Entanglement Detection with Trace Polynomials

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We provide a systematic method for nonlinear entanglement detection based on trace polynomial inequalities. In particular, this allows us to employ multipartite witnesses for the detection of bipartite states, and vice versa. We identify pairs of entangled states and witnesses for which linear detection fails, but for which nonlinear detection succeeds. With the trace polynomial formulation a great variety of witnesses arise from immanant inequalities, which can be implemented in the laboratory through the randomized measurements toolbox.

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The experimental detection of entanglement is an ongoing challenge [1–3], for which a key tool are entanglement witnesses [4]. These detect some entangled states by virtue of having a negative expectation value, separating them from the set of fully separable states, i.e., from convex combinations of product states. While every entangled state can be detected by some witness, the construction of witnesses is not a straightforward task [4] and frequently relies on making use of specific structure in the state to be detected [5–12].

Nonlinear entanglement detection has become a recent focus of attention due to the development of the randomized measurement toolbox [13], making local unitary invariants like partial transpose moments experimentally accessible through single-copy measurements [14,15]. However, the currently available techniques are limited and a systematic development of nonlinear witnesses is desirable. The aim of this Letter is to provide such a systematic method that is not only suitable for the randomized measurement framework, but also makes a broader use of known constructions. The basic task we study is the following: given an entanglement witness W, can it be employed also in a nonlinear fashion as to detect entanglement in multiple copies $\varrho^{\otimes k}$?

Here we answer this question in the affirmative. In particular we show that (i) having access to multiple copies of the state, it is possible to detect entanglement locally, where the size of states and witnesses can be different (Observation 1 and Figs. 2 and 3); (ii) that there exist pairs of states and witnesses for which linear entanglement detection fails, but for which nonlinear detection succeeds (Observation 2); and (iii) that there is a large class of nonlinear witnesses arising from trace polynomial inequalities (Fig. 4). These can analytically be treated in the group ring $\mathbb{C}S_n$ and are experimentally accessible through randomized measurements [13].

Nonlinear entanglement witnesses.—A quantum state is called separable if it can be written as a convex combination of product states,

$$\varrho = \sum_i p_i \varrho_i^{(1)} \otimes \varrho_i^{(2)} \otimes \ldots \otimes \varrho_i^{(n)}, \tag{1}$$

and entangled otherwise. Entanglement detection with a witness works in the following way: to detect an entangled state ϱ_{ent}, a witness is an observable W such that $\mathrm{tr}(W\varrho_{\mathrm{ent}}) < 0$ holds, while $\mathrm{tr}(W\varrho_{\mathrm{SEP}}) \geq 0$ for all separable states $\varrho_{\mathrm{SEP}} \in \mathrm{SEP}$. In this way, W acts as a hyperplane, separating a subset of entangled states from the rest. We call this linear detection, in contrast to nonlinear detection introduced below. This is illustrated in Fig. 1.

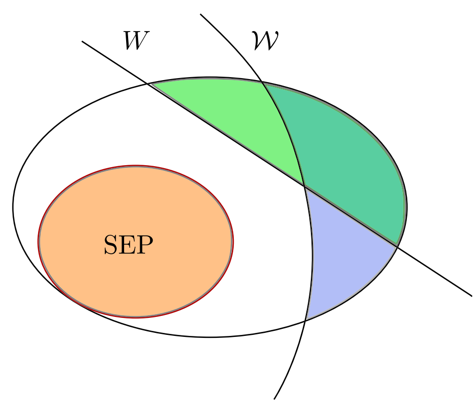

FIG. 1. Linear versus nonlinear entanglement detection. A linear witness W defines a hyperplane in the state space, separating some entangled states (green area) from the rest; a nonlinear witness $\mathcal{W}$ cuts the state space analogously in a nonlinear hypersurface, thus detecting a different set of entangled states (blue area).

