Lie Algebraic Quantum Phase Reduction

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(Received 1 December 2022; revised 10 August 2023; accepted 22 January 2024; published 27 February 2024)

We introduce a general framework of phase reduction theory for quantum nonlinear oscillators. By employing the quantum trajectory theory, we define the limit-cycle trajectory and the phase according to a stochastic Schrödinger equation. Because a perturbation is represented by unitary transformation in quantum dynamics, we calculate phase response curves with respect to generators of a Lie algebra. Our method shows that the continuous measurement yields phase clusters and alters the phase response curves. The observable clusters capture the phase dynamics of individual quantum oscillators, unlike indirect indicators obtained from density operators. Furthermore, our method can be applied to finite-level systems that lack classical counterparts.

DOI: 10.1103/PhysRevLett.132.093602

Introduction.—The last decade has witnessed a remarkable shift in the interest in synchronization, extending from classical dynamics to the quantum regime [1–4]. Numerous studies have been reported on the synchronization of nonlinear oscillators that show quantum effects, such as spins [5,6], optomechanical systems [7,8], cold atoms [9,10], quantum heat engines [11–14], and (discrete or continuous) time crystals [15–17]. In fact, synchronization in quantum systems is critical for considerable advances in future quantum technologies, including quantum communication and cryptography [18,19]. For example, recent studies have shown that quantum synchronization helps addressing important security issues in quantum key distribution protocols [20]. Therefore, exploring synchronization in the quantum regime holds great technological promise. In this direction, theoretical models of limit cycles (i.e., self-sustained oscillators adaptable to weak perturbations) have been proposed in open quantum systems, such as quantum van der Pol oscillators [21–25] and spin oscillators [26]. Furthermore, several experimental reports have demonstrated quantum synchronization of limit cycles in laboratory settings [6,17,27,28].

Against this background, we propose a quantum phase-reduction theory for continuous measurement to describe quantum limit cycles in phase dynamics. The phase reduction theory [29,30] reduces the multidimensional dynamics of a weakly perturbed limit cycle to one-dimensional phase dynamics. By continuously monitoring the environment to which oscillatory systems are coupled, quantum trajectories of the system come to obey a stochastic Schrödinger equation (SSE) [31–33]. When the effect of quantum noise is sufficiently weak, these trajectories fluctuate around a deterministic trajectory. However, since a perturbation in quantum limit-cycle dynamics differs from that in classical dynamics and is represented by a unitary transformation, we calculate the phase response to a perturbation within the Lie-algebraic framework. Thus, we can derive a quantum phase equation from a Lindblad equation that describes a weakly perturbed dissipative system. Note that the proposed approach reproduces the conventional phase-reduction theory in the classical limit. Using quantum van der Pol oscillators, we show the proposal approach recovers the definitions of the limit-cycle trajectory, the phase, the perturbation, and the phase response curve (PRC) of the conventional phase-reduction theory. Whereas Ref. [34] relies on the semiclassical approximation, we develop a fully quantum phase-reduction approach. Thus, our approach captures the dynamics of quantum oscillators, even in the deep quantum regime and physically corresponds to the continuous measurement scheme. Moreover, it is applicable to quantum oscillators that lack classical counterparts, such as qubits and spins. In the quantum regime, the trajectories of quantum states are obtained by continuous measurements, where the measurement itself affect the dynamics. Our approach captures this measurement backaction and reveals that the measurement yields phase clusters and alters the PRC in the quantum regime, through simulations of quantum van der Pol oscillators. The resulting clusters visualize phase dynamics unique to individual quantum oscillators and cannot be captured by the indirect indicators obtained from density operators.

