Extending the Known Region of Nonlocal Boxes that Collapse Communication Complexity

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Nonsignaling boxes ($\mathcal{NS}$) are theoretical resources defined by the principle of no-faster-than-light communication. They generalize quantum correlations and some of them are known to collapse communication complexity (CC). However, this collapse is strongly believed to be unachievable in nature, so its study provides intuition on which theories are unrealistic. In the present Letter, we find a better sufficient condition for a nonlocal box to collapse CC, thus extending the known collapsing region. In some slices of $\mathcal{NS}$, we show this condition coincides with an area outside of an ellipse.

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Entanglement is a fascinating relation linking pairs of particles. It was experimentally confirmed in the late 20th century [1–3], and it has the striking property of "nonlocality": two entangled particles, although being very distantly separated, provide strongly correlated results when their state is measured, yet the result of those measurements could not be known ahead of time [4,5].

Nevertheless, this powerful nonlocality described by quantum mechanics is limited by Tsirelson's famous bound [6]. It is then natural to wonder if there could exist a more general theory than quantum mechanics to accurately describe the world, with more powerful nonlocality than quantum entanglement. To that end, the common framework is the one of nonlocal boxes (NLBs) [7]. An NLB is a theoretical tool that generalizes the notions of shared randomness, quantum correlation, and nonsignaling correlation. As drawn in Fig. 1, an NLB has two input bits and two outputs bits. Alice has access only to the left side and Bob to the right side. Immediately after inputting x in the box, Alice receives a, whether or not Bob has already input his bit y. More formally, an NLB is characterized by a conditional distribution $P(a,b|x,y)$ that satisfies the nonsignaling conditions [7,8] $\sum_{\tilde{b}} P(a,\tilde{b}|x,0) = \sum_{\tilde{b}} P(a,\tilde{b}|x,1)$ and $\sum_{\tilde{a}} P(\tilde{a},b|0,y) = \sum_{\tilde{a}} P(\tilde{a},b|1,y)$ for all $a,b,x,y \in \{0,1\}$. Denote by $\mathcal{NS}$ the set of all NLBs, which is an eight-dimensional convex set with finitely many extremal points [9,10].

Among the most famous boxes, there is the PR box, introduced by Popescu and Rohrlich [7], taking value $1/2$ if $a \oplus b = xy$, and 0 otherwise, where the symbol $\oplus$ denotes the sum modulo 2. Note that this box is designed to perfectly win at the Clauser-Horne-Shimony-Holt (CHSH) game [5]. In this Letter, we will also use the PR' box, taking value $1/2$ if $a \oplus b = (x \oplus 1)(y \oplus 1)$, and 0 otherwise, which perfectly wins at CHSH′ [11] [same game as CHSH but with the rule $a \oplus b = (x \oplus 1)(y \oplus 1)$]. In addition, we will use the fully random box I, taking value $1/4$ for all inputs and outputs, and the shared randomness box SR, taking value $1/2$ if $a = b$, and 0 otherwise, independent of the entries x and y. Note that SR is nothing more than a shared random bit. A bar above a box means that the behavior is the opposite one: $\bar{P} \coloneqq 1 - P$.

The notion of communication complexity (CC) was introduced by Yao [12] and was widely studied in the late 20th century [13,14]. It can be viewed as a game as presented in Fig. 1. The communication complexity $\mathrm{CC}(f)$ of f is defined as the minimal amount of communication bits required to win at this game for any question

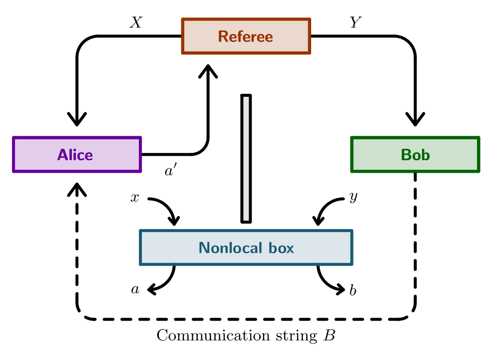

FIG. 1. Communication complexity game. Lowercase letters a, a', b, x, y are bits and capital letters are strings: $X \in \{0,1\}^n$, $Y \in \{0,1\}^m$, and $B \in \{0,1\}^k$. Let $f\colon \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ be known. Once the game starts, Alice and Bob are spacelike separated and the referee sends them the respective strings X and Y. The goal is that Alice answers a bit a' such that $a' = f(X,Y)$. To achieve it, Bob is allowed to send some communication bits to Alice, but these bits are costly so he wants to send as few as possible. They may also use as many copies as they want of an NLB.

