Effective Field Theories on the Jet Bundle

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We develop a generalized field space geometry for higher-derivative scalar field theories, expressing scattering amplitudes in terms of a covariant geometry on the all-order jet bundle. The incorporation of spacetime and field derivative coordinates solves complications due to higher-order derivatives faced by existing approaches to field space geometry. We identify a jet bundle analog to the field space metric that, besides field redefinitions, exhibits invariance under total derivatives. The invariance consequently extends to its amplitude contributions and the canonical covariant geometry.

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Introduction.—Scattering amplitudes in quantum field theory are invariant under field redefinitions [1–3], a property aptly framed in terms of coordinate independence in differential geometry. Indeed, it is well-understood that the field space manifold, endowed with a Riemannian metric, identifies amplitudes with covariant tensors like its curvature [4,5]. This formalism finds many uses in the study of effective field theories [6–11], but suffers a significant shortcoming: the Riemannian geometry is unable to naturally accommodate operators involving higher-order derivatives on fields, so that prior manipulation of the Lagrangian is necessary to fit such operators into the Riemannian framework.

To be concrete, consider a theory of scalar fields $\phi^i(x^\mu)$ on the spacetime manifold $\mathcal{T}$ endowed with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Just as x^μ charts out $\mathcal{T}$, ϕ^i charts out a manifold $\mathcal{M}$, the field space manifold, with dimension equal to the number of flavors. Field redefinitions $\phi^i(\tilde{\phi}^j)$ without derivatives are then coordinate transformations on $\mathcal{M}$, under which first field derivatives $\partial_\mu \phi^i \equiv \phi^i_\mu$ transform like a tensor on $\mathcal{M}$:

$$\phi^i_\mu = \frac{\partial \phi^i}{\partial \tilde{\phi}^j} \tilde{\phi}^j_\mu. \tag{1}$$

Thus, given a Lagrangian $\mathcal{L}$ comprising

$$V(\phi^a) + \eta^{\mu\nu} g_{ij}(\phi^a)\phi^i_\mu \phi^j_\nu + \eta^{\mu\nu}\eta^{\rho\sigma} c_{ijkl}(\phi^a)\phi^i_\mu \phi^j_\nu \phi^k_\rho \phi^l_\sigma, \tag{2}$$

the coefficients V, g_{ij}, and c_{ijkl} transform as tensors. In particular, g_{ij} is positive-definite in a unitary theory and hence a Riemannian metric, giving rise to a Levi-Civita connection ∇ on $\mathcal{M}$ with an associated curvature tensor R^i_{jkl}. Scattering amplitudes can then be expressed using covariant derivatives of these tensors under ∇, making field redefinition invariance manifest [10]. However, consider a term involving $\partial_\mu \partial_\nu \phi^i \equiv \phi^i_{\mu\nu}$:

$$\mathcal{L} \supset \eta^{\mu\nu} h_i(\phi^a) \phi^i_{\mu\nu}. \tag{3}$$

Second field derivatives do not transform like tensors:

$$\phi^i_{\mu_1\mu_2} = \frac{\partial^2 \phi^i}{\partial \tilde{\phi}^j \partial \tilde{\phi}^k} \tilde{\phi}^j_{\mu_1} \tilde{\phi}^k_{\mu_2} + \frac{\partial \phi^i}{\partial \tilde{\phi}^j} \tilde{\phi}^j_{\mu_1\mu_2}, \tag{4}$$

a complication that makes g_{ij} a tensor no longer:

$$\tilde{g}_{kl} = \frac{\partial \phi^i}{\partial \tilde{\phi}^k}\frac{\partial \phi^j}{\partial \tilde{\phi}^l} g_{ij} + \frac{\partial^2 \phi^i}{\partial \tilde{\phi}^k \partial \tilde{\phi}^l} h_i, \quad \tilde{h}_k = \frac{\partial \phi^i}{\partial \tilde{\phi}^k} h_i, \tag{5}$$

and derails the path to covariant amplitudes [12].

To understand why covariance on $\mathcal{M}$ is too limited to treat a general Lagrangian, recall that the tangent space at a point $\phi^i \in \mathcal{M}$ consists of vectors φ^i indicating the directions in which one can tangentially pass through ϕ^i. Precisely speaking, φ^i represents an equivalence class of tangent curves $\phi'^i(t)$ with $\phi'^i(0) = \phi^i$, whose first derivatives $d\phi'^i/dt|_{t=0}$ equal φ^i. Cotangent spaces are the dual spaces of tangent ones, and tensors are built from products of tangent and cotangent spaces at each point. Observe that this construction is based on one-dimensional tangent curves and not $(\dim \mathcal{T})$-dimensional fields. When identifying tensors like V and c_{ijkl}, we are abusing the fact that ϕ^i_μ transforms identically to φ^i. Neither ϕ^i_μ nor $\mathcal{L}$ nor any of its summands is a tensor on $\mathcal{M}$, even if their transformation laws resemble one. Therefore, working on $\mathcal{M}$ only, we are forced to pick out by hand its covariant quantities within $\mathcal{L}$. A more principled approach is to make the whole Lagrangian a covariant object living on some larger manifold, and use that larger covariant geometry to derive scattering amplitudes.