Derivation.—In open quantum dynamics, quantum limit-cycle oscillators are usually described by a Lindblad equation [35,36]. Let $\rho(t)$ be a density operator at time t whose time evolution is governed by

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{k=1}^{M} \mathcal{D}[L_k]\rho, \tag{1}$$

where H is a Hamiltonian operator and L_k are jump operators, and $\mathcal{D}[O]$ is the dissipator defined by $\mathcal{D}[O]\rho \equiv O\rho O^\dagger - (1/2)(O^\dagger O\rho + \rho O^\dagger O)$. To obtain a general phase-reduction approach that can be applied to quantum limit cycle models, we do not specify the jump operators L_k. Note that a Lindblad equation describes a density operator, not the dynamics of the measurable quantum state. The latter are described using quantum trajectory theory [37], which describes the stochastic evolution of a pure state of the system $|\psi\rangle$, obtained by continuously monitoring the environment. In the homodyne detection scheme, the continuous measurement can be experimentally implemented mainly by one of two approaches: detection of homodyne currents by physical detectors [32] and continuous application of weak Gaussian measurements [38]. In fact, quantum trajectories have been observed with various physical platforms, such as superconducting devices [39–41], trapped ions [42], and mechanical resonators [43,44]. In the homodyne detection scheme, the evolution can be described by the following diffusive SSE in the Stratonovich form [31–33,45].

$$d|\psi\rangle = \left[-iH_{\mathrm{eff}} + \sum_{k=1}^{M} \frac{1}{2}\langle L_k^\dagger L_k\rangle + \langle X_k\rangle\left(L_k - \frac{\langle X_k\rangle}{2}\right) + \frac{1}{4}\left(-2L_k^2 + \langle L_k^2\rangle + \langle L_k^{\dagger 2}\rangle\right)\right]|\psi\rangle dt + \sum_{k=1}^{M}\left(L_k - \frac{\langle X_k\rangle}{2}\right)|\psi\rangle \circ dW_k(t), \tag{2}$$

where $\circ$ denotes the Stratonovich calculus, $H_{\mathrm{eff}} \equiv H - (i/2)\sum_{k=1}^{M} L_k^\dagger L_k$ is a non-Hermitian operator (i.e., an effective Hamiltonian), and $X_k \equiv L_k + L_k^\dagger$ are quadratures of the system. Here, $\langle O\rangle$ denotes the expectation value of O with respect to state $|\psi\rangle$, i.e., $\langle O\rangle \equiv \langle\psi|O|\psi\rangle$. Random variables dW_k are Wiener increments that satisfy $\mathbb{E}[dW_k] = 0$, and $\mathbb{E}[dW_k^2] = dt$, where $\mathbb{E}[\cdot]$ denotes the average over all possible trajectories. The homodyne currents J_k are defined as $J_k(t) \equiv \langle X_k\rangle + \xi_k(t)$, where $\xi_k(t) \equiv dW_k/dt$. In general, a limit-cycle trajectory in quantum dynamics and the phase along it is not well defined. In the classical stochastic dynamics of limit cycles, a stochastic differential equation is represented by adding noise terms to a given deterministic differential equation [48–51]. In contrast, quantum dynamics are stochastic in nature and the deterministic equation is not given. To realize a quantum phase reduction, the deterministic limit cycle and the phase along it should be defined. The classical deterministic limit-cycle dynamics corresponds to an equation obtained by removing noise terms from a stochastic differential equation in the Stratonovich form. As an analog of classical cases, we propose here to remove noise terms from an SSE in the Stratonovich form and define the resulting equation as the deterministic limit-cycle dynamics:

$$d|\psi\rangle = \left[-iH_{\mathrm{eff}} + \sum_{k=1}^{M} \frac{1}{2}\langle L_k^\dagger L_k\rangle + \langle X_k\rangle\left(L_k - \frac{\langle X_k\rangle}{2}\right) + \frac{1}{4}\left(-2L_k^2 + \langle L_k^2\rangle + \langle L_k^{\dagger 2}\rangle\right)\right]|\psi\rangle dt. \tag{3}$$

An SSE is usually represented and calculated in the Ito form for computational and statistical convenience. It is worth emphasizing that noise terms should be removed from an SSE in the Stratonovich interpretation, rather than in the Ito interpretation for the following reasons. The first is related to the chain rule of differentiation calculation. In fact, the phase reduction requires a coordinate transformation between a state vector and a phase coordinate. The transformation is performed via the chain rule of differentiation, which holds only in the Stratonovich form (not in the Ito form). The second reason is related to norm preservation. To ensure that the limit cycles represent physically observable trajectories of pure states, it is essential to satisfy norm preservation. Note that the norm of Eq. (3) is preserved, $d\||\psi\|| = 0$, where $\||\psi\|| \equiv \sqrt{\langle\psi|\psi\rangle}$. Therefore, Eq. (3) stands on its own as pure-state dynamics, which is not the case for the Ito interpretation. Even when considering an arbitrary stochastic calculus, the norm preservation is satisfied only in the Stratonovich calculus [45]. It is a nontrivial property that an SSE with noise terms removed also stands as pure-state dynamics because, unlike the case for classical dynamics, the deterministic dynamics Eq. (3) is not given.

