Bounding the Amount of Entanglement from Witness Operators

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We present an approach to estimate the operational distinguishability between an entangled state and any separable state directly from measuring an entanglement witness. We show that this estimation also implies bounds on a variety of other well-known entanglement quantifiers. This approach for entanglement estimation is then extended to both the measurement-device-independent scenario and the fully device-independent scenario, where we obtain nontrivial but suboptimal bounds. The procedure requires no numerical optimization and is easy to compute. It offers ways for experimenters to not only detect, but also quantify, entanglement from the standard entanglement witness procedure.

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Introduction.—Entanglement [1] is a fundamental quantum resource promising advantages over the classical resources in many information tasks ranging from quantum computation and communication [2] to metrology [3,4]. A fundamental problem is to decide if a given state is entangled or not. If affirmative, a natural problem is then to quantify the amount of entanglement present in the state. However, both these problems are typically nondeterministic polynomial-time hard [5–7], and that has led to the development of a variety of partial entanglement criteria [8,9] and computational methods [10].

Practical entanglement detection must be reasonably efficient, i.e., use far fewer local measurements than those needed for state tomography while still maintaining an adequate detection power [11–13]. The leading tool for such purposes is entanglement witnesses (EWs) [14], which are observable quantities that are positive for all separable states and negative for some entangled states. Every entangled state can be detected by some suitably chosen entanglement witness [1], although identifying it is not straightforward.

Assuming that a device precisely measures the observables prescribed in an EW is usually insufficient. This can for instance be ascribed either to the realistic influence of imperfect control [15,16] or to the presence of uncharacterized devices. A well-known scenario to overcome such issues is the measurement-device-independent (MDI) scenario [17,18]. EWs designed for the MDI scenario work with uncharacterized measurement devices but require the introduction of trusted, precisely controlled, quantum sources. All entangled states can be detected in this way [17]. An even stronger framework is the device-independent (DI) scenario, in which all assumptions on the quantum measurement devices are removed. EWs designed for the DI scenario are commonly known as Bell inequalities, and thus verify entanglement from quantum nonlocality [19–21].

While the violation of EWs in these three scenarios is sufficient to detect entanglement, a natural question is how the magnitude of the violation can be used to quantify the entanglement in the state. When devices are trusted, entanglement quantification from EWs has for instance been approached using the Legendre transform [22–26]. For MDI-EWs, entanglement quantification often comes with costly computational methods [27–29]. In the DI scenario, little is known about entanglement quantification. Notable exceptions again require dedicated computational methods that grow rapidly with the complexity of the scenario [30].

In this Letter, we introduce a comprehensive method for estimating entanglement, which requires no numerical optimization and is easy to compute. The main idea is to suitably normalize any given entanglement witness so that its violation magnitude can be used directly to bound the trace distance between a given entangled state and its best separable approximation. In turn, this entanglement quantifier can be used to bound a variety of well-known entanglement measures. We then extend this idea from the scenario of trusted devices to both the MDI scenario and the DI scenario, for which the bounds are typically not tight. The method is inherently analytical. For trusted devices and the MDI scenario, the quantification can be easily computed. In the DI scenario, the most demanding computation needed is an upper bound on the largest quantum violation of the considered EW. Finally, we showcase how the approach can go further, by quantifying entanglement with respect to a given Schmidt number and estimating the depth of entanglement in multipartite systems.

Trusted devices.—We focus first on the bipartite scenario. When devices are trusted, consider that we are given a standard EW in the form of a Hermitian operator $\mathcal{W}$ [1,31]. By definition, we have $w(\rho) \equiv \mathrm{Tr}(\rho\mathcal{W}) \geq 0$ for all separable states and $w < 0$ for some entangled states, where and henceforth we omit the state if not introducing ambiguity. Our goal is to bound from below the distinguishability between ρ and any separable state in terms of a negative w, and then use the bound to estimate other entanglement measures. This entanglement measure corresponds to the smallest trace distance between ρ and any separable state whose set is specified by Ω, that is,

$$E_{\mathrm{tr}}(\rho) = \min_{\varrho\in\Omega} D_{\mathrm{tr}}(\varrho, \rho), \tag{1}$$

where $D_{\mathrm{tr}}(\varrho, \rho) = \frac{1}{2}\mathrm{Tr}|\varrho - \rho|$ is the trace distance, with $\mathrm{Tr}|A| = \mathrm{Tr}\sqrt{AA^\dagger}$. The measure E_{tr} is motivated by having a natural physical interpretation: it stands in one-to-one correspondence with the largest success probability of discriminating between the state ρ and any possible separable state via a quantum measurement.