In this Letter, we realize the more principled approach by adding to the field space manifold all degrees of freedom necessary to accommodate spacetime and higher-derivative field coordinates, so that any Lagrangian can be embedded into what are known as "jet bundles." Intuitively, these are bundles over spacetime whose fibers comprise fields and their spacetime derivatives up to a given order [13]. Consequently, any scattering amplitude can be written in terms of covariant objects thereon known as distinguished (d) tensors. As minimal degrees of freedom are added, the countable dimensionality of this framework offers an advantage over alternative approaches that index fields by position or momentum [18,19].

After a concise introduction to jet bundles, we construct d-tensor derivatives on the first-order jet bundle and use them to obtain covariant expressions for the tree-level scattering amplitudes of the Lagrangian in Eq. (2). We then illustrate the relative advantages of jet bundles at second order, where the resulting geometry automatically encodes invariance under total derivatives, before extending the construction of d-tensor derivatives to arbitrary order. Along the way we develop a number of original results, including a geometry covariant under induced fiberwise transformations on an all-order jet bundle with a multidimensional base space, the systematic covariantization of amplitude contributions from any field theory operator, and a geometric interpretation of invariance under total derivatives for the two-derivative part of the theory. Detailed derivations of all-order results, covariant expressions for scattering amplitudes of the most general four-derivative scalar Lagrangian, and the connection with the closely related notion of Lagrange spaces [20] are discussed in the Supplemental Material [21]. The jet bundle formalism promises to significantly expand the relevance of geometric methods to effective field theories.

Jet bundles.—We begin with the essential ingredients required to construct covariant amplitudes on the jet bundle. The first important ingredient is the *multi-index*, which suitably collects field derivatives to form derivative coordinates. As partial derivatives commute, field derivatives are uniquely specified by the number of partial derivatives in each spacetime coordinate. To this end, define a multi-index $\Lambda(\tau)$ as a tuple of $(\dim \mathcal{T})$ non-negative integers, where τ is a spacetime index. The order is given by $|\Lambda| \equiv \sum_\tau \Lambda(\tau)$. Denote

$$\frac{\partial^{|\Lambda|}}{\partial x^\Lambda} \equiv \prod_\tau \left(\frac{\partial}{\partial x^\tau}\right)^{\Lambda(\tau)}, \qquad \phi^i_\Lambda \equiv \frac{\partial^{|\Lambda|}\phi^i}{\partial x^\Lambda}. \tag{6}$$

We take ϕ^i_Λ with $|\Lambda| = 0$ to mean ϕ^i. The subscript μ in ϕ^i_μ can be interpreted as an order-one multi-index $\Lambda(\tau)$ with one for $\tau = \mu$ and zero otherwise. At higher orders, we may for convenience write $\phi_{\mu_1\mu_2}$ or $\phi_{\mu_2\mu_1}$ in place of ϕ_Λ where $\Lambda(\tau) = 1$ if $\tau = \mu_1$ or μ_2 and 0 otherwise.

A Lagrangian is a function in ϕ^a and ϕ^a_Λ. To fit it into an enlargement of $\mathcal{M}$, we need to incorporate derivatives in spacetime of arbitrary order $|\Lambda|$. This can be done through a natural extension of the tangent space construction, which as we have seen in the introduction generates first derivatives in a single time dimension. Let us attach the spacetime manifold to $\mathcal{M}$ to form the trivial fiber bundle $E = \mathcal{T} \times \mathcal{M}$ with an associated projection $\pi: E \to \mathcal{T}$ [25]. A field $\phi^i(x^\mu)$ is then simply a section of E. For $\mathsf{q} \geq 1$, we say that two sections ϕ^i and ϕ'^i have the same q jet at a spacetime point $x^\mu \in \mathcal{T}$ if

$$\left.\frac{\partial^{|\Lambda|}\phi^i}{\partial x^\Lambda}\right|_{x^\mu} = \left.\frac{\partial^{|\Lambda|}\phi'^i}{\partial x^\Lambda}\right|_{x^\mu} \quad \text{for all } 0 \leq |\Lambda| \leq \mathsf{q}, \tag{7}$$

i.e., their derivatives match at x^μ up to qth order. Having the same q jet is an equivalence relation between sections, and the set of such equivalence classes over all spacetime points is known as the qth-order jet bundle $J^{\mathsf{q}}(\pi)$ [14,26]. This is a manifold charted by $(x^\mu, \phi^i, \phi^i_\Lambda)$ where the derivative coordinates run up to $|\Lambda| \leq \mathsf{q}$, with fiber bundle structures over $\mathcal{T}$, E, and $J^{\mathsf{r}}(\pi)$ for $\mathsf{r} < \mathsf{q}$ given by the projections