When Eq. (3) satisfies $\lim_{t\to\infty}|\langle\psi(t)|\psi(t+T)\rangle| = 1$ for a period T, $|\psi\rangle$ has a limit-cycle solution $|\psi_0\rangle$ to which $|\psi\rangle$ converges. Since $U(1)$ has no physical effect on the state $|\psi\rangle$ [52], the $U(1)$ transformation has no effect on the phase. We define the phase on a quantum limit cycle using the deterministic trajectory $|\psi_0\rangle$. There are several schemes for the phase reduction in classical stochastic systems [49,50,53]. For simplicity, we derive the phase equation by following the procedure in [49]. The phase θ is defined along the limit-cycle solution $|\psi_0\rangle$ using Eq. (3) as to change at a constant frequency $\omega = 2\pi/T$. Furthermore, by virtue of the convergence to the limit-cycle solution $|\psi_0\rangle$, the phase θ outside of it is defined by an isochron under Eq. (3) as $\Theta(|\psi(t)\rangle) \equiv \Theta(\lim_{n\to\infty}|\psi(t+nT)\rangle)$, where the phase function $\Theta(|\psi\rangle)$ represents the phase at the state $|\psi\rangle$. Here, we assume that the perturbation is sufficiently weak, i.e., the state $|\psi\rangle$ is in the vicinity of the limit-cycle solution $|\psi_0\rangle$.

It should be mentioned that our definition of the PRC differs from that of the classical counterpart. While the state is defined in the Euclidean space for the classical limit cycle, the unitary group in the Hilbert space defines the state of a quantum limit cycle. Therefore, the corresponding bases are the generators of the unitary group $U(N)$ [54].

They can be decomposed into the generators of $U(1)$ and those of the special unitary group $SU(N)$. $U(1)$ represents the scalar multiplication, while $SU(N)$ is a unitary group with a determinant $\det[U] = 1$. For example, the generators of $SU(2)$ correspond to Pauli matrices. By the definition of the phase, $U(1)$ has no effect on it. Thus, only $SU(N)$ should be considered for the phase dynamics. In quantum limit cycles, the perturbation is represented by an infinitesimal unitary transformation and the PRC is calculated for it. Based on a Lie algebra, an arbitrary infinitesimal unitary transformation is represented by the Taylor expansion as $U|\psi\rangle = \exp\left(\sum_{l=1}^{N^2-1} -ig_l E_l - ig_0 I\right)|\psi\rangle \simeq |\psi\rangle - \sum_{l=1}^{N^2-1} ig_l E_l|\psi\rangle - ig_0|\psi\rangle$, where E_l are generators of $SU(N)$, I is the identity matrix, and real coefficients g_l satisfy $|g_l| \ll 1$. The PRCs for the generators E_l are represented as

$$Z_l(\theta) \equiv \lim_{g_l \to 0} \frac{\Theta[\exp(-ig_l E_l)|\psi_0(\theta)\rangle] - \Theta[|\psi_0(\theta)\rangle]}{g_l}, \tag{4}$$

where $|\psi_0(\theta)\rangle$ represents the state $|\psi\rangle$ on the limit-cycle solution $|\psi_0\rangle$ with phase θ. Equation (4) describes the partial derivative with respect to a unitary transformation by generator E_l. This formulation defines the quantum PRC. For the case of high-dimensional systems, e.g., semiclassical systems, PRCs with respect to $N^2 - 1$ generators of a Lie algebra demand large computational resource. In such a case, we can calculate PRC either by a direct method with respect to an arbitrary Hamiltonian or an adjoint method in the Euclidean space [45]. While the SSE [Eq. (2)] and Eq. (3) are described as non-Hermitian dynamics, owing to their nonlinearity, they can also be represented as Hermitian dynamics [45]. Thus, the stochastic terms in an SSE can be represented by traceless Hermitian operators as $d|\psi\rangle = -i\sum_{k=1}^{M} H_k|\psi\rangle \circ dW_k$ [45], where traceless Hermitian operators H_k are defined by

$$H_k \equiv i(L_k - \langle L_k\rangle)|\psi\rangle\langle\psi| + \text{H.c.} \tag{5}$$