strings X and Y. One can see that $\mathrm{CC}(f) \leq m$ always, and that $\mathrm{CC}(f) \geq 1$ if f is not constant in Y. There exists also a probabilistic version of CC [15], in which Alice is allowed to make some mistakes: $\mathrm{CC}^p(f)$ (for some $p \in [0,1]$) is defined as the minimal amount of communication bits required to win with probability $\geq p$, for any X and Y. Note that $\mathrm{CC}^{1/2}(f) = 0$ for any f, using the strategy in which Alice always answers a uniformly random bit $a' \in \{0,1\}$. We say that CC "collapses" (or that it is trivial) when a single bit of communication is enough and that the error is bounded, i.e., when there exists $p > 1/2$ such that for all f we have $\mathrm{CC}^p(f) \leq 1$. This is strongly believed to be impossible in nature since it would imply the absurdity that a single bit of communication is sufficient to distantly compute any f [15–18].

Thus, the study of such a collapse helps one to understand why some correlations are not allowed in quantum mechanics. In the past two decades, some boxes were shown to be collapsing, i.e., to collapse CC, see Fig. 2. However, there is still a major open question: do the other NLBs also collapse communication complexity? In the present Letter, after generalizing the BBLMTU protocol [15] (named after the authors' initials), we find a new sufficient condition that analytically extends the region of collapsing boxes, thus partially answering the open question.

Protocols.—We define by induction a sequence of protocols $(\mathcal{P}_k)_{k\geq 0}$ generalizing the BBLMTU protocol [15], the main difference being that we add local uniformity.

Local uniformization: We say that a box $P \in \mathcal{NS}$ is "locally uniform" if, on each player's side, the box always outputs uniformly random bits: $P(a|x) = 1/2$ and $P(b|y) = 1/2$ for any $a, b, x, y \in \{0,1\}$, where $P(a|x) \coloneqq \sum_b P(a,b|x,y)$ is independent of y by nonsignaling and similarly for $P(b|y)$. The local uniformity will be useful many times in later computations. However, some boxes P are *not* locally uniform, e.g., $P = [(PR + P_0)/2] \in \mathcal{NS}$, where P_0 is the box that always answers (0,0) independent of the entries (x, y). This is why Alice and Bob use a "trick": from P and a shared random bit r, they simulate another box $\tilde{P} \in \mathcal{NS}$ by adding r to the outputs of P. That way, the new box $\tilde{P}$ is indeed locally uniform, and importantly, it has the same bias η_{xy} as the initial box P for all x, y,

$$P(a \oplus b = xy|x,y) = \tilde{P}(a' \oplus b' = xy|x,y) = \frac{1+\eta_{xy}}{2},$$

where $\eta_{xy} \in [-1,1]$ is defined as $\eta_{xy} \coloneqq 2P(a \oplus b = xy|x,y) - 1 = 2\sum_c P(c, c \oplus xy|x,y) - 1$.

Protocol $\mathcal{P}_0$: Fix a Boolean function $f\colon \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ and strings $X \in \{0,1\}^n$ and $Y \in \{0,1\}^m$. The goal of the protocol is to perform a "distributed computation" of f [16], i.e., to find bits $a, b \in \{0,1\}$ known by Alice and Bob, respectively, such that

$$a \oplus b = f(X,Y). \tag{1}$$

Assume Alice and Bob share uniformly random variables $Z \in \{0,1\}^m$ and $r \in \{0,1\}$. Upon receiving her string X, Alice produces a bit $a \coloneqq f(X,Z) \oplus r$. As for Bob, if he receives a string Y that is equal to Z, then he sets $b \coloneqq r$; otherwise, he generates a local random variable r_B and sets $b \coloneqq r_B$. Now, separating the cases $Y = Z$ and $Y \neq Z$, the distributed computation (1) is achieved with probability

$$p_0 \coloneqq \mathbb{P}(1) = \frac{1}{2^m} + \frac{1}{2}\left(1 - \frac{1}{2^m}\right) = \frac{1}{2} + \frac{1}{2^{m+1}} > \frac{1}{2}.$$