$$\begin{cases} \pi_{\mathsf{q}}\colon J^{\mathsf{q}}(\pi) \to \mathcal{T} \\ (x^\mu, \phi^i, \phi^i_\Lambda) \mapsto (x^\mu) \end{cases}, \qquad \begin{cases} \pi_{\mathsf{q},0}\colon J^{\mathsf{q}}(\pi) \to E \\ (x^\mu, \phi^i, \phi^i_\Lambda) \mapsto (x^\mu, \phi^i) \end{cases},$$
$$\begin{cases} \pi_{\mathsf{q},\mathsf{r}}\colon J^{\mathsf{q}}(\pi) \to J^{\mathsf{r}}(\pi) \\ (x^\mu, \phi^i, \phi^i_\Lambda) \mapsto (x^\mu, \phi^i, \phi^i_\Theta) \end{cases}, \tag{8}$$

where $|\Theta| \leq \mathsf{r}$. Having enlarged $\mathcal{M}$ to $J^{\mathsf{q}}(\pi)$, it is now possible to embed any Lagrangian with qth-or-lower field derivatives—it is simply a function $\mathcal{L}\colon J^{\mathsf{q}}(\pi) \to \mathbb{R}$.

The second key ingredient is the notion of a d tensor. For our purposes, a d tensor is a jet bundle object $T^{\mu i \ldots}_{j\nu\ldots}(x^\alpha, \phi^a, \phi^a_\Lambda)$ that transforms under a field redefinition $\phi^i(\tilde{\phi}^j)$ as if it were a tensor on $\mathcal{M}$ [27]:

$$\tilde{T}^{\mu i \ldots}_{j\nu\ldots} = \left(\frac{\partial\tilde{\phi}^i}{\partial\phi^k}\cdots\right) T^{\mu k\ldots}_{l\nu\ldots}\left(\frac{\partial\phi^l}{\partial\tilde{\phi}^j}\cdots\right). \tag{9}$$

Examples of d tensors include $\mathcal{L}$ and ϕ^i_μ.

A route to obtaining scattering amplitudes from $\mathcal{L}$ is to organize its pieces by derivative count and read off Feynman rules from the coefficients, e.g., V, g_{ij}, and c_{ijkl} in Eq. (2). This is equivalent to differentiating $\mathcal{L}$ by field derivatives and evaluating on the null section $\{\phi^i_\Lambda = 0\}$ of the jet bundle. However, the process must now be carried out covariantly, which requires d-tensor derivatives.

Let us begin with the simplest case $\mathsf{q} = 1$, i.e., the first-order jet bundle. The chain rule induces the following transformations of tangent vectors of field and derivative coordinates:

$$\frac{\partial}{\partial\tilde{\phi}^i} = \frac{\partial\phi^j}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j} + \frac{\partial\phi^j_\rho}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_\rho}, \qquad \frac{\partial}{\partial\tilde{\phi}^i_\mu} = \frac{\partial\phi^j}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_\mu}. \tag{10}$$

Evidently, not all are d tensors. A nonlinear connection N on $J^1(\pi)$ makes the d-tensor combination

$$\frac{\delta}{\delta\phi^i} \equiv \frac{\partial}{\partial\phi^i} - N^j_{\rho i}\frac{\partial}{\partial\phi^j_\rho}. \tag{11}$$

The basis $\{\delta/\delta\phi^i, \partial/\partial\phi^i_\mu\}$ spans horizontal (field) and vertical (derivative) distributions in $TJ^1(\pi)$. The coefficient $N^j_{\rho i}$ must transform as

$$N^k_{\rho l}\frac{\partial\phi^l}{\partial\tilde{\phi}^i} = \tilde{N}^j_{\rho i}\frac{\partial\phi^k}{\partial\tilde{\phi}^j} - \frac{\partial\phi^k_\rho}{\partial\tilde{\phi}^i} = \tilde{N}^j_{\rho i}\frac{\partial\phi^k}{\partial\tilde{\phi}^j} - \frac{\partial^2\phi^k}{\partial\tilde{\phi}^i\partial\tilde{\phi}^j}\tilde{\phi}^j_\rho. \tag{12}$$