Because of the trace-orthogonal property of the Lie algebra, traceless Hermitian operators H_k can be decomposed into a linear combination of $SU(N)$ generators as $H_k = \sum_{l=1}^{N^2-1} g_{k,l}E_l$, where the coefficients $g_{k,l}$ are defined by $g_{k,l} \equiv \mathrm{Tr}[H_k E_l]$. Therefore, the following quantum phase equation is derived from the chain rule

$$\frac{d\theta}{dt} = \omega + \sum_{k=1}^{M}\sum_{l=1}^{N^2-1} Z_l(\theta) g_{k,l}(\theta) \circ \xi_k(t), \tag{6}$$

where $g_{k,l}(\theta)$ is evaluated at $|\psi\rangle = |\psi_0(\theta)\rangle$ on the limit cycle. The phase equation (6) in the Stratonovich form can be converted into an equivalent equation in the Ito form [55]

$$\frac{d\theta}{dt} = \omega + \frac{1}{2}\sum_{k=1}^{M} \frac{dY_k(\theta)}{d\theta} Y_k(\theta) + \sum_{k=1}^{M} Y_k(\theta)\xi_k(t), \tag{7}$$

where $Y_k(\theta) = \sum_{l=1}^{N^2-1} Z_l(\theta) g_{k,l}(\theta)$. As long as the quantum dynamics is represented by the SSE [Eq. (2)], arbitrary weak perturbations can be considered in our framework [45]. In the following, we shall elaborate on the difference between our approach and that in Ref. [34], which is the extant phase-reduction approach for quantum systems. In a semiclassical approximation, Ref. [34] reduces quantum dynamics to a classical one based on a quasiprobability distribution [32,56], and applies the conventional phase-reduction theory to it. In contrast, based on a Lie algebra, our approach proposes the original framework of phase reduction theory directly applicable to the pure state $|\psi\rangle$ of quantum limit cycles. To explain the difference in detail, we examine the quantum van der Pol oscillator defined by

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma_{1g}\mathcal{D}[a^\dagger]\rho + \gamma_{1d}\mathcal{D}[a]\rho + \gamma_{2d}\mathcal{D}[a^2]\rho, \tag{8}$$

where $H = a^\dagger a$ is the Hamiltonian and a and $a^\dagger$ are annihilation and creation operators, respectively. The quantum van der Pol model describes the limit-cycle dynamics at a quantum scale. In quantum systems, the measurement outcomes are stochastic in nature. Thus, the position $x = (1/\sqrt{2})(a + a^\dagger)$ and the momentum $p = -(i/\sqrt{2})(a - a^\dagger)$ are evaluated through their expectation values as $\langle x\rangle_\rho$ and $\langle p\rangle_\rho$, respectively, where $\langle O\rangle_\rho \equiv \mathrm{Tr}[O\rho]$. In the classical limit, $\langle a^\dagger a\rangle_\rho \gg 1$ (i.e., the system is at a macroscopic scale), Eq. (8) gives the equation as follows:

$$\frac{d\alpha}{dt} = -i\alpha + \frac{\epsilon}{2}\alpha - \gamma_{2d}|\alpha|^2\alpha, \tag{9}$$

