Dynamical Resource Theory of Informational Nonequilibrium Preservability

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Information is instrumental in our understanding of thermodynamics. Their interplay has been studied through completely degenerate Hamiltonians whereby the informational contributions to thermodynamic transformations can be isolated. In this setting, all states other than the maximally mixed state are considered to be in informational nonequilibrium. An important yet still open question is how to characterize the ability of quantum dynamics to preserve informational nonequilibrium. Here, the dynamical resource theory of informational nonequilibrium preservability is introduced to begin providing an answer to this question. A characterization of the allowed operations is given for qubit channels and the n-dimensional Weyl-covariant channels—a physically relevant subset of the general channels. An operational interpretation of a state discrimination game with Bell state measurements is given. Finally, an explicit link between a channel's classical capacity and its ability to preserve informational nonequilibrium is made.

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Introduction.—Resources are precious. Their value arises from their limitation, incentivizing them to be efficiently utilized and maintained. Formally, an object is considered to be a resource if it can be used by some agent to overcome the physical constraints of a system. A resource therefore allows an agent to obtain an otherwise impossible advantage.

Within the quantum regime, the study of the limitations experienced by such an agent is the central aim of quantum resource theories. They provide a rigorous mathematical framework with which the ability of quantum objects to supply an operational advantage—subject to some physical constraints—can be compared [1]. Thus far, they have proved a fruitful way to approach problems concerning a number of different quantum phenomena, with resource theories of entanglement [2], athermality [3–5], and measurement, [6] to name a few.

Initially, static resource theories (SRTs) were the sole focus. In such SRTs, the primary objects in question are quantum states, and the resource is some property of these states. There is now an effort to expand resource theories beyond states to the dynamical regime [7–16], with the aim to assess and compare the ability of different quantum operations to be resourceful, therefore providing some advantage. Within these dynamical resource theories (DRTs), the resourceful objects are quantum operations with some property of the quantum operation being considered a resource.

One such important property is the ability of operations to preserve the static resource present in the state upon which they act. These "resource preservability theories," introduced in [10], are built upon SRTs and apply structure to their set of allowed operations. In particular, not all allowed operations are equal. Some operations will completely preserve the resource, while others will completely destroy it. How well an allowed operation preserves the static resource can itself be considered a type of dynamic resource, and this is the focus of such DRTs.

Note that while intimately connected, the resource considered in the DRT and underlying SRT are fundamentally different and apply to different classes of objects—quantum channels and quantum states, respectively.

The desire for resource preservation arises naturally in any resource theory. Specifically, in the context of thermodynamics, it arises due to nonequilibrium states being considered a resource. For a given state, both its energy and the information an agent has about it will determine how resourceful it is [17]. The contribution arising from information can be studied in isolation by considering trivial Hamiltonians, where all energy levels are degenerate. In this regime, any thermodynamic transformation that occurs must arise solely from an information theoretic origin, given there can be no change in energy. The maximally mixed state is the thermal state in this picture, and all other states are considered to be in "informational nonequilibrium." An understanding of this specific case can then be used to infer results about the general case, in which energetic considerations are necessary.

The study of informational nonequilibrium has been succinctly formalized through the resource theory of informational nonequilibrium [18–20]. Such SRTs often question the existence of transformations between states without any concern for the details of said transformations. This initially led to our understanding of thermodynamics having little to do with dynamics [5]; however, insight has since been gained into the dynamical aspects of thermodynamic resource theories. For example, the largest set of feasible dynamics has been investigated [21], and their consistency with the laws of thermodynamics [22] and practically relevant constraints assessed [23,24]; their behavior in the macroscopic limit has been considered [25], and their analog in the continuous variable regime studied [26,27]. Despite this, there has been no insight into how to describe the ability of a given dynamics to preserve informational nonequilibrium. Understanding what properties prevent thermalization will not only enrich our foundational understanding of thermodynamics but also shed further light on the role of thermodynamics in information processing and transmission. In addition, most quantum dynamics used in practical applications, such as quantum computation [28] and quantum memories [29], benefit from this lack of thermalization, which is a prevalent type of decoherence. An improved understanding of these dynamics could therefore further our control over these systems.