In the Riemannian formulation of amplitudes on $\mathcal{M}$, covariant derivatives are supplied by the affine connection ∇, whose Christoffel symbols $\gamma^i_{jk}(\phi^a)$ transform appropriately to cancel the nontensorial parts of composed partial derivatives. In extending this covariant framework to jet bundles, we are thus motivated to construct N from an affine connection on $\mathcal{M}$. It is not essential that ∇ be Levi-Civita, and we make no assumption other than it be symmetric. Then the following choice suffices [28]:

$$N^j_{\rho i} = \gamma^j_{ik}\phi^k_\rho. \tag{13}$$

We have now constructed single d-tensor derivatives. But we will need an additional affine connection $\boldsymbol{\nabla}$ on $J^1(\pi)$, which is N linear in the sense that its parallel transport respects the horizontal-vertical decomposition by N [29]:

$$\boldsymbol{\nabla}_{\frac{\delta}{\delta\phi^k}}\frac{\delta}{\delta\phi^j_\Lambda} = F^i_{jk}\frac{\delta}{\delta\phi^i_\Lambda}, \qquad \boldsymbol{\nabla}_{\frac{\partial}{\partial\phi^k_\rho}}\frac{\partial}{\partial\phi^j_\Lambda} = C^{i\rho}_{jk}\frac{\partial}{\partial\phi^i_\Lambda}, \tag{14}$$

where $|\Lambda| \geq 0$. The coefficients F^i_{jk} and $C^{i\rho}_{jk}$ are h- and v-Christoffel symbols, transforming like Christoffel symbols and tensors on $\mathcal{M}$, respectively. We can choose

$$F^i_{jk} = \gamma^i_{jk}, \qquad C^{i\rho}_{jk} = 0. \tag{15}$$

This yields h- and v-covariant derivatives on any d tensor T:

$$(\boldsymbol{\nabla}T)^{i\dots}_{j\dots} = T^{i\dots}_{j\dots/k}d\phi^k + T^{i\dots}_{j\dots}|^\rho_k\delta\phi^k_\rho, \tag{16a}$$

$$\boldsymbol{\nabla}_k T^{i\dots}_{j\dots} \equiv T^{i\dots}_{j\dots/k} = \frac{\delta T^{i\dots}_{j\dots}}{\delta\phi^k} + F^i_{mk}T^{m\dots}_{j\dots} - F^m_{jk}T^{i\dots}_{m\dots} + \dots, \tag{16b}$$

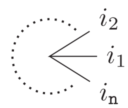
$$\boldsymbol{\nabla}^\rho_k T^{i\dots}_{j\dots} \equiv T^{i\dots}_{j\dots}|^\rho_k = \frac{\partial T^{i\dots}_{j\dots}}{\partial\phi^k_\rho}, \tag{16c}$$

allowing us to compose d-tensor derivatives.

Equipped with the d tensor $\mathcal{L}$ and a covariant way to differentiate it on $J^1(\pi)$, we can proceed to assemble covariant amplitudes.

Covariant amplitudes.—Consider the Lagrangian in Eq. (2). We call $\bar{\phi}^i = \arg\min V(\phi^a)$ and $\{\phi^i = \bar{\phi}^i, \phi^i_\mu = 0\}$ the vacuum on $\mathcal{M}$ and $J^1(\pi)$, respectively; the latter lies on the null section of $J^1(\pi)$. Evaluation of geometric quantities in either space at the corresponding vacuum will be denoted with an overline. To compute tree-level amplitudes, we expand V, g_{ij}, and c_{ijkl} about $\bar{\phi}^i$, indicating partial derivatives with commas. In coordinates that diagonalize $\bar{V}_{,ij}$ and $\bar{g}_{ij}$, the scalar field masses can be read off from their ratios $\bar{V}_{,ij} = -2\bar{g}_{ij}m_i^2$. The Feynman rules in momentum space are then

$$i\text{—}j = \frac{i\bar{g}^{ij}}{2(p^2 - m_i^2)}, \tag{17a}$$



$$= i\left[\bar{V}_{,\dots} - 2\sum_{1\leq\mathtt{a}<\mathtt{b}\leq\mathtt{n}}(p_\mathtt{a}\cdot p_\mathtt{b})\bar{g}_{i_\mathtt{a}i_\mathtt{b},\dots} + 8\sum_{\mathtt{a}<\mathtt{b},\mathtt{a}<\mathtt{c}<\mathtt{d}}(p_\mathtt{a}\cdot p_\mathtt{b})(p_\mathtt{c}\cdot p_\mathtt{d})\bar{c}_{i_\mathtt{a}i_\mathtt{b}i_\mathtt{c}i_\mathtt{d},\dots}\right], \tag{17b}$$

where the ellipses represent indices from i_1 to $i_\mathtt{n}$ that are not explicitly written.

At tree level, any momentum p appearing in a Feynman diagram can be traded for Mandelstam variables, which are external kinematic data that can be set aside. Rewriting any m_i^2 as $-\bar{g}^{ii}\bar{V}_{,ii}/2$, all that remain in the amplitude are partial derivatives of V, g, or c evaluated at the vacuum, which we desire to convert to covariant expressions.