Because of the shared random bit r, note that the bit a is locally uniform: $\mathbb{P}(a|X) = 1/2$ for all a, X, and similarly for b. In total, this protocol uses $m+1$ shared random bits.

Protocol $\mathcal{P}_1$: As in $\mathcal{P}_0$, we fix f, X, and Y, and we try to obtain the distributed computation (1) with a better probability $p_1 > p_0$. To that end, we realize four steps.

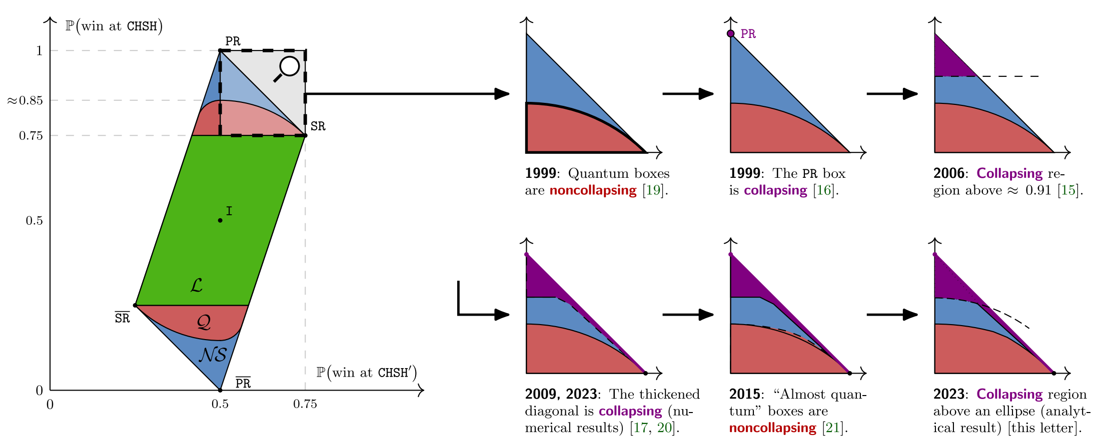

FIG. 2. Historical overview of collapsing boxes, drawn in the slice of $\mathcal{NS}$ passing through PR and SR and I. Red and purple represent, respectively, the noncollapsing and the collapsing boxes. In blue is drawn the region of boxes for which we do not know yet if they collapse communication complexity. See Refs. [22–31] for impossibility results and others.

(a) We use the protocol $\mathcal{P}_0$ independently three times and obtain three pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ such that

$$a_i \oplus b_i = \begin{cases} f(X,Y) & \text{with prob. } p_0, \\ f(X,Y) \oplus 1 & \text{with prob. } 1-p_0, \end{cases}$$

for $i = 1, 2, 3$. Note that this is a repetition code that will be decoded in (b) using a majority vote.

(b) The majority function $\mathrm{Maj}\colon \{0,1\}^3 \to \{0,1\}$ is the function that outputs the most-appearing bit in its entries, i.e., $\mathrm{Maj}(\alpha,\beta,\gamma) = \mathbb{1}_{\alpha+\beta+\gamma\geq 2}$, where $\mathbb{1}$ is the indicator function. The equality

$$f(X,Y) = \mathrm{Maj}(a_1 \oplus b_1, a_2 \oplus b_2, a_3 \oplus b_3) \tag{2}$$

occurs if and only if at least two of the equations $f(X,Y) = a_i \oplus b_i$ $(i = 1, 2, 3)$ hold. Denote $e_i \coloneqq a_i \oplus b_i \oplus f(X,Y)$, and notice that $e_i = 0$ if and only if $a_i \oplus b_i = f(X,Y)$ for fixed i, so that Eq. (2) is equivalent to $\mathrm{Maj}(e_1, e_2, e_3) = 0$. But the e_i's are independent and $\mathbb{P}(e_i = \alpha) = p_0^{1-\alpha}(1-p_0)^\alpha$ for $\alpha = 0,\ 1$, so equality (2) holds with probability

