Momentum Shell in Quarkyonic Matter from Explicit Duality: A Dual Model for Cold, Dense QCD

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(Received 6 July 2023; revised 26 October 2023; accepted 20 February 2024; published 11 March 2024)

We present a model of cold QCD matter that bridges nuclear and quark matter through the duality relation between quarks and baryons. The baryon number and energy densities are expressed as functionals of either the baryon momentum distribution, f_{B}, or the quark distribution, f_{Q}, which are subject to the constraints on fermions, $0 \leq f_{\mathrm{B,Q}} \leq 1$. The theory is ideal in the sense that the confinement of quarks into baryons is reflected in the duality relation between f_{Q} and f_{B}, while other possible interactions among quarks and baryons are all neglected. The variational problem with the duality constraints is formulated and we explicitly construct analytic solutions, finding two distinct regimes: a nuclear matter regime at low density and a quarkyonic regime at high density. In the quarkyonic regime, baryons underoccupy states at low momenta but form a momentum shell with $f_{\mathrm{B}} = 1$ on top of a quark Fermi sea. Such a theory describes a rapid transition from a soft nuclear equation of state to a stiff quarkyonic equation of state. At this transition, there is a rapid increase in the pressure.

DOI: 10.1103/PhysRevLett.132.112701

Introduction.—Understanding cold, dense QCD matter is a difficult problem. Slightly above nuclear saturation density, the importance of many-body forces complicates the physical picture. The distinction between baryon and quark degrees of freedom is not clear-cut and in fact a proper description should allow a *dual* simultaneous description of both quarks and baryons. This Letter attempts such a dual description.

We construct such a dual model on the basis of very simple principles. This model is analytically solvable. It has its consequences that the high-density phase is "quarkyonic." The notion of quarkyonic matter has emerged from studies of cold, dense QCD in the limit of a large number of colors, $N_{\mathrm{c}} \to \infty$, where the quark screening effects to color confinement are suppressed by a factor $1/N_{\mathrm{c}}$. At a finite quark chemical potential μ_{Q}, the confinement persists to $\mu_{\mathrm{Q}} \sim \sqrt{N_{\mathrm{c}}}\Lambda_{\mathrm{QCD}}$ ($\Lambda_{\mathrm{QCD}} \simeq 300$ MeV: dynamical scale in QCD) until quarks Debye screen gluons, while quarks establish the Fermi sea at much lower density, with $\mu_{\mathrm{Q}} \sim \Lambda_{\mathrm{QCD}}$. At $\Lambda_{\mathrm{QCD}} \ll \mu_{\mathrm{Q}} \ll \sqrt{N_{\mathrm{c}}}\Lambda_{\mathrm{QCD}}$, there must be quark matter with the confinement. This paradoxical feature was resolved by assuming baryons as effective degrees of freedom on top of a quark Fermi sea [1] (see, however, Ref. [2]). This dual feature, with quarks and baryons in different domains of momenta for a single phase of matter, is suitable to describe continuous evolution from nuclear matter to weakly coupled quark matter [3–8].

Models of quarkyonic matter are shown to reveal an equation of state (EOS) that rapidly stiffens from low to high density [9–15] (see also Refs. [16–21]) and therefore has features consistent with the observed properties of neutron stars [22–35]. Notably, these features encompass a rapid rise in the sound speed [36–49] and the vanishing of the trace anomaly signifying the conformal nature of dense matter [50–55] (see also Ref. [56]).

In this Letter, we present dynamical descriptions for quarkyonic matter by making full use of the duality between baryons and quarks [57,58] (see also Refs. [59,60]). The baryon and energy densities are expressed by either baryonic or quark degrees of freedom, characterized by the occupation probability of momentum states, $f_{\mathrm{B}}(k)$ and $f_{\mathrm{Q}}(q)$, for baryons and quarks, respectively (the letters k and q are used exclusively for baryon and quark momentum, respectively). Since baryons are composed of quarks, there is a relationship between f_{B} and f_{Q}, allowing a dual description without double counting of quark contributions.