To do so, we deploy the following trick. In the normal coordinates of a symmetric ∇ at the vacuum on $\mathcal{M}$, one can replace partial derivatives on any tensor $t^{i\dots}_{j\dots}(\phi^a)$ with covariant ones [10,20]:

$$\bar{t}^{i\dots}_{j\dots,k_1\dots k_\mathtt{n}} \to \nabla_{(k_1}\dots\nabla_{k_\mathtt{n})}\bar{t}^{i\dots}_{j\dots} + \mathcal{O}(tR), \tag{18}$$

incurring additional covariant terms that involve the curvature tensor R^i_{jkl} of ∇, suppressed above. Now returning to $J^1(\pi)$, it is also easy to verify that

$$\left(\boldsymbol{\nabla}_{k_1}\dots\boldsymbol{\nabla}_{k_\mathtt{n}}T^{i\dots}_{j\dots}\right)\Big|_{\phi^a_\alpha=0} = \nabla_{k_1}\dots\nabla_{k_\mathtt{n}}\left(T^{i\dots}_{j\dots}\Big|_{\phi^a_\alpha=0}\right), \tag{19}$$

for any d tensor $T(\phi^a, \phi^a_\alpha)$. We can hence replace t in Eq. (18) with any such T that agrees with t on the null section, ∇ with $\boldsymbol{\nabla}$, and R with the hh-curvature d tensor $\mathcal{R}$ of $\boldsymbol{\nabla}$ [30]:

$$\mathcal{R}^i_{jkl} \equiv d\phi^i\left(\left(\left[\boldsymbol{\nabla}_{\delta/\delta\phi^k}, \boldsymbol{\nabla}_{\delta/\delta\phi^l}\right] - \boldsymbol{\nabla}_{[\delta/\delta\phi^k, \delta/\delta\phi^l]}\right)\frac{\delta}{\delta\phi^j}\right) = \frac{\delta\gamma^i_{jl}}{\delta\phi^k} + \gamma^i_{mk}\gamma^m_{jl} - (k\leftrightarrow l) = R^i_{jkl}. \tag{20}$$

The result is a manifestly covariant expression at the vacuum of $J^1(\pi)$ that, in normal coordinates, equals the desired partial derivative of $t(\phi^a)$ at the vacuum of $\mathcal{M}$.

For a $\mathsf{q}=1$ Lagrangian, the coefficients on $\mathcal{M}$ that appear in the Feynman rules are always tensors, and their d-tensor counterparts are simply partial derivatives of $\mathcal{L}$ by ϕ^i_μ. For example,

$$V = \mathcal{L}\Big|_{\phi^a_\alpha=0}, \quad g_{ij} = \frac{\eta_{\mu\nu}}{8}\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\Big|_{\phi^a_\alpha=0},$$
$$c_{ijkl} = \frac{5\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}}{576}\frac{\partial^4\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu\partial\phi^k_\rho\partial\phi^l_\sigma}\Big|_{\phi^a_\alpha=0}. \tag{21}$$

We can thus use the trick to write all nonkinematic terms in the scattering amplitude as d tensors in normal coordinates. But since the total amplitude is covariant, the ensuing covariant expression must actually hold in any coordinates, and we are done.

This procedure is largely the same as in [20]. Denoting the n-point amplitude as $\mathcal{A}_{1\ldots\mathtt{n}}$ where $1, \ldots, \mathtt{n}$ label the external legs, we find

$$\mathcal{A}_{123} = \prod_{\mathrm{e}=1}^{3}\sqrt{\frac{\bar{g}^{\mathrm{ee}}}{2}}\left\{\bar{V}_{/(123)} - \frac{1}{2}\sum_{\mathrm{a}=1}^{3}\bar{V}_{/\mathrm{a}j}\bar{g}^{jk}(2\bar{g}_{k(\mathrm{b}/\mathrm{c})} - \bar{g}_{\mathrm{bc}/k})\right\}, \tag{22a}$$

$$\begin{aligned}\mathcal{A}_{1234} = \prod_{\mathrm{e}=1}^{4}\sqrt{\frac{\bar{g}^{\mathrm{ee}}}{2}}\Bigg\{&\left[-\frac{1}{2}A_{12j}\frac{\bar{g}^{jk}}{s_{12}-m_j^2}A_{34k} + (s_{13}, s_{14}\,\text{channels})\right] + \bar{V}_{/(1234)} + \sum_{\mathrm{a}=1}^{4}m_\mathrm{a}^2[3\bar{g}_{\mathrm{a}(\mathrm{b}/\mathrm{cd})} + \bar{R}_{(\mathrm{bcd})\mathrm{a}}]\\ &+ 4\sum_{\mathrm{a}<\mathrm{b}}m_\mathrm{a}^2 m_\mathrm{b}^2\bar{c}_{1234} - \sum_{\mathrm{a}<\mathrm{b}}s_{\mathrm{ab}}\left[\bar{g}_{\mathrm{ab}/(\mathrm{cd})} + \frac{1}{3}\bar{R}_{\mathrm{a}(\mathrm{cd})\mathrm{b}} + \frac{1}{3}\bar{R}_{\mathrm{b}(\mathrm{cd})\mathrm{a}} + 2(m_\mathrm{c}^2 + m_\mathrm{d}^2)\bar{c}_{1234}\right] + 2(s_{12}^2 + s_{13}^2 + s_{14}^2)\bar{c}_{1234}\Bigg\},\end{aligned}$$