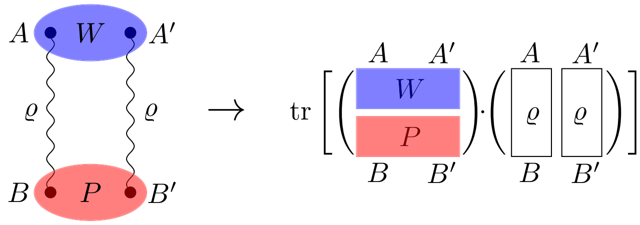

FIG. 2. Sketch of the entanglement concentration scheme. Expressions of the type $\langle W \otimes P\rangle_{\varrho^{\otimes 2}}$ can be obtained by first measuring two copies of Bob's subsystems (in red), and then two copies of Alice's subsystems (in blue). When $\varrho = |\psi\rangle\langle\psi|$ is a pure state such that $|\psi\rangle = \mathbb{1} \otimes S|\phi^+\rangle$ with S invertible, Bob can teleport his part of ϱ to Alice by using $P = |\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = \mathbb{1} \otimes S^{\dagger -1}|\phi^+\rangle$. Then standard linear witness evaluation $\mathrm{tr}(W|\psi\rangle\langle\psi|)$ is obtained as a particular case of the nonlinear method proposed in this Letter.

How can one find nonlinear witnesses $\mathcal{W}$? A range of methods are based on spin-squeezing inequalities and purity inequalities [16–19], nonlinear corrections to linear witnesses [20], and multicopy scenarios which show surprising entanglement activation properties [21]. In a multicopy scenario, one asks that a witness satisfies $\mathrm{tr}(\mathcal{W}\varrho_{\mathrm{ent}}^{\otimes k}) < 0$, while $tr(\mathcal{W}\varrho_{\mathrm{SEP}}^{\otimes k}) \geq 0$ for all separable states $\varrho_{\mathrm{SEP}} \in \mathrm{SEP}$. Setting $k = 1$ then recovers the standard use of witnesses.

Our approach here is to take a tensor product of linear witnesses,

$$\mathcal{W} = W_1 \otimes \ldots \otimes W_n. \tag{2}$$

Naturally, the expectation values need to be computed in a manner such that the dimensions of $\mathcal{W}$ and $\varrho^{\otimes k}$ match (see Fig. 2 for a detailed explanation). For example, for a tripartite state ϱ_{ABC}, take the tensor product of three bipartite witnesses,

$$\mathcal{W} = U_{AA'} \otimes V_{BB'} \otimes W_{CC'}. \tag{3}$$

Then, the expression

$$\langle \mathcal{W}\rangle_{\varrho^{\otimes 2}} = \mathrm{tr}(\mathcal{W}(\varrho_{ABC} \otimes \varrho_{A'B'C'})) \tag{4}$$

is non-negative if ϱ is separable. At first sight, it may not be clear why Eq. (4) can detect entanglement, since the witnesses act along the cuts $A|A'$, $B|B'$, and $C|C'$ and $\varrho_{ABC}^{\otimes 2}$ is separable in the cut $ABC|A'B'C'$. This apparent contradiction is resolved by realizing that the tensor product of witnesses for $A|A'$, $B|B'$, $C|C'$ is not necessarily a witness for $ABC|A'B'C'$. The following shows that this method can indeed work:

Observation 1.—Bipartite witnesses can be used to detect multipartite entangled states nonlinearly.