To this end, we observe that $D_{\mathrm{tr}}(\rho, \varrho) \geq \frac{1}{2}|\mathrm{Tr}((\varrho - \rho)\mathcal{W})| \geq -(w/2)$ if $-\mathbb{1} \leq \mathcal{W} \leq \mathbb{1}$. Naturally, there are many ways of satisfying the latter condition. A simple one is to divide $\mathcal{W}$ by its largest modulus eigenvalue denoted by $\lambda \equiv \max\{|\lambda_\pm|\}$, where $\lambda_\pm$ are, respectively, the largest and smallest eigenvalues of $\mathcal{W}$. However, an often better choice is to consider the following Hermitian operator:

$$\mathcal{W}' = \frac{2\mathcal{W} - (\lambda_+ + \lambda_-)\mathbb{1}}{\lambda_+ - \lambda_-} \equiv 2\mathcal{W}_c - \frac{(\lambda_+ + \lambda_-)\mathbb{1}}{\lambda_+ - \lambda_-}, \tag{2}$$

where the nontrivial part is denoted by $2\mathcal{W}_c \equiv [2\mathcal{W}/(\lambda_+ - \lambda_-)]$, and a smaller involved numerator $[(\lambda_+ - \lambda_-)/2]$ than λ would ensure a larger lower bound on E_{tr}, from which Theorem 1 follows.

Theorem 1 (quantitative normalized EW).—For an EW $\mathcal{W}$ that detects an entangled state ρ by a negative witness value, it holds

$$E_{\mathrm{tr}}(\rho) \geq -w_c, \tag{3}$$

where $w_c \equiv \mathrm{Tr}(\mathcal{W}_c\rho)$, and we have this theorem as $E_{\mathrm{tr}}(\rho) = \frac{1}{2}\mathrm{Tr}|\varrho_{\mathrm{opt}} - \rho| \geq \frac{1}{2}|\mathrm{Tr}[(\varrho_{\mathrm{opt}} - \rho)\mathcal{W}']| = |\mathrm{Tr}[(\varrho_{\mathrm{opt}} - \rho)\mathcal{W}_c]| \geq -w_c$ with ϱ_{opt} denoting the (closest) separable state thus $\mathrm{Tr}(\mathcal{W}_c\varrho_{\mathrm{opt}}) \geq 0$.

Thus, we can estimate $E_{\mathrm{tr}}(\rho)$ directly from the renormalized entanglement witness $\mathcal{W}_c$. In addition, we can use $E_{\mathrm{tr}}(\rho)$ itself to bound several other well-known entanglement quantifiers.

Focusing on other distance-based entanglement measures [32], prominent examples are the smallest relative entropy, E_{re}, or infidelity, E_{if}, between an entangled state and a separable state [32]. Other relevant examples are convex roof measures, such as the entanglement of formation [33], $\tilde{E}_{\mathrm{F}}$, the concurrence [34], $\tilde{E}_{\mathrm{C}}$, and the geometric measure of entanglement [35], $\tilde{E}_{\mathrm{G}}$. Also relevant are the robustness, E_{rob}, and generalized robustness, E_{ROB}, of entanglement [36]. The definitions of these measures are summarized in [37]. In [37] we show how all these quantifiers of entanglement can be bounded in terms of E_{tr}. Hence, they can be bounded directly from the observed value of the renormalized EW as

$$E_{\mathrm{re}}, \tilde{E}_{\mathrm{F}} \geq \frac{2}{\ln 2} w_c^2, \qquad E_{\mathrm{if}}, \tilde{E}_{\mathrm{G}} \geq w_c^2,$$
$$\tilde{E}_{\mathrm{C}} \geq -\sqrt{2}w_c, \qquad E_{\mathrm{rob,ROB}} \geq \frac{-w_c}{1 + w_c}. \tag{4}$$

Note that these relations, as well as Eq. (3), still remain if we replace the set of separable states with other relevant convex sets on which we can define analogous witness operators. A particularly interesting example is the set of states with a limited Schmidt number [40]. This would permit the quantification of an entangled state with respect to any other entangled state of a limited entanglement dimension.