The simplest description one can imagine of such matter is that except for the confinement of quarks into baryons, interactions of quarks and baryons can be ignored. Such an idealized matter we will call "IdylliQ" (ideal dual quarkyonic) matter. We formulate a variational problem to minimize the energy density functional $\varepsilon_{\mathrm{B}}[f_{\mathrm{B}}] = \varepsilon_{\mathrm{Q}}[f_{\mathrm{Q}}]$ for a given baryon density n_{B}, and determine the distribution f_{B} and f_{Q}. Even such an idealized model is nontrivial due to the duality relation and quantum mechanical constraints $0 \leq f_{\mathrm{Q,B}} \leq 1$. The quark substructure constraint is crucial for rapid evolution of stiffness from nuclear to quark matter [57,58]. For this IdylliQ model, we establish the momentum shell of baryons on top of the quark Fermi sea as schematically shown in Fig. 1. IdylliQ theory gives a novel

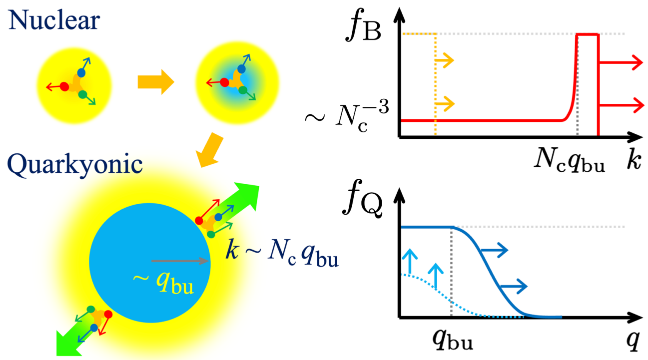

FIG. 1. Evolution of $f_{\rm B}$ and $f_{\rm Q}$ from the nuclear (dotted) to quarkyonic regime (solid). The saturation of quark states drives baryons into the relativistic regime. Arrows in the rightmost panels indicate increasing $\mu_{\rm B}$.

alternative explanation for the previously proposed picture of quarkyonic matter [1,9].

Duality.—We posit the duality relation between $f_{\rm Q}$ for quarks with *a given color* and $f_{\rm B}$ for baryons as (notation: $\int_k \equiv \int \{[d^d\boldsymbol{k}]/[(2\pi)^d]\}$) [57,58]

$$[f_{\rm Q}(q)]_{f\sigma} = \sum_{i=n,p,\cdots} \sum_{\sigma'=\uparrow,\downarrow} \int_k \left[\varphi\left(\boldsymbol{q} - \frac{\boldsymbol{k}}{N_c}\right)\right]^{i\sigma'}_{f\sigma} [f_{\rm B}(k)]_{i\sigma'}, \quad (1)$$

where φ is a single quark momentum distribution with the flavor f and spin σ in a single baryon state of a species i and spin σ'. Collecting quark contributions from each baryon leads to quark distributions in dense matter. In this Letter, we limit ourselves to symmetric nuclear matter and include a spin-isospin degeneracy factor 4 in the expressions of thermodynamic quantities, but elsewhere we drop the spin-flavor indices f, σ. The extension for multiflavors and multibaryon species will be discussed in the forthcoming papers.

The normalization is $\int_q \varphi(q) = 1$. The dual expression of the baryon number readily follows from Eq. (1) as

$$n_{\rm B} = 4\int_k f_{\rm B}(k) = 4\int_q f_{\rm Q}(q). \quad (2)$$

The energy densities in terms of baryons and quarks are

$$\varepsilon_{\rm B}[f_{\rm B}] = 4\int_k E_{\rm B}(k) f_{\rm B}(k),$$
$$\varepsilon_{\rm Q}[f_{\rm Q}] = 4\int_q E_{\rm Q}(q)[N_c f_{\rm Q}(q)]. \quad (3)$$

Remember $f_{\rm Q}$ is defined for a fixed color, $f_{\rm Q} \equiv f_{\rm Q}^R = f_{\rm Q}^G = f_{\rm Q}^B$ with which $n_{\rm B} = n_{\rm Q}^R = n_{\rm Q}^G = n_{\rm Q}^B$. A single baryon is assumed to have the energy contributions summed from N_c-confined quarks, $E_{\rm B}(k) = N_c \int_q E_{\rm Q}(q)\varphi(\boldsymbol{q} - \boldsymbol{k}/N_c)$. Then a duality relation follows, $\varepsilon = \varepsilon_{\rm B}[f_{\rm B}] = \varepsilon_{\rm Q}[f_{\rm Q}]$. As quarks are confined in a spatial domain of the baryon size $\sim\Lambda_{\rm QCD}^{-1}$, quarks can be energetic and $\varphi(q)$ is spread to momenta of $\sim\Lambda_{\rm QCD}$. The *mechanical* pressure inside of a baryon is large.

In this Letter, going from low to high densities we keep using the same φ determined in vacuum. Our main target here is the transient regime from baryonic to quark matter, where using φ for localized quarks may not be a bad approximation. The structural changes in baryons, such as swelling, would possibly increase the low momentum components of φ, but such modifications merely shift the onset of quark matter formation to lower density.