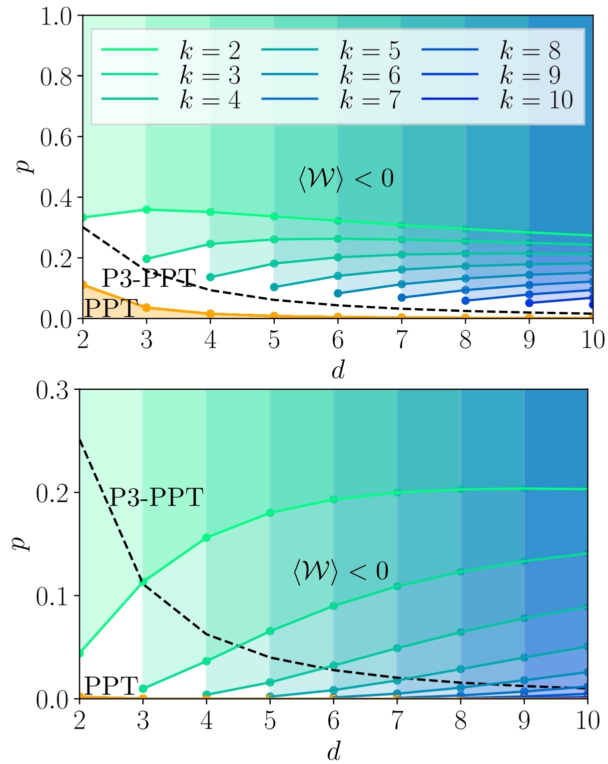

FIG. 3. Nonlinear detection of noisy four- (top) and ten-partite (bottom) GHZ states. Shown are the detection curves with $k = 2, 3, \ldots, 10$ copies of noisy n-partite GHZ states shared among an even number of parties n with local dimension d (top $n = 4$; bottom $n = 10$). We assume $d \geq k$ as otherwise the antisymmetrizer vanishes. All states with positive partial transpositions (PPT) are in the orange region, and hence states outside are entangled; the green-blue regions are states that are detected by Eq. (10) with $W = \mathbb{1} - k!P_{1^k}$ and $\mathcal{W} = W \otimes P_{1^k}^{\otimes n-1}$, even though $\mathrm{tr}(W\varrho) \geq 0$. Entangled states that are not detected by $\mathcal{W}$ are in the white region, suggesting that detection gains robustness with the number of copies. The dashed line denotes the detection threshold by the P3-PPT criterion proposed in [15].

To see this take the two-qubit witnesses [22]

$$W = \mathbb{1} - X \otimes X - Z \otimes Z, \qquad V = |\phi^+\rangle\langle\phi^+|^\Gamma, \tag{5}$$

where Γ is the partial transpose and $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$; and the Greenberger-Horne-Zeilinger state $\varphi = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$ with $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Then

$$\mathrm{tr}((W_{AA'} \otimes W_{BB'} \otimes V_{CC'})(\varphi_{ABC}{}^{\otimes 2})) = -1/2. \tag{6}$$

An example with $k = n$ is the detection of the Bell state $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, where the witnesses of Eq. (5) give $\langle W \otimes V\rangle_{\psi^{+\otimes 2}} = -1/2$.

Entanglement concentration.—If in Eq. (2) we replace some witnesses W_i by positive semidefinite operators P_i, the expectation value with respect to k copies of an n-qudit separable state remains non-negative. The following observation shows that this way, a witness with non-negative expectation value with respect to a certain entangled state can detect its entanglement.

Observation 2.—There exist pairs of states ϱ and witnesses W for which $\mathrm{tr}(W\varrho) \geq 0$, but for which there is $P \geq 0$ such that

$$\mathrm{tr}((W \otimes P)\varrho^{\otimes k}) < 0. \tag{7}$$

In short: linear entanglement detection with W fails, but nonlinear detection with $W \otimes P$ succeeds.

Consider the n-qudit Greenberger-Horne-Zeilinger state $|\mathrm{GHZ}\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |i\rangle^{\otimes n}$ affected by white noise,

$$\varrho = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-p}{d^n}\mathbb{1}, \tag{8}$$

whose experimental generation and detection plays a central role in quantum computing and communication tasks [23–25]. Take the n-qudit witness

$$W = \mathbb{1} - n! P_{1^n}, \tag{9}$$

where P_{1^n} is the projector onto the fully antisymmetric subspace of n qudits. The state ϱ has a nonpositive partial transpose (and thus is entangled) for $p > 1/(1 + d^{n-1})$ [26,27].

While $\mathrm{tr}(W\varrho) \geq 0$, given $\mathcal{W} = (\mathbb{1} - k! P_{1^k}) \otimes P_{1^k}^{\otimes n-1}$ one verifies that

$$\mathrm{tr}(\mathcal{W}\varrho^{\otimes k}) < 0 \tag{10}$$

for a range of values of p (see Fig. 3).

With this perspective, one can generalize Eq. (2) and evaluate n observables $O_i \in L[(\mathbb{C}^d)^{\otimes k}]$ on k copies of $\varrho \in L[(\mathbb{C}^d)^{\otimes n}]$ as

$$\mathrm{tr}[(O_1 \otimes \ldots \otimes O_n)\varrho^{\otimes k}], \tag{11}$$

thus certifying that ϱ is entangled if the result is negative and the operators O_i are either positive semidefinite operators or witnesses (see Fig. 2 for an example). This way of combining positive operators and witnesses can be understood as follows: suppose Alice and Bob share two copies of an entangled state ϱ_{AB} and $\varrho_{A'B'}$. If Bob measures $\mathbb{1}_{AA'} \otimes P_{BB'}$, then with probability $\mathrm{tr}(P\varrho_B \otimes \varrho_{B'})$, the state on Alice's side reads