We exemplify the result on the well-known fidelity witness [41–43] assuming the form $\mathcal{W} = c\hat{\mathbb{1}} - |\phi\rangle\langle\phi|$, where $c \equiv \max_{\varrho\in\Omega}\mathrm{Tr}(\varrho|\phi\rangle\langle\phi|)$. We have $\lambda_- = c - 1$ and $\lambda_+ = c$ and $\lambda_+ - \lambda_- = 1$, hence $\mathcal{W}_c = \mathcal{W}$. The magnitude of a negative witness value immediately gives a lower bound on E_{tr}. For example, let $|\phi\rangle$ be the maximum entangled state in d-dimensional space, namely, $\rho = |\phi\rangle\langle\phi|$ with $|\phi\rangle = (\sum_i |ii\rangle/\sqrt{d})$. We have $c = (1/d)$, the related witness reads $\mathcal{W} = (\mathbb{1}/d) - |\phi\rangle\langle\phi|$. This leads to the tight bound $E_{\mathrm{tr}}(\phi) \geq 1 - (1/d)$. To investigate the accuracy of the bounds on the other entanglement measures, we have computed the ratios, R, between the bounds (4) and the exact values. We find $R_{\mathrm{E_{tr}}} = R_{\mathrm{E_{rob,ROB}}} = 1$, $R_{\mathrm{E_{re}},\tilde{\mathrm{E}}_{\mathrm{F}}} = [2(d-1)^2/d^2 \ln 2 \log_2 d]$, $R_{\mathrm{E_{if}},\tilde{\mathrm{E}}_G} = 1$, $R_{\tilde{\mathrm{E}}_{\mathrm{C}}} = \sqrt{1 - (1/d)}$. Thus, some cases are tight (when the ratios $R = 1$) while others are proper lower bounds. Such an estimation immediately applies to the data from current experiments, for example, let $|\phi\rangle = \frac{1}{4}(|0000\rangle - |1001\rangle - |0110\rangle - |1111\rangle)$ be a four-qubit cluster state, for which $c = \frac{1}{2}$. With a measured fidelity between $|\phi\rangle$ and a concerned ρ as $\mathrm{Tr}(\rho|\phi\rangle\langle\phi|) = 0.85$ [44], one immediately obtains a lower bound on $E_{\mathrm{Tr}}(\rho) \geq 0.35$.

Measurement-device-independent scenario.—Every EW in the standard (or trusted) scenario can be converted into an EW in the MDI scenario [17,18]. Specifically, the original EW can always be decomposed into the form $\mathcal{W} = \sum_{s,t} \alpha_{s,t} \tau_s^{\mathrm{T}} \otimes \omega_t^{\mathrm{T}}$, with T denoting the transpose and

$\alpha_{s,t}$ being some real coefficients for some states τ_s and ω_t. In the MDI scenario, two parties, Alice and Bob, receive one share of ρ each and then individually prepare ancillary quantum states τ_s and ω_t, according to privately selected classical inputs s and t, respectively. Each of them performs an uncharacterized measurement $\{A_a\}$ and $\{B_b\}$ on the joint system-ancilla state. The correlations become $p(a,b|s,t) = \mathrm{Tr}[(A_a \otimes B_b)(\tau_s \otimes \rho_{AB} \otimes \omega_t)]$.

Note that $\sum_{s,t}\alpha_{s,t}p(a,b|s,t)=\mathrm{Tr}[(A_a\otimes B_b)(\mathcal{W}^{\mathrm{T}}\otimes\rho_{AB})]$, as shown in [18], if the first outcome in each measurement corresponds to a projection onto the maximally entangled state, the value of the MDI-EW $I_{\mathcal{W}}(\rho_{AB}) = \sum_{s,t}\alpha_{s,t}p(1,1|s,t)$ reduces to the value of the original EW, up to a positive constant. Hence, it follows that for any separable state, $I_{\mathcal{W}}(\rho_{AB}) \geq 0$, whereas $I_{\mathcal{W}}(\rho_{AB}) < 0$ implies entanglement. Thus, at the cost of introducing trusted sources for preparing τ_s and ω_t, entanglement can now be certified with uncharacterized measurements.

This initial MDI protocol, collecting the correlation for the first outcome in each side, is highly inefficient for estimating entanglement. We slightly optimize the protocol by collecting all pairs of outcomes (a,b) and calculate the quantity $\sum_{s,t}\alpha_{s,t}p(a,b|s,t) \equiv w_{a,b}(\rho_{AB})$ for each of them (the original MDI-EW $I_{\mathcal{W}} = w_{1,1}$). As one is free to label outcomes of measurements, if there exists one pair of outcome a, b such that $w_{a,b} < 0$. One can relabel (a,b) as (1,1), and entanglement is verified according to the original MDI-EW. By summing all the $w_{a,b} < 0$, we define a MDI witness value as

$$I'_{\mathcal{W}}(\rho_{AB}) = \sum_{a,b|w_{a,b}<0} w_{a,b}(\rho_{AB}).$$

We remark that $I'_{\mathcal{W}}$ is nonlinear since the picked out $w_{a,b}$s depend on the state ρ. It is clear that, for any separable state, $w_{a,b}(\rho_{AB}) \geq 0$ and $I'_{\mathcal{W}}(\rho_{AB}) = 0$.