Minimization of energy functional.—With duality (1) as a constraint, we calculate the energy density ε for a given $n_{\rm B}$. We consider energy functionals

$$\varepsilon = \varepsilon_{\rm B}[f_{\rm B}]|_{n_{\rm B}} = \varepsilon_{\rm Q}[f_{\rm Q}]|_{n_{\rm B}}, \quad (4)$$

and minimize them by optimizing $f_{\rm B}$ or $f_{\rm Q}$ while holding $n_{\rm B}$ fixed. A novelty in our optimization program is that the solutions are determined not only by the stationary condition $\delta\varepsilon/\delta f = 0$ but also by the boundary conditions $f_{\rm B,Q} = 0$ or 1. The thermodynamic energy density is obtained by substituting the optimized distributions, $\varepsilon_{\rm EOS}(n_{\rm B}) = \varepsilon_{\rm B}[f_{\rm B}^*]|_{n_{\rm B}} = \varepsilon_{\rm Q}[f_{\rm Q}^*]|_{n_{\rm B}}$.

In practice, one can find the $f_{\rm B}^*$ and $f_{\rm Q}^*$ by minimizing

$$\tilde{\varepsilon} = \varepsilon_{\rm B}[f_{\rm B}] - \lambda_{\rm B} n_{\rm B} = \varepsilon_{\rm Q}[f_{\rm Q}] - \lambda_{\rm Q} n_{\rm Q}, \quad (5)$$

where $\lambda_{\rm B} = N_c \lambda_{\rm Q}$. It is tempting to identify the λ's as chemical potentials and $\tilde{\varepsilon}$ as the thermodynamic functional. Unfortunately they do not satisfy the thermodynamic relations if solutions are partly determined by the boundary conditions. Hence, we use $\tilde{\varepsilon}$ only to find $f_{\rm B}^*$ and $f_{\rm Q}^*$, and use them in computations of $\varepsilon_{\rm EOS}(n_{\rm B})$.

Global constraints.—The constraints in our theory appear *global*, as $f_{\rm Q}$ at a given momentum depends on $f_{\rm B}$ for the entire momentum range. The variation leads to

$$\frac{\delta\tilde{\varepsilon}}{\delta f_{\rm B}(k)} = E_{\rm B}(k) - \lambda_{\rm B}, \qquad \frac{\delta\tilde{\varepsilon}}{\delta f_{\rm Q}(q)} = E_{\rm Q}(q) - \lambda_{\rm Q}. \quad (6)$$

At momenta with $\delta\tilde{\varepsilon}/\delta f_{\rm B,Q} < 0$, greater $f_{\rm B,Q}$ reduces $\tilde{\varepsilon}$ and grows toward the boundary $f_{\rm B,Q} = 1$, while $\delta\tilde{\varepsilon}/\delta f_{\rm B,Q} > 0$ drives $f_{\rm B,Q}$ to the other boundary, $f_{\rm B,Q} = 0$. We would get the optimized distributions

$$f_{\rm B}^{\rm var}(k) = \Theta(k_{\rm F} - k), \qquad f_{\rm Q}^{\rm var}(q) = \Theta(q_{\rm F} - q), \quad (7)$$

where $k_{\rm F}$ and $q_{\rm F}$ are determined through $\lambda_{\rm B} = E_{\rm B}(k_{\rm F})$ and $\lambda_{\rm Q} = E_{\rm Q}(q_{\rm F})$.

The above solutions are not usable everywhere. For instance, $f_{\rm Q}^{\rm var}$ at large momenta is incompatible with the sum rule (1); at large momenta ($q \gg k/N_c$), the scaling should be $f_{\rm Q}(q) \sim n_{\rm B}\varphi(q)$. Another problem is that, if we keep using $f_{\rm B}^{\rm var}$ in the regime $\Lambda_{\rm QCD} \ll k_{\rm F} \ll N_{\rm c}\Lambda_{\rm QCD}$, then $f_{\rm Q}(0) \sim n_{\rm B}\varphi(0) \sim k_{\rm F}^3/\Lambda_{\rm QCD}^3$, violating $f_{\rm Q} \le 1$ at $q = 0$. Our problem is to patch the candidates of solutions, found from variational calculations and the boundary conditions, into the form consistent with the duality constraints, and then to minimize the energy.