where $A_{\mathrm{ab}i} = \bar{V}_{/(\mathrm{ab}i)} - \frac{1}{2}\bar{V}_{/\mathrm{a}j}\bar{g}^{jk}(2\bar{g}_{k(\mathrm{b}/i)} - \bar{g}_{\mathrm{b}i/k}) - \frac{1}{2}\bar{V}_{/\mathrm{b}j}\bar{g}^{jk}(2\bar{g}_{k(\mathrm{a}/i)} - \bar{g}_{\mathrm{a}i/k}) + s_{\mathrm{ab}}(2\bar{g}_{i(\mathrm{a}/\mathrm{b})} - \bar{g}_{\mathrm{ab}/i})$, (22b)

and so on. For brevity, apart from $\mathcal{A}$, A, and Mandelstam variables $s_{\mathrm{ab}} = (p_\mathrm{a} + p_\mathrm{b})^2$, any $\mathtt{a}$ or explicit number appearing in a subscript should be interpreted as flavor index i_a, and any tensor on $\mathcal{M}$ as its d-tensor replacement in Eq. (21). The indices $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$, and $\mathtt{d}$ are all distinct, and the $\sqrt{\bar{g}^{\mathrm{ee}}/2}$ are wave function normalization factors.

Of course, instead of the covariant geometry of $\boldsymbol{\nabla}$ on the null section of $J^1(\pi)$, that of ∇ on $\mathcal{M}$ would have sufficed to covariantize scattering amplitudes if we terminate the replacement procedure at Eq. (18). However, since $\mathcal{L}$ does not fit on $\mathcal{M}$, we then have to endow $\mathcal{M}$ with tensors V, g, and c manually extracted from $\mathcal{L}$. By enlarging the manifold, we have eliminated the need for additional structures besides $\mathcal{L}$ and $\boldsymbol{\nabla}$ in expressing amplitudes covariantly, even if intermediate steps might involve other temporary objects. Moreover, the covariant geometry of the order-one jet bundle paves the way to higher field derivatives, which field space alone cannot handle systematically. This brings us to the next order—$\mathsf{q}=2$.

Invariance under total derivatives.—The construction of d-tensor derivatives on the second-order jet bundle proceeds in close analogy with the first-order case. The tangent vectors of $J^2(\pi)$ transform as [31,32]

$$\frac{\partial}{\partial\tilde{\phi}^i} = \frac{\partial\phi^j}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j} + \frac{\partial\phi^j_\rho}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_\rho} + \frac{\partial\phi^j_{\rho_1\rho_2}}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_{\rho_1\rho_2}},$$
$$\frac{\partial}{\partial\tilde{\phi}^i_\mu} = \frac{\partial\phi^j}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_\mu} + \frac{\partial\phi^j_{\rho_1\rho_2}}{\partial\tilde{\phi}^i_\mu}\frac{\partial}{\partial\phi^j_{\rho_1\rho_2}}, \qquad \frac{\partial}{\partial\tilde{\phi}^i_{\mu_1\mu_2}} = \frac{\partial\phi^j}{\partial\tilde{\phi}^i}\frac{\partial}{\partial\phi^j_{\mu_1\mu_2}}. \tag{23}$$

We require a suitable nonlinear connection N that enables us to combine them into d tensors

$$\frac{\delta}{\delta\phi^i} = \frac{\partial}{\partial\phi^i} - N^j_{\rho i}\frac{\partial}{\partial\phi^j_\rho} - N^j_{\rho_1\rho_2 i}\frac{\partial}{\partial\phi^j_{\rho_1\rho_2}}, \tag{24a}$$

$$\frac{\delta}{\delta\phi^i_\mu} = \frac{\partial}{\partial\phi^i_\mu} - N^{j\mu}_{\rho_1\rho_2 i}\frac{\partial}{\partial\phi^j_{\rho_1\rho_2}}. \tag{24b}$$

The coefficients are all denoted by N, but can be distinguished by index structure. The answer, as will be apparent later, is to set

$$N^{j}_{\rho i} = \gamma^{j}_{ik}\phi^{k}_{\rho}, \qquad N^{j\mu}_{\rho_1\rho_2 i} = 2N^{j}_{(\rho_1|i}\delta^{\mu}_{|\rho_2)} = 2\gamma^{j}_{ik}\phi^{k}_{(\rho_1}\delta^{\mu}_{\rho_2)},$$
$$N^{j}_{\rho_1\rho_2 i} = (\gamma^{j}_{ik,l} - \gamma^{j}_{mk}\gamma^{m}_{il})\phi^{k}_{(\rho_1}\phi^{l}_{\rho_2)} + \gamma^{j}_{ik}\phi^{k}_{\rho_1\rho_2}. \tag{25}$$

An N-linear connection then follows from a simple modification of Eq. (14), with $F^{i}_{jk} = \gamma^{i}_{jk}$ and $C^{i\Theta}_{jk} = 0$ for $|\Theta| \geq 1$.