Within this Letter, we define and characterize the dynamical resource theory of informational nonequilibrium preservability to begin addressing this question. The informational nonequilibrium of a state will henceforth be referred to as the static resource to distinguish it from the dynamic resource.

The resource theory of informational nonequilibrium.— The set of free states of a resource theory are those states that contain no resource. In the SRT of informational nonequilibrium, the only free state is the maximally mixed state, $\mathbb{I}/d_S$. Hence, the set of free states is $\{\Upsilon_S := \mathbb{I}_S/d_S\}$, where the subscript labels the subsystem and d is the dimension.

All dynamics considered are quantum channels: completely positive trace-preserving linear maps. In a resource theory, dynamics is captured by the allowed operations. These are resource nongenerating physical manipulations that can be applied arbitrarily many times at no cost. Within the resource theory of informational nonequilibrium, these are taken to be the set of "noisy operations," $\mathfrak{O}^{\mathrm{st}}$, given by

$$\mathcal{D}(\rho_S) = \mathrm{Tr}_{\mathrm{E}'}[U_{\mathrm{SE}}(\rho_{\mathrm{S}} \otimes \Upsilon_{\mathrm{E}})U_{\mathrm{SE}}^{\dagger}] \in \mathfrak{O}^{\mathrm{st}}, \tag{1}$$

where U_{SE} is a unitary operator on a joint system SE [30]. Physically, this means coupling the system S with a bath E in informational equilibrium, evolving the joint system under a global unitary, and then discarding some part of the joint system and environment, E′. The resource theory of informational nonequilibrium will be referred to as $\mathcal{R}^{\mathrm{st}}$.

The existence of a noisy operation between two states ρ and σ is equivalent to the existence of a "unital" channel, $\mathcal{N}$, such that $\sigma = \mathcal{N}(\rho)$ (a unital channel leaves the identity invariant) [19]. If ρ and σ are of equal dimension, this unital channel is a "mixed unitary channel."

The dynamical resource theory of informational nonequilibrium preservability.—Any SRT induces a DRT of static resource preservability [10]. The DRT of informational nonequilibrium is built upon $\mathcal{R}^{\mathrm{st}}$. It aims to apply structure to the set of noisy operations based on the channel's ability to preserve nonequilibrium. Formally, a noisy operation $\mathcal{N} \in \mathfrak{O}^{\mathrm{st}}$ is considered resourceful if it can output some nonequilibrium state, $\mathcal{N}(\rho) \neq \Upsilon_{\mathrm{S}}$ for some ρ. The free objects of this resource theory are those channels that remove all of the resource from states upon which they act. In this case, since the free set of the SRT is a single state, the only free channel in the DRT is the state preparation channel of Υ_{S}, $\Lambda(\cdot) := \mathrm{Tr}(\cdot)\Upsilon_{\mathrm{S}}$.

Allowed operations of a dynamical resource theory are "superchannels"—linear mappings of quantum channels to quantum channels—that can only decrease the resource content of the channel [31]. Superchannels can be realized in the lab with a preprocessing and postprocessing channel connected to a memory system. Using this form and the structure of resource nongenerating superchannels presented in [10], superchannels in the free set, $\mathfrak{O}^{\mathrm{dyn}}$, have the following form:

$$\Pi(\mathcal{N}) = \sum_{\kappa} p_{\kappa}\mathcal{E}_{\kappa} \circ \mathcal{N} \circ \mathcal{P}_{\kappa} \in \mathfrak{O}^{\mathrm{dyn}}, \tag{2}$$

where shared randomness, described by the random variable κ and a probability distribution p_{κ}, is permitted between the preprocessing and postprocessing noisy operations $\mathcal{P}_{\kappa}$, $\mathcal{E}_{\kappa}$. It can be verified that these superchannels map noisy operations to noisy operations, with Λ only being mapped to itself. Further details on the allowed superchannels are given in Supplemental Material A [32].