We now show that the witness value $I'_{\mathcal{W}}(\rho_{AB})$ can be used to bound the entanglement measure $E_{\mathrm{tr}}(\rho)$. To this end, we observe that, for two Hermitian matrices M and N, $\mathrm{Tr}|M|\,\mathrm{Tr}|N| = \mathrm{Tr}|M \otimes N|$. Since $\mathrm{Tr}|\varrho_{\mathrm{opt}} - \rho| = (1/\mathrm{Tr}|\mathcal{W}^{\mathrm{T}}|)\mathrm{Tr}|\mathcal{W}^{\mathrm{T}} \otimes (\varrho_{\mathrm{opt}} - \rho)|$, then we have

$$\begin{aligned} 2\mathrm{Tr}|\mathcal{W}^{\mathrm{T}}|E_{\mathrm{tr}}(\rho) &= \mathrm{Tr}|\mathcal{W}^{\mathrm{T}} \otimes (\varrho_{\mathrm{opt}} - \rho)| \\ &\geq \sum_{a,b}|\mathrm{Tr}[\mathcal{W}^{\mathrm{T}} \otimes (\varrho_{\mathrm{opt}} - \rho)\cdot(A_a \otimes B_b)]| \\ &\geq |\sum_{a,b|w_{a,b}(\rho)<0}(w_{a,b}(\rho) - w_{a,b}(\varrho_{\mathrm{opt}}))| \\ &\quad + |\sum_{a,b|w_{a,b}(\rho)\geq 0}(w_{a,b}(\rho) - w_{a,b}(\varrho_{\mathrm{opt}}))| \\ &= \sum_{a,b|w_{a,b}(\rho)<0} 2[w_{a,b}(\varrho_{\mathrm{opt}}) - w_{a,b}(\rho)] \\ &\geq -2I'_{\mathcal{W}}(\rho), \end{aligned} \tag{5}$$

where ϱ_{opt} denotes the closest separable state, and in the second equality we have used $\sum_{a,b} w_{a,b}(\rho) = \sum_{a,b} w_{a,b}(\varrho_{\mathrm{opt}}) = \mathrm{Tr}(\mathcal{W}^{\mathrm{T}})$, which implies that $\sum_{a,b|w_{a,b}(\rho)<0}[w_{a,b}(\rho) - w_{a,b}(\varrho_{\mathrm{opt}})] = \sum_{a,b|w_{a,b}(\rho)\geq 0}[w_{a,b}(\rho) - w_{a,b}(\varrho_{\mathrm{opt}})]$. Then we arrive at

$$E_{\mathrm{tr}}(\rho) \geq -\frac{I'_{\mathcal{W}}(\rho)}{\mathrm{Tr}|\mathcal{W}^{\mathrm{T}}|}. \tag{6}$$

We consider the two-qubit Werner state $(1-v)\mathbb{1}/4 + v|\phi^-\rangle\langle\phi^-|$ with $(0 \leq v \leq 1)$ detected with EW $\mathcal{W} = \mathbb{1}/2 - |\phi^-\rangle\langle\phi^-|$ where $|\phi^-\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$, for which, $\mathrm{Tr}|\mathcal{W}^{\mathrm{T}}| = 2$. Here, we employ the Bell state measurement $\{|\phi^\pm\rangle\langle\phi^\pm|, |\psi^\pm\rangle\langle\psi^\pm|\}$, where $|\phi^\pm\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$ and $|\psi^\pm\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$. The state is entangled when $v > 1/3$, then we have $w_{a,b}(\rho) = [(1 - 3v)/16] < 0$ when $a = b$, otherwise, $w_{a,b}(\rho) = [(1+v)/16] > 0$. Thus we obtain an estimation $E_{\mathrm{tr}}(\rho_v) \geq -[(1-3v)/8]$, which is a half of the real value when $v = 1$. Estimations for other measures can be done via Eq. (4).

Device-independent scenario.—In the DI scenario, we make no assumptions on Alice's and Bob's devices. Consider n parties, called $\{\mathcal{A}_1, \cdots \mathcal{A}_n\}$. Party $\mathcal{A}_i$ privately selects an input s_i from a finite classical alphabet and performs a measurement $\{\mathcal{M}_{a_i|s_i}\}$ which has outcome a_i. The probability of obtaining outcomes $\mathbf{a} = \{a_1, \ldots, a_n\}$ when the inputs are $\mathbf{s} = \{s_1, \ldots, s_n\}$ is denoted $p(\mathbf{a}|\mathbf{s})$. A DI-EW, also known as a Bell inequality, when appropriately normalized for our purposes, takes the form