A new feature arises at second order—besides field redefinitions, the Lagrangian can be modified by a total derivative without changing the resulting amplitudes. To illustrate this, it suffices to examine a minimal example, the most general two-derivative Lagrangian

$$\mathcal{L} = V(\phi^a) + \eta^{\mu\nu} g_{ij}(\phi^a)\phi^i_\mu \phi^j_\nu + \eta^{\mu\nu} h_i(\phi^a)\phi^i_{\mu\nu}, \tag{26}$$

in which integration by parts shuffles g_{ij} and h_i. The two pieces really represent a single kinetic term,

$$G_{ij} = g_{ij} - h_{(i,j)}, \tag{27}$$

which should be positive-definite. We can follow the same procedure as before to covariantize amplitudes, once we figure out the tensors $t(\phi^a)$ in the Feynman rules to be replaced with d tensors $T(\phi^a, \phi^a_\Lambda)$ obtained from $\mathcal{L}$, invoking Eq. (19) with $\alpha \to \Lambda$. Picking out V and h_i from $\mathcal{L}$ is simple because they are indeed tensors,

$$\mathcal{L}\Big|_{\phi^a_\Lambda = 0} = V, \qquad \frac{1}{4}\eta_{\mu\nu}\frac{\delta\mathcal{L}}{\delta\phi^i_{\mu\nu}}\Big|_{\phi^a_\Lambda=0} = h_i, \tag{28}$$

but since g_{ij} is no longer a tensor, the naive guess for g_{ij} yields an extraneous term,

$$\frac{1}{8}\eta_{\mu\nu}\frac{\delta^2\mathcal{L}}{\delta\phi^{(i}_\mu \delta\phi^{j)}_\nu}\Big|_{\phi^a_\Lambda=0} = g_{ij} - \gamma^k_{ij}h_k. \tag{29}$$

The nontensorial term in the transformation law for γ^k_{ij} precisely cancels the nontensorial term for g_{ij} in Eq. (5). Eliminating the extraneous term, we should take the difference with the covariant derivative of h_i, but that simply returns the d tensor for G_{ij}:

$$\frac{1}{4}\eta_{\mu\nu}\left(\frac{1}{2}\frac{\delta^2\mathcal{L}}{\delta\phi^{(i}_\mu\delta\phi^{j)}_\nu} - \boldsymbol{\nabla}_{(i}\frac{\delta\mathcal{L}}{\delta\phi^{j)}_{\mu\nu}}\right)\Big|_{\phi^a_\Lambda=0} = G_{ij}. \tag{30}$$

Evidently, the jet bundle geometry automatically identifies a single kinetic term even if we start with two pieces. The amplitudes thence derived from Eq. (30) exhibit explicit invariance under both field redefinitions and total derivatives.

Thus far, we have remained agnostic on the choice of field space connection that generates the jet bundle connections. But now, there is a natural candidate: the Levi-Civita connection of the kinetic term Eq. (30). This choice is canonical as it follows solely from the Lagrangian, independent of the arbitrary γ used to derive it. It is also meaningful due to its significance in the Lagrangian, so that any covariant physics arising from the kinetic term is encapsulated by the hh curvature [33]. Such a choice is the covariant generalization of that in [34] and agrees with the established Riemannian framework when the Lagrangian is actually first-order. Despite now tying the jet bundle connection to a Lagrangian, the resultant covariant geometry will nevertheless remain invariant under total derivatives, as it should be for a framework that extracts local physics.

To all derivative orders.—The last obstacle is extending the covariant geometry of the jet bundle to arbitrary order. This can be done using a generalization of prolongation on higher-order tangent bundles [35,36], with complications due to multi-indices in $\dim\mathcal{T} > 1$. Here, we state the results, leaving a proof to the Supplemental Material [21].

Let $\mathsf{q} \geq \mathsf{r} \geq 0$ and abbreviate the order-q subscript $\mu_1 \ldots \mu_\mathsf{q}$ as μ_Q with an uppercase Q. By $\mu_{\mathsf{Q}-1}$, we mean $\mu_1\ldots\mu_{\mathsf{q}-1}$. Also write $\delta^{\rho_\mathsf{Q}}_{\mu_\mathsf{Q}}$ for the Kronecker delta that indicates multi-index equality $\rho_\mathsf{Q} = \mu_\mathsf{Q}$. Define

$$\Gamma_{\mu_0} = \phi^i_{\mu_0}\frac{\partial}{\partial\phi^i} + \phi^i_{\mu_0\mu_1}\frac{\partial}{\partial\phi^i_{\mu_1}} + \phi^i_{\mu_0\mu_1\mu_2}\frac{\partial}{\partial\phi^i_{\mu_1\mu_2}} + \ldots. \tag{31}$$