$$\begin{aligned} \mathbb{P}(2) &= \sum_{\substack{\alpha,\beta,\gamma\in\{0,1\} \\ \text{such that } \mathrm{Maj}(\alpha,\beta,\gamma)=0}} \mathbb{P}(e_1=\alpha)\mathbb{P}(e_2=\beta)\mathbb{P}(e_3=\gamma) \\ &= \sum_{\substack{\alpha,\beta,\gamma\in\{0,1\} \\ \text{such that } \mathrm{Maj}(\alpha,\beta,\gamma)=0}} p_0^{3-\alpha-\beta-\gamma}(1-p_0)^{\alpha+\beta+\gamma}. \end{aligned}$$

(c) Now, we try to distributively compute the majority function. Observe that

$$\begin{aligned} &\mathrm{Maj}(a_1 \oplus b_1, a_2 \oplus b_2, a_3 \oplus b_3) \\ &\quad = \mathrm{Maj}(a_1, a_2, a_3) \oplus \mathrm{Maj}(b_1, b_2, b_3) \oplus r_1 s_1 \oplus r_2 s_2, \end{aligned}$$

where $r_1 \coloneqq a_1 \oplus a_2$, $s_1 \coloneqq b_2 \oplus b_3$, $r_2 \coloneqq a_2 \oplus a_3$, and $s_2 \coloneqq b_1 \oplus b_2$. To distributively compute the two products $r_j s_j$ $(j = 1,\ 2)$, Alice and Bob use two copies of their locally uniform box $\tilde{P}$, see Fig. 3. They obtain pairs of bits (a_1', b_1') and (a_2', b_2') such that $a_j' \oplus b_j' = r_j s_j$ with bias η_{r_j,s_j}. Consider the events $E_{\alpha,\beta,\gamma} \coloneqq (e_1 = \alpha,\ e_2 = \beta,\ e_3 = \gamma)$ and $F_{\delta,\varepsilon,\zeta,\theta} \coloneqq (r_1 = \delta,\ r_2 = \varepsilon,\ s_1 = \zeta,\ s_2 = \theta)$, where the greek letters are in $\{0,1\}$. On the one hand, under $E_{\alpha,\beta,\gamma}$ and $F_{\delta,\varepsilon,\zeta,\theta}$, we see that the equality

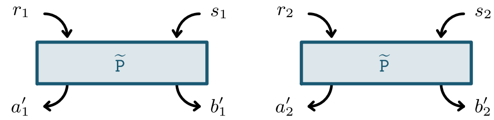

FIG. 3. Distributively compute the products $r_1 s_1$ and $r_2 s_2$ with probability bias η_{r_1,s_1} and η_{r_2,s_2}, respectively.

$$r_1 s_1 \oplus r_2 s_2 = (a_1' \oplus b_1') \oplus (a_2' \oplus b_2') \tag{3}$$

holds if and only if both of the equations $r_j s_j = a_j' \oplus b_j'$ hold $(j = 1,\ 2)$ or if none of them hold (because errors cancel out: $1 \oplus 1 = 0$). Hence, this equality holds with a bias $\eta_{\delta,\zeta}\eta_{\varepsilon,\theta}$,

$$\mathbb{P}\Big(\text{Eq. (1)}|E_{\alpha\beta\gamma}, F_{\delta\varepsilon\zeta\theta}\Big) = \frac{1+\eta_{\delta,\zeta}\eta_{\varepsilon,\theta}}{2}, \tag{4}$$

(conditional to knowing X and Y as well). On the other hand, seeing that the definitions of r_j and s_j lead to the relations $s_1 = r_2 \oplus e_2 \oplus e_3$ and $s_2 = r_1 \oplus e_1 \oplus e_2$, and using the independence of the a_i's and their local uniform distribution in $\mathcal{P}_0$, direct computations yield that

$$\mathbb{P}(F_{\delta,\varepsilon,\zeta,\theta}|E_{\alpha,\beta,\gamma}) = \frac{1}{4}\mathbb{1}_{\zeta=\beta\oplus\gamma\oplus\varepsilon}\mathbb{1}_{\theta=\alpha\oplus\beta\oplus\delta}. \tag{5}$$