where $\alpha \equiv (\langle x\rangle_\rho + i\langle p\rangle_\rho)/\sqrt{2}$ and $\epsilon \equiv \gamma_{1g} - \gamma_{1d}$ corresponds to the difference between one-particle gain and loss rates. Differentiating the real part of Eq. (9) with respect to time and substituting the imaginary part of Eq. (9) into it, the classical van der Pol model is recovered up to $O(\epsilon^2)$ as $\langle\ddot{x}\rangle_\rho + \langle x\rangle_\rho = \epsilon\{1 - (\langle x\rangle_\rho^2 + \langle\dot{x}\rangle_\rho^2)/A_c^2\}\langle\dot{x}\rangle_\rho + O(\epsilon^2)$, where $A_c \equiv \sqrt{\epsilon/\gamma_{2d}}$ [25]. For the semiclassical approximation, the previous work in [34] can be applied only to systems near the classical limit $\gamma_{1g} \gg \gamma_{2d}$. In contrast, our approach can be applied to an arbitrary regime, including the deep quantum regime $\gamma_{2d} \gg \gamma_{1g}$. Similarly, our approach differs from Ref. [57], which is a feedback control scheme to enhance synchronization by applying the semiclassical phase reduction to a homodyne detection scheme.

Thus far, we have been concerned with regimes ranging from the semiclassical to the quantum regime. Historically, the phase reduction theory was demonstrated in the context of classical deterministic dynamics. In the following,

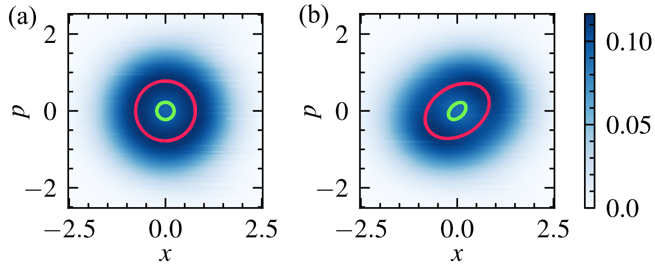

FIG. 1. Wigner function and limit-cycle trajectories of quantum and semiclassical phase-reduction approaches in the two-parameter settings quantum regime, (a) $(\Delta, \Omega, \eta e^{i\lambda}, \gamma_{1g}, \gamma_{1d})/\gamma_{2d} = (1, 0, 0, 0.1, 0)$ and (b) $(\Delta, \Omega, \eta e^{i\lambda}, \gamma_{1g}, \gamma_{1d})/\gamma_{2d} = (1, 0, -0.2, 0.1, 0)$. Color intensity is proportional to the quasiprobability of the Wigner function. The limit-cycle trajectory of the quantum phase reduction (red line) passes through the high quasiprobability region of the Wigner function, while that of the semiclassical phase reduction (green line) does not. The fidelity $F(\rho_1, \rho_2) \equiv \mathrm{Tr}[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^2$ between the true density operator and those reconstructed from the phase distribution are (a) 0.958, (b) 0.979 in our method, and (a) 0.812 in the semiclassical method.

we show that our approach reduces to the conventional phase-reduction theory in the classical limit. In the classical limit, the state $|\psi\rangle$ is considered coherent and satisfies $a|\psi\rangle = \alpha|\psi\rangle$, $a^\dagger|\psi\rangle = \alpha^*|\psi\rangle + |x\rangle$, and $|\alpha| \gg 1$, where $|x\rangle \equiv a^\dagger|\psi\rangle - \alpha^*|\psi\rangle$. Substituting these conditions into Eq. (2), we obtain

$$\frac{d\alpha}{dt} = \left[-i\alpha + \frac{\epsilon}{2}\alpha - \gamma_{2d}|\alpha|^2\alpha\right] + \sqrt{\gamma_{1g}} \circ \xi(t). \quad (10)$$

In Eq. (10), the deterministic term, which equals Eq. (9), is $O(|\alpha|^2\alpha)$ whereas the stochastic term is $O(1)$. Hence, in the classical limit, the dynamics can be considered as deterministic and its limit cycle is equivalent to the classical one. In the classical limit, the proposed and semiclassical methods give the same limit cycle not only for quantum van der Pol but also in general cases [45]. Moreover, this equivalence applies also to the perturbation and phase response. In the conventional method, the perturbation is represented by basis vector dx in the Euclidean space. It can be reproduced by the momentum operator p in the Hilbert space as follows: By the unitary perturbation $d|\psi\rangle/dt = -ip|\psi\rangle$, the derivative of the expectation value of the position is unity, i.e., $d\langle x\rangle/dt = -i\langle[x, p]\rangle = 1$. The same argument holds for dp. Because the same perturbation can be reproduced, the PRC in the conventional method can also be replicated similarly in the classical limit. The conventional PRC is defined as $Z_{\mathrm{cl}}(\theta) \equiv \{\Theta[\alpha(\theta) + dx] - \Theta[\alpha(\theta)]\}/dx$, and it can be reconstructed by the unitary transformation as $Z(\theta) = \lim_{h\to 0}\{\Theta[\exp(-ihp)|\psi_0(\theta)\rangle] - \Theta[|\psi_0(\theta)\rangle]\}/h$.