$$\mathrm{Tr}(\mathcal{W}_\beta \rho) < 0, \tag{7}$$

where $\mathcal{W}_\beta = \beta_c \mathbb{1} - \mathcal{B}$, and $\mathcal{B} = \sum_{\mathbf{a},\mathbf{s}} a_{\mathbf{a},\mathbf{s}} \hat{\mathcal{M}}_{a_1|s_1} \otimes \cdots \otimes \hat{\mathcal{M}}_{a_n|s_n}$ is Bell's quantity, and β_c is the maximum value of Bell's quantity over separable states and also the local-hidden-variable bound relevant to $\mathcal{B}$. We can now use the EW $\mathcal{W}_\beta$ to estimate DI entanglement. To this end, following an argument that closely parallels that outlined for trusted devices in Eqs. (2) and (3) leads to

$$E_{\mathrm{tr}}(\rho) \geq -\mathrm{Tr}(\rho \mathcal{W}_{\mathrm{cDI}}), \tag{8}$$

where we have defined the renormalized EW as

$$\mathcal{W}_{\mathrm{cDI}} = \frac{\mathcal{W}_\beta}{\langle\mathcal{B}\rangle_+ - \langle\mathcal{B}\rangle_-}, \tag{9}$$

where $\langle\mathcal{B}\rangle_\pm$ is the largest and smallest eigenvalues of the Bell operator $\mathcal{B}$ when evaluated over all possible measurements. In other words, it is the largest and smallest Bell values in quantum theory. In the case that these so-called Tsirelson bounds cannot be determined analytically, they can still be efficiently bounded by means of semidefinite programming [45].

As an illustration, let us consider the paradigmatic Clauser-Horne-Shimony-Holt inequality [46]. Its Bell operator, expressed in the two respective observables of Alice and Bob, reads $\mathcal{B}_{\rm chsh} = A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1$, for which $\beta_c = 2$. Our DI-EW becomes $\mathcal{W}_{\rm chsh} = 2\mathbb{1} - \mathcal{B}_{\rm chsh}$. Since the Tsirelson bounds $\langle\mathcal{B}\rangle_+ = -\langle\mathcal{B}\rangle_- = 2\sqrt{2}$, we have $\mathcal{W}_{\rm cDI-chsh} = (1/4\sqrt{2})\mathcal{W}_{\rm chsh}$. Hence, the estimated entanglement is at least

$$E_{\rm tr}(\rho) \geq \frac{\langle\mathcal{B}_{\rm chsh}\rangle_\rho - 2}{4\sqrt{2}}. \tag{10}$$

This lower bound is unfortunately not tight. For example, when $\langle\mathcal{B}_{\rm chsh}\rangle_\rho = 2\sqrt{2}$, the lower bound is $\frac{1}{2} - (\sqrt{2}/4)$, instead of the true value $\frac{1}{2}$.

To improve our bound, we can first rewrite a DI-EW as $[(\max_{\varrho\in\Omega}\langle\mathcal{B}\rangle_\varrho - \langle\mathcal{B}\rangle_\rho)/(\langle\mathcal{B}\rangle_+ - \langle\mathcal{B}\rangle_-)]$. A maximal CHSH violation, $\langle\mathcal{B}_{\rm chsh}\rangle_\rho = 2\sqrt{2}$, is well-known to self-test anti-communting qubit measurements for both Alice and Bob. For such measurements, there are well-known uncertainty relations $\langle A_0\rangle^2 + \langle A_1\rangle^2 \leq 1$ and $\langle B_0\rangle^2 + \langle B_1\rangle^2 \leq 1$. These relations imply a bound on $\max_{\varrho\in\Omega}\langle\mathcal{B}\rangle_\varrho$. Specifically, for a product state $\varrho = \varrho_A \otimes \varrho_B$, one has the correlations as $\langle A_iB_j\rangle_\varrho = \langle A_i\rangle_{\varrho_A}\langle B_j\rangle_{\varrho_B}$ and $\langle\mathcal{B}_{\rm chsh}\rangle_\varrho = \sum_{i,j=0,1}(-1)^{ij}\langle A_i\rangle_{\varrho_A}\langle B_j\rangle_{\varrho_B}$. The uncertainty relations then imply $\max_{\varrho\in\Omega}\langle\mathcal{B}_{\rm chsh}\rangle_\varrho = \sqrt{2}$, which is strictly less than the classical bound $\beta_c = 2$ involved in Eq. (10), thus improving the lower bound in Eq. (10) from $\frac{1}{2} - (\sqrt{2}/4)$ to $\frac{1}{4}$.