Therefore, summing the products of (4) and (5) over all $\delta,\varepsilon,\zeta,\theta\in\{0,1\}$, we obtain

$$\mathbb{P}\Big(\text{Eq. (1)}|E_{\alpha\beta\gamma}\Big) = \sum_{\delta,\varepsilon\in\{0,1\}} \frac{1+\eta_{\delta,\beta\oplus\gamma\oplus\varepsilon}\eta_{\varepsilon,\alpha\oplus\beta\oplus\delta}}{8}. \tag{6}$$

Hence, we obtain a distributed computation of the majority function as follows:

$$\mathrm{Maj}(a_1 \oplus b_1, a_2 \oplus b_2, a_3 \oplus b_3) = \underbrace{(\mathrm{Maj}(a_1,a_2,a_3) \oplus a_1' \oplus a_2')}_{=:\tilde{a}} \oplus \underbrace{(\mathrm{Maj}(b_1,b_2,b_3) \oplus b_1' \oplus b_2')}_{=:\tilde{b}}, \tag{7}$$

with probability $\sum_{\delta,\varepsilon}(1+\eta_{\delta,\beta\oplus\gamma\oplus\varepsilon}\eta_{\varepsilon,\alpha\oplus\beta\oplus\delta})/8$.

(d) Using steps (b) and (c), we obtain that the equality

$$f(X,Y) = \tilde{a} \oplus \tilde{b} \tag{8}$$

holds if and only if both (2) and (7) hold or if none of them hold. This happens with probability

$$p_1 \coloneqq \mathbb{P}(9) = \mathbb{P}((2) \wedge (7)) + \mathbb{P}(\neg(2) \wedge \neg(7))$$
$$= \sum_{\alpha,\beta,\gamma,\delta,\varepsilon \in \{0,1\}} p_0^{3-\alpha-\beta-\gamma}(1-p_0)^{\alpha+\beta+\gamma} \frac{1+(-1)^{\mathrm{Maj}(\alpha,\beta,\gamma)}\eta_{\delta,\beta\oplus\gamma\oplus\varepsilon}\eta_{\varepsilon,\alpha\oplus\beta\oplus\delta}}{8}.$$

where the sign $+$ from Eq. (6) was changed here into $(-1)^{\mathrm{Maj}(\alpha,\beta,\gamma)}$ because $\mathbb{P}(\neg(7)) = \sum_{\delta,\varepsilon}(1-\eta_{\delta,\beta\oplus\gamma\oplus\varepsilon}\eta_{\varepsilon,\alpha\oplus\beta\oplus\delta})/8$, and this case exactly corresponds to the case where $\mathrm{Maj}(e_1, e_2, e_3) = 1$. Hence, we constructed a protocol $\mathcal{P}_1$ based on $\mathcal{P}_0$, and its probability of achieving (1) is p_1. We will find in the next section a sufficient condition for which $p_1 > p_0$. In total, this protocol uses $3(m+2)-1$ shared random bits and two copies of P.

Protocol $\mathcal{P}_{k+1}$ ($k \geq 1$): We proceed as in $\mathcal{P}_1$. We build $\mathcal{P}_{k+1}$ after performing $\mathcal{P}_k$ three times. In total, the protocol $\mathcal{P}_{k+1}$ uses $3^{k+1}(m+2)-1$ shared random bits and $3^{k+1}-1$ copies of P, and it distributively computes f with probability

$$p_{k+1} = \sum_{\alpha,\beta,\gamma,\delta,\varepsilon \in \{0,1\}} p_k^{3-\alpha-\beta-\gamma}(1-p_k)^{\alpha+\beta+\gamma} \frac{1+(-1)^{\mathrm{Maj}(\alpha,\beta,\gamma)}\eta_{\delta,\beta\oplus\gamma\oplus\varepsilon}\eta_{\varepsilon,\alpha\oplus\beta\oplus\delta}}{8}.$$