Solvable model.—What makes the dual theory nontrivial is its global nature. The difficulty lies in the reconstruction of $f_{\rm B}$ from a given $f_{\rm Q}$. At high density we have good reasoning to choose $f_{\rm Q} = 1$ for some interval of q. But it is difficult to tell which $f_{\rm B}$ gives $f_{\rm Q} = 1$ while not violating $f_{\rm Q} \le 1$ anywhere. To uncover the general features of the dual theory, we choose a specific φ that reduces the global problem to the one determining a couple of global constants. We choose

$$\varphi_{3\rm d}(\boldsymbol{q}) = \frac{2\pi^2}{\Lambda^3}\frac{e^{-q/\Lambda}}{q/\Lambda}, \tag{8}$$

which is the inverse of a linear differential operator $\hat{L} = -\nabla_q^2 + (1/\Lambda^2)$. Applying this operator to the sum rule (1), we find the local relation between $f_{\rm B}$ and $f_{\rm Q}$ ($d = 3$),

$$f_{\rm B}(N_{\rm c}q) = \frac{\Lambda^2}{N_{\rm c}^d}\hat{L}[f_{\rm Q}(q)]. \tag{9}$$

Here, we assume $E_{\rm B}(k) = \sqrt{k^2 + M_{\rm B}^2}$ with $M_{\rm B}$ being a constant as we focus on the deconfining aspect. The single quark energy can also be determined as

$$E_{\rm Q} = \sqrt{q^2 + M_{\rm Q}^2}\left(1 - \frac{(d-1)\Lambda^2}{q^2 + M_{\rm Q}^2} - \frac{M_{\rm Q}^2\Lambda^2}{(q^2 + M_{\rm Q}^2)^2}\right), \tag{10}$$

where $M_{\rm Q} \equiv M_{\rm B}/N_{\rm c}$. We note $E_{\rm Q}'(q) > 0$ everywhere.

We must examine which $f_{\rm Q}$ satisfies $f_{\rm B} = 0$ or $f_{\rm B} = 1$. For this purpose we introduce $y_\pm(q) = {\rm e}^{\pm q/\Lambda}/q$, which satisfies $\hat{L}[y_\pm] = 0$. The boundary $f_{\rm B}(N_{\rm c}q) = 0$ can be obtained as $f_{\rm Q}^{f_{\rm B}=0}(q) = c_+y_+(q) + c_-y_-(q)$. Meanwhile $f_{\rm B}(N_{\rm c}q) = 1$ can be obtained as $f_{\rm Q}^{f_{\rm B}=1}(q) = N_{\rm c}^d + d_+y_+(q) + d_-y_-(q)$. The constants $c_\pm$ and $d_\pm$ must be chosen to keep $0 \le f_{\rm Q}(q) \le 1$.

Now we have exhausted candidates of local solutions for $f_{\rm Q}$: the boundary values $f_{\rm Q} = 0$, 1 and those dual to $f_{\rm B} = 0, 1$. The question is how to patch them. At momenta where two different solutions meet, the second derivative in $\hat{L}$ and the condition $0 \le f_{\rm B} \le 1$ demands $f_{\rm Q}$ to be continuous up to the first derivative. For example, acting $\hat{L}$ on a function $f_{\rm Q}^{\rm bu} = \eta(q)\Theta(q_{\rm bu} - q)$ generates the terms $\eta'(q_{\rm bu})\delta(q - q_{\rm bu})$ and $\eta(q_{\rm bu})\delta'(q - q_{\rm bu})$. To cancel such delta's violating the condition $f_{\rm B} \le 1$, we have to add a function $f_{\rm Q}^{\rm joint} = \xi(q)\Theta(q - q_{\rm bu})$ with $\xi(q_{\rm bu}) = \eta(q_{\rm bu})$ and $\xi'(q_{\rm bu}) = \eta'(q_{\rm bu})$. We construct such a $f_{\rm Q}^{\rm joint}$ using solutions $f_{\rm Q}^{f_{\rm B}=1}$ and $f_{\rm Q}^{f_{\rm B}=0}$.

Transitions from baryonic to quark matter.—We go over from a dilute baryonic matter to a dense quark matter by patching the candidates of local solutions.

In dilute matter one can simply use the ideal baryon gas $f_{\rm B}^{\rm idB}(k) = \Theta(k_{\rm F} - k)$ and its dual expression $f_{\rm Q}^{\rm idB}$. This regime continues until $f_{\rm Q}^{\rm idB}$ reaches the upper bound. It occurs first at $q = 0$ when $k_{\rm F} \simeq \sqrt{2/N_{\rm c}}\Lambda$ or $n_{\rm B}/n_0 = 2.58 \times (\Lambda/0.4~{\rm GeV})^3$ for $N_{\rm c} = 3$ ($n_0 \simeq 0.16~{\rm fm}^{-3}$: normal nuclear density). Note that this implies the saturation density is parametrically small compared to the QCD scale Λ^3.