Estimating entanglement of k-depth.—We apply our approach also to quantify the depth of entanglement in multipartite systems. A multipartite pure state $|\Psi\rangle$ is said to be k-producible if the subsystems can be partitioned into m pairwise disjoint and nonempty subsets, denoted by $\{\mathcal{A}_1, \cdots \mathcal{A}_m\}$, such that state $|\Psi\rangle$ can be written as a product as $|\psi_1\rangle_{\mathcal{A}_1} \otimes \cdots \otimes |\psi_m\rangle_{\mathcal{A}_m}$ and each set $\mathcal{A}_i$ contains at k elements. Similarly, a mixed state is called k-producible if it can be decomposed into a mixture of k-producible pure states. If a state is not k-producible, it is said that its entanglement depth is at least $k+1$. We write $\mathcal{W}_{\mathcal{P}_k}$ for a witness of k-depth entanglement; it is an observable such that the expectation value $w_{c,\mathcal{P}_k} \equiv \mathrm{Tr}(\mathcal{W}_{\mathcal{P}_k}\rho) \geq 0$. If ρ is k-producible state and $w_{c,\mathcal{P}_k} < 0$ for some state with entanglement depth more than k.

Let $\Omega_{\mathcal{P}_k}$ be the set of all states with entanglement depth at most k. In analogy with the bipartite case, we consider the distinguishability between a given state ρ, and any state in $\Omega_{\mathcal{P}_k}$, namely

$$E_{{\rm tr};\mathcal{P}_k}(\rho) \equiv \min_{\varrho\in\Omega_{\mathcal{P}_k}} D_{\rm tr}(\rho, \varrho) \geq -w_{c,\mathcal{P}_k}, \tag{11}$$

which can be estimated immediately with some known multipartite entanglement witnesses. Two cases are provided in what follows.

We exemplify this for a standard entanglement witness tailored for a noisy tripartite W state $\rho = (v\mathbb{1}/8) + (1-v)|\Psi\rangle\langle\Psi|$ where $|\Psi\rangle = (1/\sqrt{3})(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$. Standard multipartite EWs can be constructed by solving the so-called multipartite separability eigenvalue equations [47,48]. In Ref. [47] examples of EWs of k-producible entanglement for this state are given as $\mathcal{W}_{\mathcal{P}_1} = \frac{4}{9}\cdot\mathbb{1} - |\Psi\rangle\langle\Psi|$, and $\mathcal{W}_{\mathcal{P}_2} = \frac{2}{3}\cdot\mathbb{1} - |\Psi\rangle\langle\Psi|$. For both of them, $\lambda_+ - \lambda_- = 1$. We have $w_{c,\mathcal{P}_1} = \frac{4}{9} - (7v/8)$ and $w_{c,\mathcal{P}_2} = \frac{2}{3} - (7v/8)$. Entanglement is certified when $v \geq (40/63)$ as $w_{c,\mathcal{P}_1} < 0$, and entanglement of depth three is certified when $v > (8/21)$. From our approach, we get a also quantitative statement. For instance when $v = 1$, we have $E_{{\rm tr};\mathcal{P}_1}(\rho) \geq (31/72)$ and $E_{{\rm tr};\mathcal{P}_2}(\rho) \geq (5/24)$.

Next, we consider the multipartite scenario in a device-independent setting. Multipartite DI-EWs are designed by exploiting the fact that quantum state with a deeper entanglement depth can violate some Bell's inequalities to a larger extent [19,21,49–52]. One typical such Bell's inequality is the Svetlichny inequality that reads [49,53]

$$\mathcal{B}_S^{(n)} = 2^{-\frac{n}{2}}\sum_{\mathbf{a},\mathbf{s}}(-1)^{a+\lfloor\frac{s}{2}\rfloor}\hat{\mathcal{M}}_{a_1|s_1} \otimes \cdots \otimes \hat{\mathcal{M}}_{a_n|s_n}, \tag{12}$$

where $s = \sum_i s_i$ and $a = \sum_i a_i$ are the sum of measurement setting $s_i \in \{0,1\}$ and outcome $a_i \in \{0,1\}$. The maximum expectation value of $\mathcal{B}_S^{(n)}$ with respect to the set of k-producible states is [54] $\beta_k \equiv 2^{(n-2\lfloor\lceil n/k\rceil/2\rfloor)/2}$, and the maximum and minimum values for an arbitrary quantum state are $\langle\mathcal{B}_S^{(n)}\rangle_+ = -\langle\mathcal{B}_S^{(n)}\rangle_- = 2^{(n-1)/2}$. Thus, the renormalized multipartite DI-EW of k-producible states reads

$$\mathcal{W}^{(n)}_{{\rm cDI};\mathcal{P}_k} \coloneqq \frac{\mathcal{W}^n_{{\rm DI},\mathcal{P}_k}}{2^{(n+1)/2}}, \qquad \mathcal{W}^n_{{\rm DI},\mathcal{P}_k} \coloneqq \beta_k\cdot\mathbb{1} - \mathcal{B}_S^{(n)}, \tag{13}$$

from which we have the quantitative estimation

$$E^{(n)}_{{\rm tr};\mathcal{P}_k} \geq -w^{(n)}_{{\rm cDI};\mathcal{P}_k} = -\mathrm{Tr}(\rho\mathcal{W}^{(n)}_{{\rm cDI};\mathcal{P}_k}). \tag{14}$$