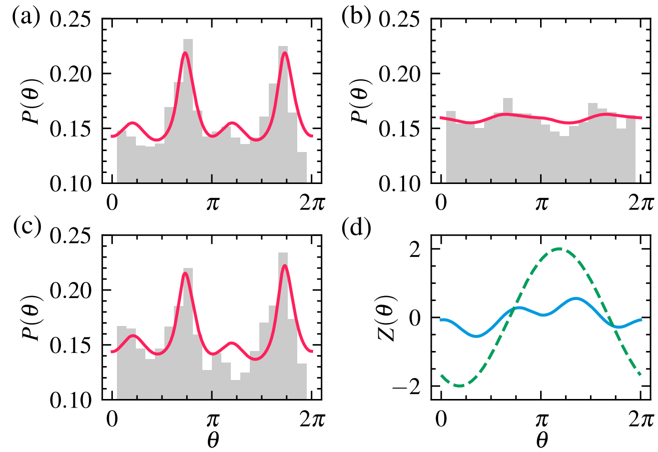

FIG. 2. Phase distribution $P(\theta)$ in steady state and PRC $Z(\theta)$. (a), (b), and (c) Phase distribution (a) in the quantum regime, (b) in the deep quantum regime, and (c) subjected to harmonic drive as weak perturbation. (d) PRCs with respect to harmonic drive. The gray histograms are computed from SSE simulations and the red lines are computed from simulations of the proposed phase equation for (a), (b), and (c). The solid blue line is obtained from the proposed method and the dashed green line is obtained from the semiclassical method for (d). The strength of the weak perturbation is $\Omega_p = 0.05$ for (c). The parameters are $(\Delta, \Omega, \eta e^{i\lambda}, \gamma_{1g}, \gamma_{1d})/\gamma_{2d} = (1, 0, 0, 0.5, 0)$ for (a), (c), and (d), $(\Delta, \Omega, \eta e^{i\lambda}, \gamma_{1g}, \gamma_{1d})/\gamma_{2d} = (1, 0, 0, 0.1, 0)$ for (b).

Example.—As an example, we consider the quantum van der Pol oscillators [Eq. (8)] in a rotating frame [22], where $H = -\Delta a^\dagger a + i\Omega(a^\dagger - a) + i\eta[a^2\exp(-i\lambda) - a^{\dagger 2}\exp(i\lambda)]$ is the Hamiltonian, $\Delta = \omega_d - \omega_0$ is the detuning between the system's natural frequency ω_0 and a harmonic drive frequency ω_d, Ω is the strength of the harmonic drive, and η and λ are the strength and phase of squeezing, respectively. In a rotating frame, the system rotates with a harmonic drive frequency ω_d. First, we numerically validate the accuracy of the approximation by comparing the derived phase equation to the semiclassical phase equation. Figure 1 shows the Wigner function in the steady state and the limit-cycle trajectory of each phase-reduction method in the quantum regime. We calculate the reconstructed density operator $\rho_{\mathrm{re}} \equiv \int d\theta P(\theta)|\psi_0(\theta)\rangle\langle\psi_0(\theta)|$, where $P(\theta)$ is a probability density function of the phase θ, for each phase-reduction method. Furthermore, we compare their fidelity level to those of the true density operator [34]. Our method provides a better approximation than the semiclassical method (see the caption of Fig. 1 for details), because it reduces a pure state to the phase without the semiclassical approximation. Note that we cannot calculate fidelity for the semiclassical method in Fig. 1(b) because diffusion matrices of a semiclassical Langevin equation are not positive-semidefinite in some points.