In the postsaturation regime, we can no longer use the ideal baryon gas picture. For the low momentum part of $f_{\rm Q}$, the only candidate is $f_{\rm Q} = 1$. We found that the solution must involve three segments (two segment models cannot satisfy the continuity at the first derivative),

$$f_{\rm Q}(q) = \Theta(q_{\rm bu} - q) + f_{\rm Q}^{f_{\rm B}=1}(q)\Theta(q_{\rm sh} - q)\Theta(q - q_{\rm bu}) + f_{\rm Q}^{f_{\rm B}=0}(q)\Theta(q - q_{\rm sh}), \tag{11}$$

where d_+ in $f_{\rm Q}^{f_{\rm B}=0}$ must be zero. Its dual baryon distribution is ($k = N_{\rm c}q, k_{\rm bu} = N_{\rm c}q_{\rm bu}, k_{\rm sh} = N_{\rm c}q_{\rm sh}$)

$$f_{\rm B}(k) = \frac{1}{N_{\rm c}^d}\Theta(k_{\rm bu} - k) + \Theta(k_{\rm sh} - k)\Theta(k - k_{\rm bu}), \tag{12}$$

which is small in the *bulk* Fermi sea at $k \le k_{\rm bu}$ but forms the baryon *shell* with the maximum height at $k_{\rm bu} < k \le k_{\rm sh}$, reproducing the form conjectured by McLerran and Reddy [9]. This shape is energetically favored as $f_{\rm B}$ and $f_{\rm Q}$ are kept as compact as possible. Figure 2 shows the forms of $f_{\rm B}^{\rm idB}$ and $f_{\rm Q}^{\rm idB}$ in the dilute regime at $n_{\rm B}/n_0 \lesssim 2.6$, and the forms of $f_{\rm B}$ (12) and $f_{\rm Q}$ (11) in the postsaturation regime at $n_{\rm B}/n_0 \gtrsim 2.6$.

With four conditions from two junction points, one can express $c_\pm$, d_-, and $\Delta_{\rm Q} = q_{\rm sh} - q_{\rm bu}$ as functions of $q_{\rm sh}$. The parameter $q_{\rm sh}$ is determined by observing that the $\delta\tilde{\varepsilon}/\delta f_{\rm B} > 0$ for $E_{\rm B} > \lambda_{\rm Q}$ introduces the energy cost unless $f_{\rm B}$ drops from 1 to 0 at $\lambda_{\rm B} = E_{\rm B}(N_{\rm c}q_{\rm sh})$. Here, we display only the equation to determine $\Delta_{\rm Q}$ as it is needed for computations of EOS. The equation to be solved is

$$\frac{\Lambda + q_{\rm bu}}{\Lambda + q_{\rm bu} - (\Lambda + q_{\rm sh})e^{-\Delta_{\rm Q}/\Lambda}} = N_{\rm c}^3. \tag{13}$$

Below, we discuss the thickness of the baryon momentum shell, $\Delta_{\rm B} = N_{\rm c}\Delta_{\rm Q}$, which is obtained as a solution of this transcendental equation. Close to the momentum $k_{\rm sh} = k_{\rm sat}$ at which $f_{\rm Q}$ saturates, we expand the

