geq n) = \mathbb{P}(M_n < S), \quad (12)$$

so that the distribution of  $N_S$  is directly deduced from that of  $M_n$ . In particular, we obtain (see SM for derivation and numerical verification [38]) the scaling with  $S$  of the first two cumulants of  $N_S$ ,

$$\langle N_S \rangle, \quad \sqrt{\text{Var}(N_S)} \propto \begin{cases} S^\mu & \mu < 1 \\ S^2 & \mu = 1 \\ \exp[(S/a)^{\mu/(1+\mu)}] & \mu > 1, \end{cases} \quad (13)$$

where  $a$  is a positive constant.

Next, the distribution of  $N_S$  gives access to that of the lifetime  $T_S$ . The lifetime  $T_S$  is given by the sum of the

inter-visit times  $\tilde{\tau}_k$  ( $k < n$ ) corresponding to  $\tau_k$  conditioned on being lesser than the time  $S$  to starve. With this the distribution of  $T_S$  reads

$$\begin{aligned} \mathbb{P}(T_S = t) &= \int_0^\infty dn \mathbb{P}\left(\sum_{k=0}^{N_S-1} \tau_k + S = t | N_S = n\right) \mathbb{P}(N_S = n) \\ &\approx \int_0^\infty dn \delta(n\langle\tilde{\tau}_\infty\rangle + S - t) \mathbb{P}(N_S = n), \end{aligned}$$

where we have used that, for large  $k$ , the distribution of  $\tilde{\tau}_k$  becomes independent of  $k$  and the sum follows the law of large numbers,  $\sum_{k=0}^{n-1} \langle\tilde{\tau}_k\rangle \sim n \lim_{k \rightarrow \infty} \langle\tilde{\tau}_k\rangle = n\langle\tilde{\tau}_\infty\rangle$  (see SM for details and numerical checks [38]). This leads to the tail distribution:

$$\mathbb{P}(T_S \geq t) \approx \mathbb{P}\left(N_S \geq \frac{t-S}{\langle\tilde{\tau}_\infty\rangle}\right). \quad (14)$$

In particular, the scaling with  $S$  of the first two cumulants of the lifetime  $T_S$  is given by

$$\langle T_S \rangle, \quad \sqrt{\text{Var}(T_S)} \propto \begin{cases} S & \mu < 1 \\ S^2 & \mu = 1 \\ \exp[(S/a)^{\mu/(1+\mu)}] & \mu > 1. \end{cases} \quad (15)$$

In Fig. 3 we validate Eqs. (14) and (15) via simulations. These results have strong consequences in the important case of a diffusing particle in dimension 1, 2, and 3: a given metabolic time  $S$  leads to radically different lifetimes depending on the space dimension (see SM Fig. S7 for comparison of the average lifetimes).

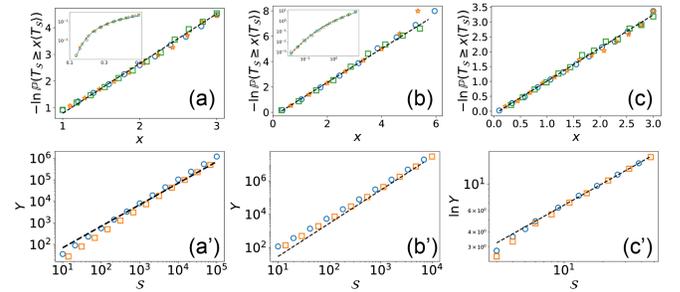


FIG. 3. Lifetime of a starving RW. (a)–(c)  $T_S$  distributions as a function of the rescaled variable  $x \equiv T_S / \langle T_S \rangle$  (the insets show the behavior at small  $x$  values) and (a')–(c') the corresponding averages (blue circles) and standard deviations (orange squares) of  $T_S$ . The black dashed lines correspond to the best fit of the theory. Different universality classes are represented by (a) RWs on a percolation cluster,  $\mu \approx 0.659$  (recurrent),  $S = 14\,667, 31\,622$ , and  $68\,129$ ; (b) nearest-neighbor 2D RWs,  $\mu = 1$  (marginal),  $S = 2335, 4832$ , and  $10^4$ ; (c) nearest-neighbor 3D RWs,  $\mu = 3/2$  (transient),  $S = 12, 17$ , and  $22$ . Increasing values of  $S$  are represented by blue circles, orange stars, and green squares.

We show in SM that the knowledge of the lifetime distribution allows one to obtain the distribution of the position of the forager at starvation, denoted by  $\vec{R}_S$  and defined for the isotropic RWs considered here by its norm  $R_S$  [38]. In particular, we find that (up to log corrections in marginal case)

$$\langle R_S \rangle, \sqrt{\text{Var}(R_S)} \propto \begin{cases} S^{1/d_w} & \mu < 1 \\ S^{2/d_w} & \mu = 1 \\ \exp\left[\frac{1}{d_w} \left(\frac{S}{a}\right)^{\mu/(1+\mu)}\right] & \mu > 1. \end{cases} \quad (16)$$

Several comments on Eqs. (13)–(16), echoing the results of the 1D nearest-neighbor starving RW [24,25] case recalled in the Introduction, are in order. (i) The fluctuations of  $N_S$ ,  $T_S$ , and  $R_S$  are relevant for all RWs classes, as the average and the standard deviation grow similarly. This highlights the importance of the distribution, Eq. (12) [and Eqs. (1)–(6)]. (ii) Averages of these observables grow algebraically for recurrent (and marginal) RWs. This generalizes the case of a regular 1D starving RW to the case of general recurrent RWs. Strikingly, the mean lifetime is linear with the starvation index  $S$  for all recurrent walks (independently of  $\mu$ ). On the other hand, the averages of  $N_S$ ,  $T_S$ , and  $R_S$  become stretch-exponentially large for transient RWs. Note that the mean-field exponential behavior is, as it should, recovered in the limit  $d_f \rightarrow \infty$ . (iii) The distributions of all these observables obey asymptotically a single-parameter scaling for any value of  $\mu$ . This extends the result known for  $d_w = 2$  in the 1D case [24,25] to general Markovian starving RWs.

We have shown that the maximum of the inter-visit times of general Markovian RWs assumes simple, universal limit distributions. We have determined the corresponding rescaled variables and the asymptotic of the limit distributions. Beyond this fundamental aspect, we have shown that these results have applications in foraging theory. They have enabled us to determine the statistics of a variety of key observables of the  $d$ -dimensional starving RW problem, and to reveal their universal features. While the universality breaks down when the inter-visit times correlations cannot be neglected, as is the case for smooth processes (such as the random acceleration process [53]; see SM [38]), our results hold for RWs on graphs with strongly inhomogeneous degree distributions [see SM for the typical example of the  $(u, v)$  flowers [54,55]].

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