$$\xi_{AA'} = \frac{\mathrm{tr}_{BB'}((\mathbb{1}_{AA'} \otimes P_{BB'})(\varrho_{AB} \otimes \varrho_{A'B'}))}{\mathrm{tr}(P\varrho_B \otimes \varrho_{B'})}, \tag{12}$$

where $\varrho_B = \mathrm{tr}_A(\varrho_{AB})$. If Alice measures $\mathrm{tr}(W\xi) < 0$ for some witness W, then ϱ is entangled. In this case, Bob's measurement has concentrated the entanglement of $\varrho^{\otimes 2}$ present in the partition $AA'|BB'$ into the subsystem AA'. This procedure is illustrated in Fig. 2.

For most pure states, this procedure of entanglement concentration on Alice's side can be understood in terms of entanglement swapping as follows. Suppose ϱ is a pure state $|\psi\rangle = \mathbb{1} \otimes S|\phi^+\rangle$ where the matrix S is invertible with $\mathrm{tr}(SS^\dagger) = 1$ and $|\phi^+\rangle = \sum_{i=0}^{d-1} |ii\rangle/\sqrt{d}$ [28]. Then, by projecting BB' onto the state $|\varphi\rangle = \mathbb{1} \otimes S^{\dagger -1}|\phi^+\rangle$, Bob ($BB'$) effectively teleports his part of $|\psi\rangle$ to Alice (AA') with nonzero probability using a second copy of the shared state $|\psi\rangle$ itself. When then Alice measures a witness W the protocol reduces to standard linear detection, evaluating $\mathrm{tr}(W|\psi\rangle\langle\psi|)$.

Trace polynomial witnesses.—How can one find suitable nonlinear witnesses that are simple to work with experimentally? Here we focus on trace polynomials, due to the fact that these are experimentally accessible through the randomized measurement toolbox [13]. A systematic construction of symmetric trace polynomials arises as follows: matrix inequalities such as the Hadamard inequality,

$$\prod_{i=1}^{n} A_{ii} \geq \det(A) \tag{13}$$

for any $n \times n$ positive semidefinite matrix A, have been a long-standing topic of investigation [29,30]. In particular, the so-called immanant inequalities generalize Eq. (13) and provide a large supply of n-partite linear witnesses of the form [31]

$$W = \sum_{\sigma \in S_n} a_\sigma \eta_d(\sigma), \tag{14}$$

where $a_\sigma \in \mathbb{C}$ and $\eta_d(\sigma)$ is the representation of $\sigma \in S_n$ permuting the n tensor factors of $(\mathbb{C}^d)^{\otimes n}$,

$$\eta_d(\sigma)|i_1\rangle \otimes \ldots \otimes |i_n\rangle = |i_{\sigma^{-1}(1)}\rangle \otimes \ldots \otimes |i_{\sigma^{-1}(n)}\rangle. \tag{15}$$

How to transform matrix inequalities to witnesses in the symmetric group algebra over the complex numbers, $\mathbb{C}S_n = \{\sum_{\sigma \in S_n} a_\sigma \sigma | a_\sigma \in \mathbb{C}\}$, is sketched in [31]. For example, from Eq. (13) one obtains the n-qudit witness of Eq. (9). A few standard immanant inequalities and their corresponding witnesses are listed in Table I.

By recycling linear witnesses or positive operators $O_i \in \mathbb{C}S_k$ arising from immanant inequalities and using them in the nonlinear way of Eq. (11), the resulting nonlinear witness $\mathcal{W} = O_1 \otimes \ldots \otimes O_n$ has interesting features. Its expectation value $\langle \mathcal{W} \rangle_{\varrho^{\otimes k}}$ is a homogeneous polynomial of degree k in ϱ, acting on the copies of subsystems as trace polynomials [32,33]. Since by the Schur-Weyl duality the action of the symmetric group S_k commutes with that of k-fold tensor products $X^{\otimes k}$, the expression $\langle \mathcal{W} \rangle_{\varrho^{\otimes k}}$ is a local unitary invariant. Similarly to

TABLE I. Immanant inequalities and their corresponding witnesses. Listed are immanant inequalities and their corresponding entanglement witnesses. Here χ_λ is the character of the irreducible representation of the symmetric group S_n labeled by the partition $\lambda \vdash n$; and P_λ are Young projectors. The Hook inequalities hold for hook tableaux with shape $\lambda = (j, 1^{n-j})$ and then $\lambda' = (j-1, 1^{n-j+1})$. The permanent dominance inequality is conjectured but has not yet been proven [29].