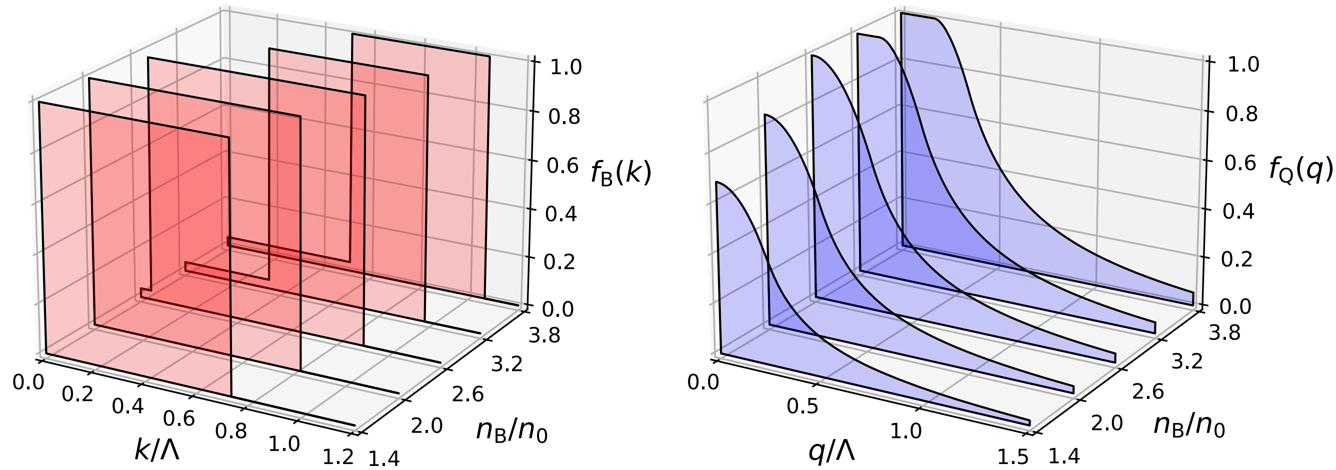

FIG. 2. Evolution of $f_{\rm B}(k)$ (left) and $f_{\rm Q}(q)$ (right) with increasing $n_{\rm B}$ for $\Lambda = 0.4$ GeV.

equation with respect to $\delta k \equiv k_{\rm sh} - k_{\rm sat}$, then the solution is approximately

$$\Delta_{\rm B} \simeq k_{\rm sh} - \sqrt{\frac{2k_{\rm sat}}{1 + k_{\rm sat}/(N_{\rm c}\Lambda)}\delta k}, \tag{14}$$

where we assumed $\delta k \ll \Lambda$ and $k_{\rm bu} \ll \Lambda$ then expanded them up to $\mathcal{O}(\delta k)$ and $\mathcal{O}(k_{\rm bu}^2)$, respectively. The $k_{\rm sh}$ derivative of $\Delta_{\rm B} \propto \sqrt{\delta k}$ diverges as $\delta k^{-1/2}$ for $\delta k \to 0$.

At large momentum, $k_{\rm sh} > \Lambda$, the shell thickness is

$$\Delta_{\rm B} \simeq \frac{\Lambda}{N_{\rm c}^2} + \frac{\Lambda^2}{N_{\rm c} k_{\rm sh}}. \tag{15}$$

It approaches constant for a large $k_{\rm sh}$.

Equations of state.—We examine the unified EOSs. Below quark saturation, the EOSs are simply those of the ideal baryon gas, $n_{\rm B}^{\rm below} = 2k_{\rm F}^3/3\pi^2$ and $\varepsilon^{\rm below} = 4\int_k E_{\rm B}(k)\Theta(k - k_{\rm F})$. Above the saturation, the bulk part of $f_{\rm B}$ is depleted,

$$n_{\rm B}^{\rm above} = 4\int_{k_{\rm bu}}^{k_{\rm sh}} \frac{d^3\boldsymbol{k}}{(2\pi)^3} + \frac{4}{N_{\rm c}^3}\int_0^{k_{\rm bu}} \frac{d^3\boldsymbol{k}}{(2\pi)^3},$$
$$\varepsilon^{\rm above} = 4\int_{k_{\rm bu}}^{k_{\rm sh}} \frac{d^3\boldsymbol{k}}{(2\pi)^3} E_{\rm B}(k) + \frac{4}{N_{\rm c}^3}\int_0^{k_{\rm bu}} \frac{d^3\boldsymbol{k}}{(2\pi)^3} E_{\rm B}(k). \tag{16}$$

Because of the depletion, the growth of $k_{\rm sh}$ increase $n_{\rm B}$ more slowly than in the presaturation regime, but the energy per particle $\varepsilon/n_{\rm B}$ grows much faster in the postsaturation regime. Accordingly the pressure $P = n_{\rm B}^2 \partial(\varepsilon/n_{\rm B})/\partial n_{\rm B}$ is large; the EOS is stiff.

It is important to examine whether thermodynamic quantities are continuous at quark saturation. At saturation $k_{\rm bu} = 0$ in Eq. (16) so that $n_{\rm B}$ and ε are continuous. Next we check whether derivatives of ε with respect to $n_{\rm B}$ are continuous. We first compute

$$2\pi^2 \frac{\partial n_{\rm B}^{\rm above}}{\partial k_{\rm sh}} = k_{\rm sh}^2 - \left(1 - \frac{1}{N_{\rm c}^3}\right) k_{\rm bu}^2 \frac{\partial k_{\rm bu}}{\partial k_{\rm sh}}. \tag{17}$$

Because of the phase space factor $k_{\rm bu}^2$, at saturation the second term specific to the postsaturation regime vanishes, leaving continuous $\partial n_{\rm B}/\partial k_{\rm sh}$. Similarly $\partial \varepsilon/\partial k_{\rm sh}$ is continuous and so are $\mu_{\rm B} = \partial\varepsilon/\partial n_{\rm B}$ and the pressure $P = \mu_{\rm B} n_{\rm B} - \varepsilon$. Note that this continuity, relying on the vanishing phase space for $k_{\rm bu} \to 0$, does not hold in $1+1$ dimensions; indeed a $1+1$ dimensional IdylliQ model yields discontinuous $\mu_{\rm B}$, which is not permitted in the thermodynamics.