The physical reasoning behind this set of superchannels being allowed is that all parts come from the allowed operations of $\mathcal{R}^{\mathrm{st}}$. Given that noisy operations only increase informational nonequilibrium, the output states of $\Pi(\mathcal{N})$ cannot contain any more static resource than the output states of $\mathcal{N}$. Superchannels in $\mathfrak{O}^{\mathrm{dyn}}$ are therefore dynamic-resource nonincreasing, as $\Pi(\mathcal{N})$ cannot preserve more informational nonequilibrium present in any state upon which it acts than $\mathcal{N}$. The dynamical resource theory of informational nonequilibrium preservability will be referred to as $\mathcal{R}^{\mathrm{dyn}}$.

All resource theories aim to quantify how resourceful a given object is. Since allowed operations cannot generate any resource, we can understand this by studying which objects can be converted into each other. In $\mathcal{R}^{\rm st}$, this characterization takes the following form: given two quantum states ρ and σ, does there exist a noisy operation mapping $\rho \rightarrow \sigma$? Answering this establishes a preorder on the set of all states based on the existence of a noisy operation between them. This preorder then allows the state's resource content to be compared. For the SRT, this convertibility question has been answered and is simple, being captured entirely by the mathematical notion of majorization [19]. Answering it for $\mathcal{R}^{\rm dyn}$ is the aim of this Letter. This will establish a preorder on the set of channels $\mathfrak{O}^{\rm st}$ based on the existence of an allowed superchannel between them. Given these are resource nonincreasing, if there exists an allowed superchannel mapping $\mathcal{N} \rightarrow \mathcal{M}$, where $\mathcal{M}, \mathcal{N} \in \mathfrak{O}^{\rm st}$, it will be known that $\mathcal{M}(\rho)$ will be closer to Υ_S then $\mathcal{N}(\rho)$ for all ρ. In other words, $\mathcal{M}$ will not preserve informational nonequilibrium better than $\mathcal{N}$.

Characterizing allowed superchannels by a Choi-state representation.—To answer the convertibility question, the Choi-Jamiołkowski isomorphism will be employed. This is a linear mapping between a quantum channel $\mathcal{N}$ and a bipartite quantum state $\mathcal{J}^{\mathcal{N}}$, given by

$$\mathcal{J}^{\mathcal{N}} = (\mathcal{I} \otimes \mathcal{N})(|\Phi\rangle\langle\Phi|), \tag{3}$$

where $|\Phi\rangle \coloneqq (1/\sqrt{d}) \sum_i |ii\rangle$. Specifically, as proven in Supplemental Material A [32], if there exists an allowed operation $\Pi \in \mathfrak{O}^{\rm dyn}$, so that $\mathcal{M} = \Pi(\mathcal{N})$ in the form of Eq. (2), for two noisy operations $\mathcal{M}$ and $\mathcal{N}$, then the corresponding Choi states are related as

$$\mathcal{J}^{\mathcal{M}} = \sum_{\kappa} p_{\kappa} (\mathcal{P}_{\kappa} \otimes \mathcal{E}_{\kappa})(\mathcal{J}^{\mathcal{N}}). \tag{4}$$

In what follows, the dynamics of a quantum system of a constant finite dimension are considered. In this case, how far a state is from informational nonequilibrium is equivalent to asking how pure that state is [19]; see the Appendix A for more details. As mentioned above, under this restriction $\mathfrak{O}^{\rm st} = \mathfrak{U}_M$, where $\mathfrak{U}_M$ is the set of mixed unitary channels.

Characterizing allowed operations for qubit systems.—Initially, qubit channels are focused on, such that our resource theory concerns qubit mixed unitary channels.

A characterization of $\mathcal{R}^{\rm dyn}$ looks to answer the following question: *given two channels $\mathcal{M}, \mathcal{N} \in \mathfrak{O}^{\rm st}$, does there exist an allowed operation $\Pi \in \mathfrak{O}^{\rm dyn}$ such that $\mathcal{M} = \Pi(\mathcal{N})$*? Using the form of allowed operations given in Eq. (4), the above question equivalently asks if there exists some convex combination of channels $\mathcal{P}_{\kappa}, \mathcal{E}_{\kappa} \in \mathfrak{O}^{\rm st}$ that act on the subsystems of $\mathcal{J}^{\mathcal{N}}$ respectively to map $\mathcal{J}^{\mathcal{N}} \rightarrow \mathcal{J}^{\mathcal{M}}$. It is shown through the following result that this depends only upon the eigenvalues of the corresponding Choi states. In what follows, $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are vectors of eigenvalues of the Choi states $\mathcal{J}^{\mathcal{M}}$ and $\mathcal{J}^{\mathcal{N}}$, respectively.