Name	Inequality	Witness
Hadamard	$\prod_i A_{ii} \geq \det(A)$	$\mathbb{1} - n! P_{1^n}$
Schur	$[\mathrm{imm}_\lambda(A)/\chi_\lambda(\mathrm{id})] \geq \det(A)$	$[P_\lambda/\chi_\lambda^2(\mathrm{id})] - P_{1^n}$
Hook	$[\mathrm{imm}_\lambda(A)/\chi_\lambda(\mathrm{id})] \geq [\mathrm{imm}_{\lambda'}(A)/\chi_{\lambda'}(\mathrm{id})]$	$[P_\lambda/\chi_\lambda^2(\mathrm{id})] - [P_{\lambda'}/\chi_{\lambda'}^2(\mathrm{id})]$
Marcus	$\mathrm{per}(A) \geq \prod_i A_{ii}$	$n! P_n - \mathbb{1}$
Permanent	$\mathrm{per}(A) \geq [\mathrm{imm}_\lambda(A)/\chi_\lambda(\mathrm{id})]$	$P_n - [P_\lambda/\chi_\lambda^2(\mathrm{id})]$

the recently introduced P3-PPT condition and other local unitary invariant quantities [14,15], trace polynomial witnesses can experimentally be measured using the randomized measurements approach. There local projective measurements in random bases allow us to estimate the expectation value of k-copy observables [13,34]. This way, trace polynomial witnesses can be evaluated with classical postprocessing by matrix multiplication, thus avoiding storing large tensor products of matrices.

Insights into the strength of this approach can be obtained using Haar integration (Theorem 4.3 in [35]). This allows us to compute the average expectation value of nonlinear trace polynomial witnesses with respect to Haar random states $|\psi\rangle \in \mathbb{C}^d$ with $d = d_1 \cdots d_n$ over the Haar measure $\mathbf{d}_\mu$,

$$\mathbb{E}[\langle \mathcal{W} \rangle_\psi] = \int_{U \in \mathbf{U}(d)} \mathrm{tr}(\mathcal{W} U |\psi\rangle\langle\psi| U^\dagger) \mathbf{d}_\mu(U). \tag{16}$$

For example, entanglement concentration of a Haar-random n-partite state $|\psi\rangle$ with n even and the witness $W = \mathbb{1} - k! P_{1^k}$ in Eq. (9) yields a negative value on average.

This approach also allows us to design separability criteria based on the spectra of the state and its reductions. For example, if a state $\varrho \in \mathbb{C}^d \otimes \mathbb{C}^d$ with $\mathrm{rank}(\varrho) < d$ has a maximally mixed marginal, then it is entangled.

Testing with Werner states.—The exponential growth of the Hilbert space with the number of subsystems limits our capability to numerically test this approach. Using trace polynomial witnesses $W \in \mathbb{C}S_k$ allows us to sample states and witnesses of nontrivial sizes with a symbolic algebra package, which is computationally cheaper than computing the expectation value of exponentially large matrices. Sampling states and witnesses symbolically relies on group ring states and can be summarized as follows. On the symmetric group S_n, a trace can be defined as