In $3+1$ dimensional IdylliQ theory, unfortunately the continuity holds only up to the first derivative. The baryon susceptibility $\chi_{\rm B}$ has a discontinuity at saturation and so does the sound speed v_s^2. The susceptibility $\chi_{\rm B}$ drops discontinuously; this dropping should not be confused with that in a second order phase transition where $\chi_{\rm B}$ jumps up. Figure 3 shows the behavior of v_s^2 as a function of $n_{\rm B}/n_0$ for $\Lambda = 0.4$ GeV. Note that in our model, v_s^2 may

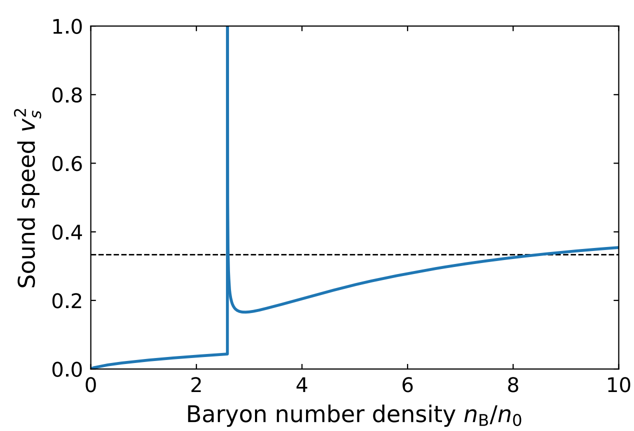

FIG. 3. Sound speed v_s^2 as a function of $n_{\rm B}/n_0$ for $\Lambda = 0.4$ GeV.

exceed the conformal value $v_s^2 = 1/3$ even at high densities where we expect it to be subconformal [3–8], depending on the value of Λ. Also, as mentioned earlier, v_s^2 is singular at saturation and the parametric dependence of the singular part, $\hat{v}_s^2$, on δk is

$$\hat{v}_s^2 \sim -\frac{k_{\mathrm{sh}} k_{\mathrm{sat}}}{M_{\mathrm{B}}^2} \frac{d\Delta_{\mathrm{B}}}{dk_{\mathrm{sh}}} \sim \frac{1}{N_{\mathrm{c}}^3} \sqrt{\frac{k_{\mathrm{sat}}}{\delta k}}, \tag{18}$$

given that $k_{\mathrm{sh}} \sim k_{\mathrm{sat}} \sim N_{\mathrm{c}}^{-1/2}\Lambda$ and $M_{\mathrm{B}} \sim N_{\mathrm{c}}\Lambda$.

Minimal corrections to IdylliQ model.—We outline how the singular behavior of Δ_{B} and v_s^2 is remedied by smoothing out the sharp edge of the baryon distribution. As the divergent part of v_s^2 (18) is proportional to $d\Delta_{\mathrm{B}}/dk_{\mathrm{sh}}$, we focus on the singular behavior of Δ_{B}. We define a function $g_{\mathrm{Q}}(k)$, which is the quark occupation at the origin corresponding to the baryon Fermi sea filled up to momentum k:

$$g_{\mathrm{Q}}(k) \equiv \int_{k'} \varphi\left(\frac{k'}{N_{\mathrm{c}}}\right) \Theta(k - k'). \tag{19}$$

Below the saturation density, $k_{\mathrm{FB}} = k_{\mathrm{sat}} - 0^+$, the condition $f_{\mathrm{Q}}(q = 0) \to 1$ is equivalent to $g_{\mathrm{Q}}(k_{\mathrm{FB}}) \to 1$.

Above the saturation density, the condition $f_{\mathrm{Q}}(q = 0) = 1$ becomes

$$g_{\mathrm{Q}}(k_{\mathrm{sh}}) - \left(1 - \frac{1}{N_{\mathrm{c}}^3}\right) g_{\mathrm{Q}}(k_{\mathrm{bu}}) = 1, \tag{20}$$

At saturation, this equation is equivalent to Eq. (13) that sets the relation between k_{sh} and Δ_{B}.