Result 1: For every given $\mathcal{M}, \mathcal{N} \in \mathfrak{O}^{\rm st}$, the following statements are equivalent: (1) $\mathcal{M} = \Pi(\mathcal{N})$ for some allowed operation $\Pi \in \mathfrak{O}^{\rm dyn}$, and (2) $\boldsymbol{\mu} = \mathbb{D}\boldsymbol{\lambda}$, where $\mathbb{D} = \sum_{n,m} p_{nm} \sigma_x^n \otimes \sigma_x^m$ with p_{nm} the elements of a probability vector and σ_x is the Pauli X.

Result 1 states that an allowed operation of $\mathcal{R}^{\rm dyn}$ exists between $\mathcal{N}$ and $\mathcal{M}$ if and only if the vector of eigenvalues, $\boldsymbol{\mu}$, of the Choi state of the channel $\mathcal{M}$ is in the convex hull of local permutations of the vector of eigenvalues, $\boldsymbol{\lambda}$, of the Choi state of the channel $\mathcal{N}$. This is reminiscent of the analogous result for $\mathcal{R}^{\rm st}$ [19] since majorization is equivalent to the existence of a doubly stochastic matrix mapping the eigenvalues of ρ and σ [38]. Here, in contrast, we do not have all doubly stochastic matrices but only a subset, the local permutations. A sketch of the proof of Result 1 is provided in the Appendix B below, and a complete derivation can be found in Supplemental Material B [32].

Mathematically, the question of the existence of an allowed operation between two channels can now be simplified to a convex optimization problem that can be easily solved on conventional computing hardware. Physically, this allows a comparison of the ability of the allowed operations of $\mathcal{R}^{\rm st}$ to preserve purity.

As an application of Result 1, we consider the existence of an allowed operation between the depolarizing, $\mathcal{D}_s^{\rm pol}$, and dephasing, $\mathcal{D}_q^{\rm ph}$, channels, parametrized by s and q, respectively (if s, $q = 1$ they are the identity channel; if s, $q = 0$ they are the completely depolarizing or dephasing channel). It is found that there exists an allowed operation of $\mathcal{R}^{\rm dyn}$ mapping $\mathcal{D}_s^{\rm pol} \rightarrow \mathcal{D}_{s'}^{\rm pol}$ if and only if $s' \leq s$, and similarly, $\mathcal{D}_q^{\rm ph} \rightarrow \mathcal{D}_{q'}^{\rm ph}$ if and only if $q' \leq q$. This confirms the obvious conclusion that increasing the probability of depolarizing or dephasing decreases the amount of preserved purity. In addition, no allowed operation exists that maps $\mathcal{D}_s^{\rm pol} \rightarrow \mathcal{D}_q^{\rm ph}$ except in the trivial case ($s = 1$), as expected. However, there does exist an allowed operation mapping $\mathcal{D}_q^{\rm ph} \rightarrow \mathcal{D}_s^{\rm pol}$ if and only if $s \leq q/(2 - q)$. See Supplemental Material B [32] for the details.

A complete set of monotones.—A monotone of a resource theory is a function $M(\cdot)$ such that $M[\mathcal{N}(\rho)] \leq M(\rho)$, where $\mathcal{N} \in \mathfrak{O}^{\rm st}$. Although, it may be the case that $M(\sigma) \leq M(\rho)$ even if no allowed operation exists between ρ and σ. *A complete set of monotones*, $\{M_i\}$, does, however, provide an alternative to the convertibility question, with an allowed operation existing such that $\rho \rightarrow \sigma$ if and only if $M_i(\sigma) \leq M_i(\rho)\ \forall\ i$. The characterization given in Result 1 admits a geometrical interpretation of an allowed operation through which a complete set of monotones can be found.