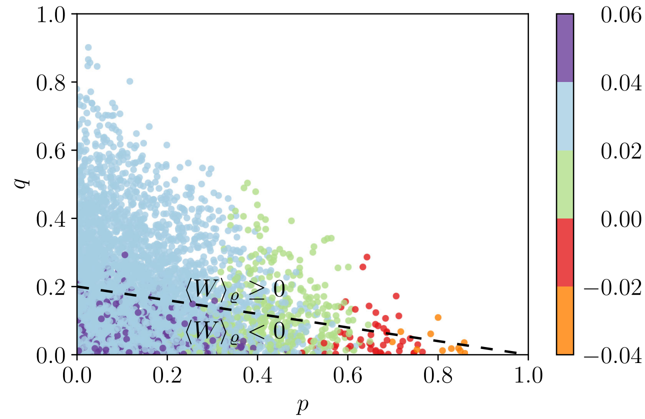

FIG. 4. Detection of three qutrit random Werner states using three copies. We represent, in colored dots, a sample of 5000 random Werner states ϱ. The horizontal and vertical axes show their projections onto the antisymmetric and symmetric subspaces, respectively, $p = \mathrm{tr}(\varrho P_{111})$ and $q = \mathrm{tr}(\varrho P_3)$, such that $\mathrm{tr}(\varrho P_{21}) = 1 - p - q$. States below (above) the analytical dashed line are detected (not detected) linearly as $\langle W \rangle_\varrho < 0$ ($\langle W \rangle_\varrho \geq 0$). The color gradient represents the expectation value $\langle \mathcal{W} \rangle_{\varrho^{\otimes 3}}$ with $\mathcal{W} = W^{\otimes 2} \otimes P_{21}$, where $W = P_3 - (P_{21}/4)$ is a witness. This suggests that states with a larger component in the antisymmetric subspace are better detected by $\mathcal{W}$.

$$\tau(\sigma) = \begin{cases} n! & \text{if } \sigma = \mathrm{id}, \\ 0 & \text{else}. \end{cases} \tag{17}$$

Reference [8] showed that given $r \in S_n$ with support only in $(\ker \eta_d)^\perp$ and $b \in \mathbb{C}S_n$, it holds that

$$n!\mathrm{tr}(Wg(d,n)\eta_d(r)\eta_d(b)) = \tau(rb), \tag{18}$$

where $\mathrm{Wg}(d,n)$ is the Weingarten operator [36,37]. This equality allows us to compute in $\mathbb{C}S_n$ expectation values of the form (11) avoiding the exponential growth of the number of parameters with the local dimension d. Let us focus on the system size $k = n = d = 3$. While a desktop computer cannot evaluate Eq. (11) in the Hilbert space $(\mathbb{C}^3)^{\otimes 9}$, the fact that the nine-qutrit state factorizes as $\varrho^{\otimes 3}$ makes this computation possible in $(\mathbb{C}S_3)^{\times 3}$ symbolically. For example, the expectation value of $\langle \mathcal{W} \rangle_{\varrho^{\otimes 3}}$ where $\mathcal{W} = W^{\otimes 2} \otimes P_{21}$ with $W = P_3 - (P_{21}/4)$, evaluated on random Werner states is shown in Fig. 4. The results suggest that the detection of random states depends on their component within each irreducible subspace.

Evaluating positive maps.—Using variations of Eq. (11) one can detect nonpositive outcomes of positive maps applied locally. For example one can evaluate the reduction criteria [38] via purity conditions [16] with Eqs. (9) and (10), and detect nonpositivity of the partial transpose Γ of a state ϱ via

$$\mathrm{tr}((\phi^+ \otimes V)(\varrho \otimes \sigma)) = \mathrm{tr}(\varrho^\Gamma \sigma) \tag{19}$$

if σ lies in a negative eigenspace of ϱ^Γ. In this sense, our method relates to the existence problem of nondecomposable tensor stable maps, which remain positive under tensor powers [39,40], as follows: there exists a nondecomposable tensor stable map, if and only if, there exists a bipartite witness W such that $W^{\otimes n}$ cannot detect any pair of n-partite states ϱ and σ. This is because a nondecomposable map Φ_W is tensor stable if and only if

$$\Phi_W{}^{\otimes n}(\rho) = \mathrm{tr}_1[W^{\otimes n}(\rho^T \otimes \mathbb{1}_d)] \geq 0 \tag{20}$$

for any n-partite state ϱ, where W is a witness for states with positive partial transpose (see Chapter 11 of [41] and references therein); which is true if and only if the expectation value of $\Phi_W{}^{\otimes n}(\rho)$ for any other n-partite state σ is non-negative.

Conclusions.—Our approach shows how entanglement witnesses can be recycled in nonlinear fashion, thus increasing the range of applicability of a given witness. In particular, this approach is suitable to randomized measurements [14,42,43], trace polynomials being naturally invariant under local unitaries.

Some open questions remain: How can one tailor nonlinear witnesses to specific states? The Keyl-Werner theorem [44] and subsequent work [45] suggest that it is possible to choose witnesses according to the spectra of the reductions of the state in hand. While here we provide a first step in this direction, a systematic approach is yet missing. Lastly, it would be interesting to make use of matrix inequalities involving elementary symmetric polynomials [29], which might be more powerful than the particular case of immanant inequalities.

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