Result.—The probability bias associated with p_{k+1} is $\mu_{k+1} \coloneqq 2p_{k+1}-1$ and it can be expressed as $\mu_{k+1} = F(\mu_k)$, with

$$F(\mu) \coloneqq \frac{\mu}{16}\Big(A + B - \mu^2(A-B)\Big),$$

where $A \coloneqq (\eta_{0,0}+\eta_{0,1}+\eta_{1,0}+\eta_{1,1})^2$ and $B \coloneqq 2\eta_{0,0}^2 + 4\eta_{0,1}\eta_{1,0} + 2\eta_{1,1}^2$, where η_{xy} was introduced above as the probability bias of the box P. Note that $0 \leq A \leq 16$ and $-8 \leq B \leq 8$ because $|\eta_{x,y}| \leq 1$ for all x, y.

Theorem 1: Sufficient condition.—Nonlocal boxes for which $A + B > 16$ collapse communication complexity.

Proof.—Assume $A + B > 16$; this inequality has three consequences.

(a) First, it gives $A - B > 16 - 2B \geq 0$ so that F admits exactly three distinct fixed points in $\mathbb{R}$,

$$\left\{0, \pm\sqrt{\frac{A+B-16}{A-B}}\right\} \eqqcolon \{0, \pm\mu_*\}.$$

(b) Second, as $(dF/d\mu)(\mu) = (1/16)(A+B-3\mu^2(A-B))$, the assumption implies that F is increasing on $[-\mu_{\max}, \mu_{\max}]$, where $\mu_{\max} \coloneqq \sqrt{[(A+B)/3(A-B)]} > 0$. Moreover, the assumption gives $(\partial F/\partial\mu)(0) > 1$, so that the fixed point 0 is repulsive.

(c) Finally, as $A + B \leq 24$, we have $A + B - 16 \leq [(A+B)/3]$. Therefore, μ_* is smaller than or equal to $\mu_{\max}$ and

$$[0, \mu_*] \subseteq [-\mu_{\max}, \mu_{\max}].$$

Now, let $P \in \mathcal{NS}$ be a box satisfying $A + B > 16$. We provide Alice and Bob with as many shared random bits and as many copies of P as they want. We show that there exists a constant $p > 1/2$ such that any arbitrary Boolean function $f: \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ can be distributively computed by Alice and Bob with probability $\geq p$, which means that communication complexity collapses. The protocol $\mathcal{P}_0$ defined above enables one to distributively compute f with probability $p_0 = (1 + 1/2^m)/2$, i.e., with bias $\mu_0 = 1/2^m > 0$. Up to adding muted variables in the entries of f, we may assume that m is large enough so that $\mu_0 \in (0, \mu_*)$. Then, combining (a), (b), and (c), we get that the sequence $(\mu_k)_k$ converges to the fixed point $\mu_* > 0$. We set $p \coloneqq (1 + \mu_*/2)/2 > 1/2$ [or replace $\mu_*/2$ by any choice of $\mu \in (0, \mu_*)$], and we know that there exists a protocol $\mathcal{P}_k$ for some k large enough such that the probability p_k of correctly distributively computing f satisfies $p_k > p$. Finally, note that p does *not* depend on f: it only depends on μ_*, which only depends on the $\eta_{x,y}$'s, which themselves only depend on P. Hence, communication complexity collapses. ■

Cases of interest.—Case 1: *PR-PR′-I*: We consider a box P that is in the slice of $\mathcal{NS}$ passing through both PR and PR' and I, studied in [11]. In this case $\eta_{0,0} = \eta_{1,1}$ and $\eta_{0,1} = \eta_{1,0}$, and the condition $A + B > 16$ of the Theorem reads as $\eta_{0,0}^2 + \eta_{0,0}\eta_{0,1} + \eta_{0,1}^2 > 2$. We make a change of coordinates using the bias of winning at CHSH $\sigma = (\eta_{0,0} + \eta_{0,1})/2$ and the one of winning at CHSH′ $\sigma' = (-\eta_{0,0} + \eta_{0,1})/2$, and we obtain

$$\sigma^2 + \frac{1}{3}\sigma'^2 > \frac{2}{3} \quad \text{or} \quad \frac{1}{3}\sigma^2 + \sigma'^2 > \frac{2}{3},$$

where the second equation holds by changing the role of σ and σ' in the first one (indeed, we may do it because flipping bits x and y allows us to go from CHSH to CHSH′). These equations give rise to the purple collapsing area drawn in Fig. 4(a). Interestingly, on the vertical axis, we find the same result as in [15]: taking $\sigma' = 0$, it is