The set of local permutations of $\boldsymbol{\lambda}$ is $\mathfrak{L} = \{\mathbb{P}_i \boldsymbol{\lambda}\colon\ \forall\ \mathbb{P}_i \in \mathfrak{P}\}$, where $\mathfrak{P}$ is the set of local qubit

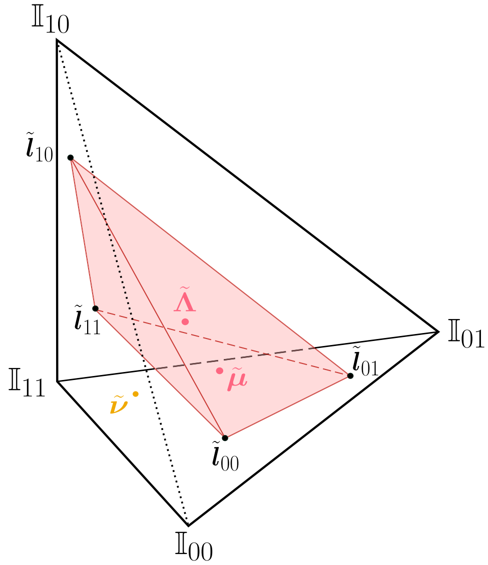

FIG. 1. A simplex in $\mathbb{R}^3$ where each point represents the Choi state of a qubit unital channel that is diagonal in the Bell basis. $\mathbb{I}_{ij}$, the extreme points of the outer simplex, are the identity channel and channels that are equivalent to the identity channel up to a Pauli operator. The set of vectors $\tilde{\mathfrak{L}}$ is plotted as points, $\tilde{\boldsymbol{l}}_{ij}$, where each point represents a local permutation of $\boldsymbol{\lambda}$. The convex hull of these vectors is shaded red with the free channel, given by $\tilde{\boldsymbol{\Lambda}}$, at the center. $\tilde{\boldsymbol{\mu}}$ sits within this convex hull, and hence, an allowed operation exists between the channel $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. $\tilde{\boldsymbol{\nu}}$ sits outside, and hence, no allowed operation exists between the channel $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$.

permutations. Result 1 can be rephrased, stating that a matrix $\mathbb{D}$ exists, such that $\boldsymbol{\mu} = \mathbb{D}\boldsymbol{\lambda}$, if $\boldsymbol{\mu}$ is in the convex hull of vectors in $\mathfrak{L}$. Each element of $\mathfrak{L}$ lives in a three-dimensional subspace of $\mathbb{R}^4$ due to normalization. The vectors $\mathfrak{L}$ within this lower dimensional subspace are given by $\tilde{\mathfrak{L}}$. Geometrically, the elements of $\tilde{\mathfrak{L}}$ form a simplex, as depicted in Fig. 1. Each point represents a qubit unital channel diagonal in the Bell basis. The points of the outer simplex are the identity channel and channels that are equivalent to the identity up to a Pauli operator. From these channels, it is possible to reach all other channels under allowed operations, as their convex hull is the whole space. The center of the simplex, $\tilde{\boldsymbol{\Lambda}}$, is the free state preparation channel, Λ. This channel can be reached by any channel under allowed operations.

Using this geometrical representation, a complete set of monotones can readily be identified. In what follows, we consider all vectors in $\tilde{\mathfrak{L}}$ to be linearly independent, which is the general case [39]. For completeness, all cases are considered in Supplemental Material C [32].

Result 2: There exists a set of $d_S^2 = 4$ linear inequalities that are a complete set of monotones of $\mathcal{R}^{\rm dyn}$ if $\tilde{\mathfrak{L}}$ is a set of linearly independent vectors.