Next, we investigate the effect of the measurement and the harmonic drive on quantum synchronization in

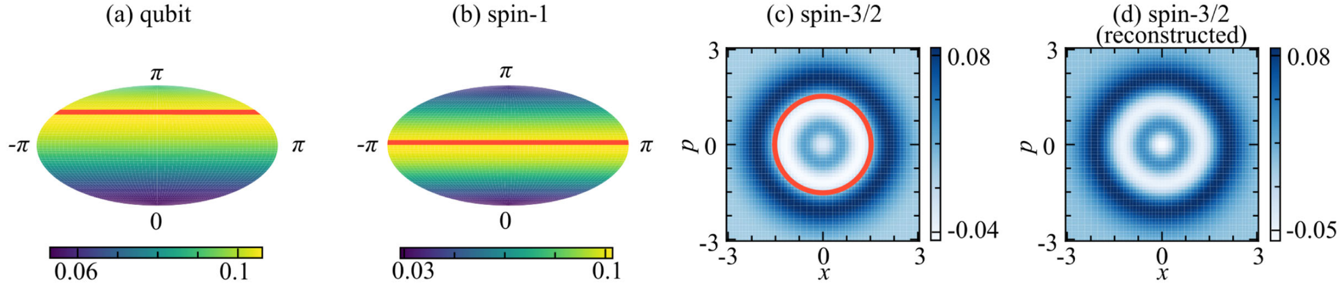

FIG. 3. Expectation values on limit-cycle trajectories and quasiprobability distributions of (a) two-level systems [58], (b) spin-1 oscillators [26], and (c) spin-3/2 oscillators; (d) reconstructed quasiprobability distribution for spin-3/2 spins from phase distribution. The quasiprobability distribution is the Husimi-Q function for (a) and (b) and the Wigner function for (c) and (d). The set of observables evaluated on limit-cycle trajectories are spin operators for (a) and (b) and position and momentum operators for (c).

the quantum regime. In contrast to classical dynamics, the measurement affects the system's trajectory in the quantum regime. For brevity of expression, we here approximate the quantum van der Pol oscillator by limiting the bosonic Fock state to the lowest N levels [24]. In Fig. 2, $N = 6$ for (a), (c), and (d), and $N = 4$ for (b). In the quantum regime, the proposed method yields a good approximation, as shown in Fig. 2(a). Although clusters diminish for the deep quantum regime in Fig. 2(b), in both cases, the measurement generates the clusters in the rotating frame in Figs. 2(a) and 2(b). Furthermore, as a weak perturbation Hamiltonian $H_p = i\Omega_p(a^\dagger - a)$, a harmonic drive is added to the Hamiltonian H and enhances synchronization in Fig. 2(c), where Ω_p is the strength of the weak perturbation. In Fig. 2(d), the PRC of the proposed method appears distorted due to the measurement, unlike that of the semiclassical method, which exhibits a sinusoidal wave pattern. Moreover, in the quantum regime, the limit cycle shrinks due to the classical approximation, resulting in larger amplitude for the PRC in the semiclassical method. Some indicators have been proposed as signatures of quantum synchronization, such as mutual information [59], quantum discord [60,61], and entanglement [62]. The observable clusters describe the synchronization dynamics of individual oscillators under the measurement backaction, unlike the indirect indicators obtained from a density operator.

Additionally, we demonstrate the applicability of our method to finite-level systems. Qubits and spins hold a central place in the field of quantum synchronization; however, they lack classical limit-cycle counterparts. We apply the proposed method to two-level systems [58], spin-1 oscillators [26] at finite temperature, and spin-3/2 oscillators, and derive the phase equations [45]. Figure 3 displays the quasiprobability distributions for each model and the expected values of observables evaluated on the limit cycles. As shown in Figs. 3(a) and 3(b), the trajectories for the qubit and spin-1 pass through regions of high probability. Since we plot expectation values, as shown in Fig. 3(c), the trajectories for spin-3/2 pass between two regions of high probability. Yet, the Wigner distribution is reconstructed from the phase distribution with fidelity $F = 0.998$ in Fig. 3(d).

Conclusion.—In this Letter, we proposed a quantum phase-reduction formulation of a Lindblad equation in a continuous measurement scheme. We consider the case of synchronization among multiple quantum oscillators with the proposed method in [63].

This work was supported by JSPS KAKENHI Grant No. JP22H03659.

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