As a simple illustration of Eq. (14), consider the correlation scoring $\langle\mathcal{B}_S^{(n)}\rangle = \langle\mathcal{B}_S^{(n)}\rangle_+$, which is attained by the GHZ state. The lower bound for the entanglement quantifier with respect to states of depth more than k becomes

$$E^{(n)}_{{\rm tr};\mathcal{P}_k} \geq \frac{1}{2}\left(1 - 2^{-2\lfloor\frac{\lceil n/k\rceil}{2}\rfloor}\right). \tag{15}$$

This lower bound asymptotically tends to the exact value for the GHZ state as $n/k \to \infty$ and tends to zero as $n/k \to 1$.

In conclusion, we have provided a simple method to estimate entanglement in the standard, MDI and DI scenarios by renormalizing witness operators. The method is tailored to the operational distinguishability measure, but it also yields nontrivial bounds on many of the well-known entanglement measures used in quantum information. This permits an experimenter to not only detect different forms of entanglement but also to quantify it, without requiring any additional measurements than those used in a standard witness-based detection scheme. Hence, this may be seen as an enhanced data analysis, which is also practically straightforward to use since our procedure requires essentially no optimization but only simple analytical expressions. Finally, we note that our approach might be interesting to consider also in the context of estimating quantum properties in other scenarios, such as steering, quantum coherence, and entanglement-assisted quantum communication.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[2] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. V. den Nest, Measurement-based quantum computation, Nat. Phys. **5**, 19 (2009).

[3] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science **306**, 1330 (2004).

[5] L. Gurvits, Classical complexity and quantum entanglement, J. Comput. Syst. Sci. **69**, 448 (2004), special Issue on STOC 2003.

[6] L. Gurvits, Classical deterministic complexity of Edmonds' problem and quantum entanglement, arXiv:quant-ph/0303055.

[7] Y. Huang, Computing quantum discord is NP-complete, New J. Phys. **16**, 033027 (2014).

[8] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[9] N. Friis, G. Vitagliano, M. Malik, and M. Huber, Entanglement certification—from theory to experiment, Nat. Rev. Phys. **1**, 72 (2019).

[10] A. Tavakoli, A. Pozas-Kerstjens, P. Brown, and M. Araújo, Semidefinite programming relaxations for quantum correlations, arXiv:2307.02551.

[11] G. Tóth and O. Gühne, Detecting genuine multipartite entanglement with two local measurements, Phys. Rev. Lett. **94**, 060501 (2005).

[12] S. Morelli, M. Huber, and A. Tavakoli, Resource-efficient high-dimensional entanglement detection via symmetric projections, Phys. Rev. Lett. **131**, 170201 (2023).

[13] P. Liu, Z. Liu, S. Chen, and X. Ma, Fundamental limitation on the detectability of entanglement, Phys. Rev. Lett. **129**, 230503 (2022).

[14] B. M. Terhal, Bell inequalities and the separability criterion, Phys. Lett. A **271**, 319 (2000).

[15] D. Rosset, R. Ferretti-Schöbitz, J.-D. Bancal, N. Gisin, and Y.-C. Liang, Imperfect measurement settings: Implications for quantum state tomography and entanglement witnesses, Phys. Rev. A **86**, 062325 (2012).

[16] S. Morelli, H. Yamasaki, M. Huber, and A. Tavakoli, Entanglement detection with imprecise measurements, Phys. Rev. Lett. **128**, 250501 (2022).

[17] F. Buscemi, All entangled quantum states are nonlocal, Phys. Rev. Lett. **108**, 200401 (2012).

[18] C. Branciard, D. Rosset, Y.-C. Liang, and N. Gisin, Measurement-device-independent entanglement witnesses for all entangled quantum states, Phys. Rev. Lett. **110**, 060405 (2013).

[19] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Device-independent witnesses of genuine multipartite entanglement, Phys. Rev. Lett. **106**, 250404 (2011).

[20] F. Baccari, D. Cavalcanti, P. Wittek, and A. Acín, Efficient device-independent entanglement detection for multipartite systems, Phys. Rev. X **7**, 021042 (2017).

[21] Y.-C. Liang, D. Rosset, J.-D. Bancal, G. Pütz, T. J. Barnea, and N. Gisin, Family of Bell-like inequalities as device-independent witnesses for entanglement depth, Phys. Rev. Lett. **114**, 190401 (2015).

[22] D. Cavalcanti and M. O. Terra Cunha, Estimating entanglement of unknown states, Appl. Phys. Lett. **89**, 084102 (2006).

[23] F. G. S. L. Brandão, Quantifying entanglement with witness operators, Phys. Rev. A **72**, 022310 (2005).