With the vectors in $\tilde{\mathfrak{L}}$ being linearly independent, their convex hull forms a tetrahedron in $\mathbb{R}^3$. If the three-dimensional representation of the channel $\boldsymbol{\mu}$, given by $\tilde{\boldsymbol{\mu}}$, lies inside this tetrahedron, it is then in the convex hull of the vectors in $\tilde{\mathfrak{L}}$. This is equivalent to $\boldsymbol{\mu}$ being in the convex hull of the vectors in $\mathfrak{L}$ and hence an allowed operation $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ existing.

Each monotone then checks on what side of the planes that encompass each of the four faces of the tetrahedron $\tilde{\boldsymbol{\mu}}$ lies on. If for all four planes $\tilde{\boldsymbol{\mu}}$ lies on the same side as $\tilde{\boldsymbol{\Lambda}}$, it must be inside the tetrahedron, and an allowed operation exists.

Characterizing allowed operations for arbitrary finite dimensional systems.—Result 1 is extended to higher dimensional systems, such that $d_S = n$, in the important specialized case that $\mathfrak{O}^{\rm st}$ is the set of “Weyl-covariant channels,” $\mathfrak{U}_{\mathcal{W}}$, where $\mathfrak{U}_{\mathcal{W}} \subsetneq \mathfrak{U}_M$. A channel $\mathcal{N}$ is Weyl-covariant if $\mathcal{N}(W_{ij}\rho W_{ij}^\dagger) = W_{ij}\mathcal{N}(\rho)W_{ij}^\dagger\ \forall\ i,j,\rho$, where W_{ij} are the discrete Weyl operators [40]. These are a physically relevant set of channels given that they contain the completely depolarizing, completely dephasing, and partial thermalization channels. More detail on these channels can be found in [40] and Supplemental Material D [32]. Our allowed operations now become a preprocessing and postprocessing Weyl-covariant channel with shared randomness between them, given by the set $\mathfrak{O}_{\mathcal{W}}^{\rm dyn} \subsetneq \mathfrak{O}^{\rm dyn}$. The DRT considered for $d_S = n$ will be referred to as $\mathcal{R}_n^{\rm dyn}$. In this picture, we get the following result.

Result 3: For every given $\mathcal{M},\mathcal{N} \in \mathfrak{O}^{\rm st} = \mathfrak{U}_{\mathcal{W}}$, the following statements are equivalent: (1) $\mathcal{M} = \Pi(\mathcal{N})$ for some allowed operation $\Pi \in \mathfrak{O}_{\mathcal{W}}^{\rm dyn}$, and (2) $\boldsymbol{\mu} = \mathbb{D}'\boldsymbol{\lambda}$, where $\mathbb{D}'$ is a convex combination of local “cyclic” permutation matrices.

A sketch of the proof of Result 3 can be found in Appendix C below, and a complete derivation can be found in Supplemental Material D [32].

Following the same logic as in the qubit case, a complete set of monotones can again be found.

Result 4: There exists a set of d_S^2 inequalities that are a complete set of monotones of $\mathcal{R}_n^{\rm dyn}$ if $\tilde{\mathfrak{L}}$ is a set of linearly independent vectors.

Characterizing allowed operations via state discrimination.—The complete set of monotones of $\mathcal{R}^{\rm dyn}$ and $\mathcal{R}_n^{\rm dyn}$ can be used to create an operational interpretation of the preorder established on the set of noisy operations.

Result 5: Let $\mathcal{N} \in \mathfrak{O}^{\rm st}$ be given. Then there exists a set of d_S^2 many bipartite states $\{\rho_{ij}\}$ making the following two statements equivalent: (1) $\mathcal{M} = \Pi(\mathcal{N})$ for some $\Pi \in \mathfrak{O}^{\rm dyn}$, and (2) for every i, j we have

$$\langle\Phi_{ij}|(\mathcal{I}\otimes\mathcal{N})(\rho_{ij})|\Phi_{ij}\rangle \geq \langle\Phi_{ij}|(\mathcal{I}\otimes\mathcal{M})(\rho_{ij})|\Phi_{ij}\rangle. \quad (5)$$

Here, $|\Phi_{ij}\rangle$ is a state from the Bell basis and $\tilde{\mathfrak{L}}$ must be a set of linearly independent vectors.