By taking the k_{sh} derivative on both sides of Eq. (20), we obtain

$$\frac{d\Delta_{\mathrm{B}}(k_{\mathrm{sh}})}{dk_{\mathrm{sh}}} = -\frac{N_{\mathrm{c}}^3}{N_{\mathrm{c}}^3 - 1} \frac{g'_{\mathrm{Q}}(k_{\mathrm{sh}})}{g'_{\mathrm{Q}}(k_{\mathrm{bu}})} + 1, \tag{21}$$

where $g'_{\mathrm{Q}}(k) = e^{-k/(N_{\mathrm{c}}\Lambda)} \times N_{\mathrm{c}} k/\Lambda^2$. Near the saturation, $g'_{\mathrm{Q}}(k_{\mathrm{bu}}) \sim k_{\mathrm{bu}} \propto \delta k^{1/2} \to 0$ yields the singularity in $d\Delta_{\mathrm{B}}(k_{\mathrm{sh}})/dk_{\mathrm{sh}}$ and $\hat{v}_s^2$. The problematic $g'_{\mathrm{Q}}(k) \propto k$ scaling comes from the $\partial\Theta(k - k')/\partial k = \delta(k - k')$ term that picks out the integrand $\sim k'^2 \varphi(k'/N_{\mathrm{c}}) \sim k'$ *exactly* at $k' = k$. A little smearing of the baryon Fermi surface cures this problem: replacing $\Theta(k - k')$ with a smooth function $g_{\mathrm{B}}(k - k')$ whose damping scale is $\sim k_{\mathrm{dif}}$, one can make $g'_{\mathrm{Q}}(k) \sim k_{\mathrm{dif}}$ finite for $k \to 0$. In turn, at saturation we have $d\Delta_{\mathrm{B}}(k_{\mathrm{sh}})/dk_{\mathrm{sh}} \sim -k_{\mathrm{sh}}/k_{\mathrm{dif}}$ and $\hat{v}_s^2 \sim M_{\mathrm{B}}^{-2} k_{\mathrm{sat}}^3/k_{\mathrm{dif}} \sim N_{\mathrm{c}}^{-3} k_{\mathrm{sat}}/k_{\mathrm{dif}}$. For the causal sound speed, the required width is $k_{\mathrm{dif}} \gtrsim N_{\mathrm{c}}^{-3} k_{\mathrm{sat}}$, much smaller than the Fermi momentum at saturation. We note that too large smearing washes out the peak structure in v_s^2 by reducing the disparity between the nuclear and quark pressure. Viewing nuclear forces as quark exchanges would explain the precursor behavior toward the quark regime. Leaving aside the details of such smoothing, our theory firmly establishes an inevitable stiffening caused by the quark substructure.

Summary and discussions.—The description IdylliQ matter we present is ultimately very simple. At low densities, there is a filled Fermi distribution of nucleons, and quarks may be thought of as degrees of freedom inside the nucleons with momenta out to $\sim\Lambda_{\mathrm{QCD}}$. At some density there is a transition characterized by the saturation of quark states. At higher density, quarks form a filled Fermi sea with an exponentially falling tail above some momentum. In the dual description, baryons underoccupy a bulk Fermi sea but form a fully filled shell at a Fermi surface. These distributions are shown in Fig. 2.

We chose the specific model (8) for φ to solve the IdylliQ theory exactly. The following findings should be universal for other choices of φ: (a) the saturation of the quark distribution f_{Q}, as demonstrated for different φ [57,58], and the underoccupation of f_{B} at lower momenta coming from the quark substructure constraint; (b) the asymptotic behavior of $f_{\mathrm{Q}}(q) \sim n_{\mathrm{B}}\varphi(q)$ at $q \to \infty$; (c) existence of the shell structure in f_{B}, energetically favored to make f_{Q} a more compact distribution. Actually these properties largely survive for the IdylliQ model perturbed by wide class of interaction functionals of $f_{\mathrm{B,Q}}$. Hence, the IdylliQ model offers a good baseline for theories of quarkyonic matter.

We thank Gordon Baym, Jean-Paul Blaizot, Kenji Fukushima, Tetsuo Hatsuda, Masakiyo Kitazawa, Rob Pisarski, Sanjay Reddy, and Volodya Vovchenko for useful conversations. Y. F. is supported by the Japan Society for the Promotion of Science (JSPS) through the Overseas Research Fellowship. The work of Y. F. and L. M. was supported by the U.S. DOE under Grant No. DE-FG02-00ER41132. The work of T. K. was supported by JSPS KAKENHI Grant No. 23K03377 and by the Graduate Program on Physics for the Universe (GPPU) at Tohoku University.

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