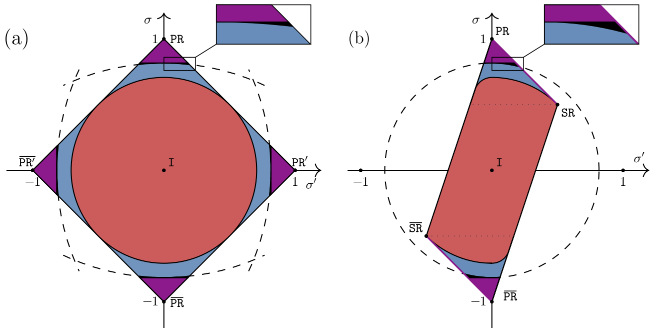

FIG. 4. Two slices of $\mathcal{NS}$. In purple is drawn the prior (analytically) known collapsing region. We extend it using Theorem 1: the black area is the new analytic collapsing region. The red area corresponds to the area of noncollapsing boxes. The blue area is the gap to be filled in red or purple (open problem). (a),(b) Represent the slices of $\mathcal{NS}$ passing through, respectively, $\{PR, PR', I\}$ (case 1, finding interest in [11]) and $\{PR, SR, I\}$ (case 2, finding interest in [17]).

enough to have $\sigma > \sqrt{2/3}$, i.e., to win at CHSH with probability $[(1+\sigma)/2] > [(3+\sqrt{6})/6] \approx 0.91$.

Case 2: $PR - SR - I$: We consider a box P that is in the slice of $\mathcal{NS}$ passing through both PR and SR and I, studied in [17]. In this case, $\eta_{0,0} = \eta_{0,1} = \eta_{1,0}$, and the condition $A + B > 16$ of the Theorem reads as $5\eta_{0,0}^2 + 2\eta_{0,0}\eta_{1,1} + \eta_{1,1}^2 > (16/3)$. We make a change of coordinates using $\sigma = (3\eta_{0,0} + \eta_{1,1})/4$ and $\sigma' = (\eta_{0,0} - \eta_{1,1})/4$, and we obtain

$$\sigma^2 + \sigma'^2 > \frac{2}{3}.$$

The induced collapsing area is represented in Fig. 4(b). The same results also hold if we replace SR by any convex combination of P_0 and P_1, which are the boxes that always output, respectively, (0,0) and (1,1) independent of the entries (x, y).

Remark.—Even in comparison with previous numerical results, our protocol finds strictly new collapsing boxes. Indeed, for instance, consider boxes in the black region of Fig. 4 that are close to the vertical axis: they are not distillable by means of the wirings of [17,20], but our result shows that they are still collapsing.

Conclusion.—After generalizing the BBLMTU protocol, we found in Theorem 1 a new sufficient condition for a box to collapse communication complexity, with the following advantages: (1) it is valid in the whole eight-dimensional convex set $\mathcal{NS}$, in contrast to the analytical result of [17] (it holds only in the segment joining PR and SR), and (2) it is completely analytical, with an explicit formula for the boundary of the new collapsing area, in contrast to previous numerical results [17,20] (as far as we know, the boundary of these two results has not yet been analytically computed). In Fig. 4, we presented two examples of new collapsing regions. Note that the importance of our result is emphasized by considering the many known impossibility results [18,25,26]. According to our present intuition of nature [15–18], a consequence is that our new collapsing boxes are unlikely to appear in nature.

Hence, we partially answer the open question, but there is still a gap to be filled: what other nonlocal boxes collapse communication complexity?

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