This can be interpreted as a state discrimination task: firstly, a referee distributes half of the bipartite state ρ_{ij}

to Alice and half to Bob. Alice then sends her half to Bob via the channel $\mathcal{N}$ or $\mathcal{M}$, and Bob makes a Bell-state measurement. Bob succeeds if he gets the measurement outcome associated with $|\Phi_{ij}\rangle$ for the state ρ_{ij}. If for all states in the set Bob's success probability is at least as high when Alice applies $\mathcal{N}$ as when Alice applies $\mathcal{M}$, an allowed operation exists between $\mathcal{N}$ and $\mathcal{M}$. Here, the individual probabilities of successful discrimination need to be considered, not just the average success probability, as is common in other state discrimination tasks [41].

This result allows the preorder to be experimentally investigated. Each state, ρ_{ij}, is constructed in a similar method to the complete set of monotones. Interestingly, there is some freedom in how this operational interpretation is formed. This is detailed with a derivation in Supplemental Material E [32].

Classical capacity quantifies informational nonequilibrium preservability.—In addition to finding complete sets of monotones, it is important to find physically motivated monotones. The "Holevo capacity"—a lower bound on the classical capacity of a quantum channel [40,42–46]—is one such physically relevant monotone of both $\mathcal{R}^{\mathrm{dyn}}$ and $\mathcal{R}^{\mathrm{dyn}}_n$.

If there exists a $\Pi \in \mathfrak{O}^{\mathrm{dyn}}$ such that $\mathcal{M} = \Pi(\mathcal{N})$, where $\mathcal{M}, \mathcal{N} \in \mathfrak{O}^{\mathrm{st}}$, then $\chi(\mathcal{N}) \geq \chi(\mathcal{M})$ where $\chi(\cdot)$ is the Holevo capacity. See Supplemental Material E [32] for a full derivation. Therefore, $\mathcal{R}^{\mathrm{dyn}}$ can be employed to compare the classical communication abilities of unital quantum channels in settings where nonequilibrium can only increase. See Appendix D below for details of how this relates to the dynamical resource theory of communication [47], and see Supplemental Material G [32] for more details on monotones.

Conclusion.—Here, the dynamical resource theory of informational nonequilibrium preservability has been characterized for both qubit mixed unitary channels and Weyl-covariant channels. Through the presented framework, the ability of two allowed operations of $\mathcal{R}^{\mathrm{st}}$ to preserve the purity of input states can be compared for all states in the space of states. This characterization acts as a proof of principle for resource preservability theories and suggests that others could be developed. The clear and immediate next step would be to extend these results to the resource theory of thermodynamics. An understanding of the ability of *thermal operations* [4] to preserve nonequilibrium could further our knowledge of the dynamical aspects of quantum thermodynamics. Such understanding could also be utilized, for example, in the study of efficient thermal processes. Moreover, a characterization of the DRT of information nonequilibrium for channels with differing input and output dimensions would be another interesting extension of the above results. Additionally, resource preservability theories could also be built upon other successful SRTs of a more pragmatic nature. For instance, the set of local operations and classical communication could be studied and their ability to preserve entanglement quantified.

Moreover, a link between the ability of a channel to preserve information nonequilibrium and classical capacity—a natural measure for classical communication—has been made. Further effort should be made to discover other physically relevant monotones of $\mathcal{R}^{\mathrm{dyn}}$ in the hope of finding additional areas in which it is applicable.

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Appendix A: Informational nonequilibrium and purity.—While similar, the resource theory of informational nonequilibrium and the resource theory of purity differ in what exactly they consider a resource. Two states that are identical on their support but embedded in different dimensional Hilbert spaces will have the same amount of resource in the resource theory of purity but a differing amount of resource in the resource theory of informational nonequilibrium. In a Hilbert space of fixed dimension, these resource theories become alternative interpretations of the same physics. Where context permits, informational nonequilibrium will be referred to as purity in this Letter. This resource theory could, therefore, also be referred to as the dynamical resource theory of purity preservability.

Appendix B: Result 1.—Here, we provide a sketch of the proof of Result 1. A complete derivation can be found in Supplemental Material B [32].