[24] M. Huber and J. I. de Vicente, Structure of multidimensional entanglement in multipartite systems, Phys. Rev. Lett. **110**, 030501 (2013).

[25] J. Eisert, F. G. S. L. Brandão, and K. M. R. Audenaert, Quantitative entanglement witnesses, New J. Phys. **9**, 46 (2007).

[26] O. Gühne, M. Reimpell, and R. F. Werner, Estimating entanglement measures in experiments, Phys. Rev. Lett. **98**, 110502 (2007).

[27] F. Shahandeh, M. J. W. Hall, and T. C. Ralph, Measurement-device-independent approach to entanglement measures, Phys. Rev. Lett. **118**, 150505 (2017).

[28] D. Rosset, A. Martin, E. Verbanis, C. C. W. Lim, and R. Thew, Practical measurement-device-independent entanglement quantification, Phys. Rev. A **98**, 052332 (2018).

[29] E. Verbanis, A. Martin, D. Rosset, C. C. W. Lim, R. T. Thew, and H. Zbinden, Resource-efficient measurement-device-independent entanglement witness, Phys. Rev. Lett. **116**, 190501 (2016).

[30] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Device-independent entanglement quantification and related applications, Phys. Rev. Lett. **111**, 030501 (2013).

[31] D. Chruściński and G. Sarbicki, Entanglement witnesses: Construction, analysis and classification, J. Phys. A **47**, 483001 (2014).

[32] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying entanglement, Phys. Rev. Lett. **78**, 2275 (1997).

[33] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).

[34] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[35] T.-C. Wei and P. M. Goldbart, Geometric measure of entanglement and applications to bipartite and multipartite quantum states, Phys. Rev. A **68**, 042307 (2003).

[36] G. Vidal and R. Tarrach, Robustness of entanglement, Phys. Rev. A **59**, 141 (1999).

[37] See Supplemental Materials at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.110204 for bounding the amount of entanglement from witness operator, which includes Refs. [38,39].

[38] A. Streltsov, Geometric measure of entanglement compared to measures based on fidelity, arXiv:0911.1796.

[39] D. Shapira, Y. Shimoni, and O. Biham, Groverian measure of entanglement for mixed states, Phys. Rev. A **73**, 044301 (2006).

[40] B. M. Terhal and P. Horodecki, Schmidt number for density matrices, Phys. Rev. A **61**, 040301(R) (2000).

[41] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Experimental detection of multipartite entanglement using witness operators, Phys. Rev. Lett. **92**, 087902 (2004).

[42] N. J. Cerf, C. Adami, and R. M. Gingrich, Reduction criterion for separability, Phys. Rev. A **60**, 898 (1999).

[43] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, Phys. Rev. A **59**, 4206 (1999).

[44] B. P. Lanyon, M. Zwerger, P. Jurcevic, C. Hempel, W. Dür, H. J. Briegel, R. Blatt, and C. F. Roos, Experimental violation of multipartite Bell inequalities with trapped ions, Phys. Rev. Lett. **112**, 100403 (2014).

[45] M. Navascués, S. Pironio, and A. Acín, Bounding the set of quantum correlations, Phys. Rev. Lett. **98**, 010401 (2007).

[46] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. **23**, 880 (1969).

[47] J. Sperling and W. Vogel, Multipartite entanglement witnesses, Phys. Rev. Lett. **111**, 110503 (2013).

[48] S. Gerke, W. Vogel, and J. Sperling, Numerical construction of multipartite entanglement witnesses, Phys. Rev. X **8**, 031047 (2018).

[49] G. Svetlichny, Distinguishing three-body from two-body nonseparability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[50] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-type inequalities to detect true n-body nonseparability, Phys. Rev. Lett. **88**, 170405 (2002).

[51] J.-D. Bancal, N. Brunner, N. Gisin, and Y.-C. Liang, Detecting genuine multipartite quantum nonlocality: A simple approach and generalization to arbitrary dimensions, Phys. Rev. Lett. **106**, 020405 (2011).

[52] S. Yu, Z.-B. Chen, J.-W. Pan, and Y.-D. Zhang, Classifying n-qubit entanglement via Bell's inequalities, Phys. Rev. Lett. **90**, 080401 (2003).

[53] J. T. Barreiro, J.-D. Bancal, P. Schindler, D. Nigg, M. Hennrich, T. Monz, N. Gisin, and R. Blatt, Demonstration of genuine multipartite entanglement with device-independent witnesses, Nat. Phys. **9**, 559 (2013).

[54] F. Baccari, J. Tura, M. Fadel, A. Aloy, J.-D. Bancal, N. Sangouard, M. Lewenstein, A. Acín, and R. Augusiak, Bell correlation depth in many-body systems, Phys. Rev. A **100**, 022121 (2019).