Proof.—Firstly, we prove that statement 1 implies statement 2. We simplify the form of the allowed operations through the following lemma.

Lemma 1.—The Choi states of qubit unital channels can always be diagonalized in the Bell basis (maximally entangled basis) under local unitaries.

Given all unitary channels are in $\mathfrak{O}^{\mathrm{st}}$, there always exists an allowed operation that can diagonalize $\mathcal{J}^{\mathcal{N}}$. The action of all qubit unital channels can, therefore, be captured through only the eigenvalues of their Choi state. Equation (4) can now be rephrased as $\boldsymbol{\mu} = \mathbb{B}\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^4$ are vectors of eigenvalues of $\mathcal{J}^{\mathcal{N}}$, $\mathcal{J}^{\mathcal{M}}$ respectively, and $\mathbb{B}$ is in the convex hull of matrices with elements $B_{nm,kl} = |\langle\Phi_{kl}|U \otimes V|\Phi_{nm}\rangle|^2$ for some general unitaries U, V. The Bell basis states are given by $|\Phi_{ij}\rangle = (\mathbb{I} \otimes W_{ij})|\Phi_{00}\rangle$, where W_{ij} is a discrete Weyl operator and $|\Phi_{00}\rangle \coloneqq |\Phi\rangle$ (see Ref. [40] and Supplemental Material A [32] for details on Weyl operators). Matrices of this type are a subset of the unistochastic matrices [49]—we coin this subset "the product-Bell unistochastic matrices." The following lemma is now employed.

Lemma 2.—Product-Bell unistochastic matrices are a convex combination of local permutation matrices.

The proof of this direction is then completed by noting that a convex combination of product-Bell unistochastic matrices remains a convex combination of local permutation matrices.

To show statement 1 implies statement 2, consider the channels $\tilde{\mathcal{N}}, \tilde{\mathcal{M}}$, whose Choi states are diagonalized in the given Bell basis with the same eigenvalues as $\mathcal{J}^{\mathcal{N}}$, $\mathcal{J}^{\mathcal{M}}$. This is guaranteed by Lemma 1. When statement 2 holds, one can write $\tilde{\mathcal{M}} = \alpha\tilde{N} + (1-\alpha)\tilde{N}[\sigma_x(\cdot)\sigma_x]$, where $\alpha = \sum_{n=m} p_{nm}$, meaning there is an allowed operation converting $\mathcal{N}$ into $\mathcal{M}$, completing the proof.

Appendix C: Result 3.—Here, we provide a sketch of the proof of Result 3. A complete derivation can be found in Supplemental Material D [32].

Proof.—The Choi states of Weyl-covariant channels are diagonal in the Bell basis, and Choi states that are diagonal in the Bell basis correspond to Weyl-covariant channels. The physical meaning of all the Choi states of qubit unital channels being diagonalizable under local unitaries (see Lemma 1 in the above appendix) can now be seen—all qubit unital channels are equal to a Weyl-covariant channel up to a preprocessing and postprocessing unitary. (This result was also recently found in [50]; see Supplemental Material A [32] for our proof.) From the diagonalization of the Choi state, the proof of Result 3 is similar to that of Result 1 seen in Appendix B above. The difference arises from $\mathbb{B}$ now being in the convex hull of matrices with elements $B_{nm,kl} = |\langle\Phi_{kl}|W_{ab}\otimes W_{cd}|\Phi_{nm}\rangle|^2$, where W_{ab} and W_{cd} are general discrete Weyl operators. This restricts the set of permutations to be only the set of local *cyclic* permutations. A complete derivation can be found in Supplemental Material D [32].

Appendix D: Classical capacity quantifying informational nonequilibrium preservability.—The dynamical resource theory of classical communication [47] has all state-preparation channels as the set of channels with no resource (the free set). This is due to these channels not being able to communicate any information. In the resource theory presented here, we have an additional thermodynamic constraint that the channels cannot drive the system out of equilibrium. This suggests that, conceptually, informational nonequilibrium preservability can be viewed as the ability to transmit classical information [47,51] subject to additional thermodynamic constraints.

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