We can assemble d-tensor derivatives

$$\frac{\delta}{\delta\phi^i_{\mu_\mathsf{R}}} = \frac{\partial}{\partial\phi^i_{\mu_\mathsf{R}}} - N^{j\mu_\mathsf{R}}_{\rho_{\mathsf{R}+1}i}\frac{\partial}{\partial\phi^j_{\rho_{\mathsf{R}+1}}} - N^{j\mu_\mathsf{R}}_{\rho_{\mathsf{R}+2}i}\frac{\partial}{\partial\phi^j_{\rho_{\mathsf{R}+2}}} - \ldots, \tag{32}$$

by recursively setting the coefficients in the dual basis of d-tensor 1-forms as

$$M^i_{\mu j} = \gamma^i_{jk}\phi^k_\mu, \qquad M^{i\rho_{\mathsf{Q}-1}}_{\mu_\mathsf{Q} j} = \mathsf{q} M^i_{(\mu_\mathsf{q}|j}\delta^{\rho_{\mathsf{Q}-1}}_{|\mu_{\mathsf{Q}-1})},$$
$$M^{i\rho_\mathsf{R}}_{\mu_\mathsf{Q} j} = \frac{\mathsf{q}}{\mathsf{q}-\mathsf{r}}\left[\Gamma_{(\mu_\mathsf{q}}M^{i\rho_\mathsf{R}}_{\mu_{\mathsf{Q}-1})j} + M^i_{(\mu_\mathsf{q}|m}M^{m\rho_\mathsf{R}}_{|\mu_{\mathsf{Q}-1})j}\right], \tag{33}$$

fixing N by duality

$$N^{i\rho_\mathsf{R}}_{\mu_\mathsf{Q} j} = M^{i\rho_\mathsf{R}}_{\mu_\mathsf{Q} j} - M^{i\sigma_{\mathsf{Q}-1}}_{\mu_\mathsf{Q} k}N^{k\rho_\mathsf{R}}_{\sigma_{\mathsf{Q}-1}j} - \cdots - M^{i\sigma_{\mathsf{R}+1}}_{\mu_\mathsf{Q} k}N^{k\rho_\mathsf{R}}_{\sigma_{\mathsf{R}+1}j}. \tag{34}$$

An N-linear connection is then given by

$$\boldsymbol{\nabla}_{\frac{\delta}{\delta\phi^k}}\frac{\delta}{\delta\phi^j_\Lambda} = \gamma^i_{jk}\frac{\delta}{\delta\phi^i_\Lambda}, \qquad \boldsymbol{\nabla}_{\frac{\delta}{\delta\phi^k_\Theta}}\frac{\delta}{\delta\phi^j_\Lambda} = 0, \tag{35}$$

where $|\Lambda| \geq 0$ and $|\Theta| \geq 1$. With such technology, we can now extract information at any derivative order from any Lagrangian covariantly, and from there derive scattering amplitudes as before. As an example, the fully general four-derivative Lagrangian is treated in the Supplemental Material [21].

Outlook.—Geometry is a potentially powerful organizing principle for diverse effective field theories, provided it can incorporate the full scope of interactions. The jet bundle generalizes the field space manifold by incorporating spacetime and field derivative coordinates of all orders. It therefore contains all degrees of freedom necessary to accommodate any scalar field theory. In this Letter, we constructed a geometry on the jet bundle that transforms covariantly, and demonstrated how the invariance of scattering amplitudes under nonderivative field redefinitions can be made manifest in terms of d tensors. In the process, we learned that an important covariant object in the Lagrangian, the kinetic term, also exhibits manifest invariance under total derivatives, yielding a new geometric perspective on amplitude physics and a canonical choice for the covariant geometry.

The establishment of a geometric framework to all derivative orders opens up the possibility of redefinitions mixing field and derivative coordinates, which nevertheless leave amplitudes invariant. Such redefinitions are relevant in practice by virtue of equation-of-motion reduction, often used to eliminate derivative operators. While the current approach produces covariant amplitudes by considering each operator in turn, derivative field redefinitions shuffle contributions between operators so that the sum remains unchanged only when momentum conservation and on-shell conditions are imposed. It remains to be seen whether the geometry of the infinite jet bundle is capable of capturing the more general invariance.

The results in this Letter may be plausibly extended to loop-level amplitudes and higher-spin theories. The employment of normal coordinates on the field space manifold to covariantize the renormalization procedure, such as in [37,38], can be translated to jet bundles to explicitly render higher-derivative operators. Beyond scalar fields, gauge bosons can be incorporated by fixing a gauge like in [39,40], or generalizing the notion of jets to principal and associated bundles; fermions may be incorporated via an extension to supermanifolds [34,41,42].

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[1] J. S. R. Chisholm, Change of variables in quantum field theories, Nucl. Phys. **26**, 469 (1961).

[2] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, Change of variables and equivalence theorems in quantum field theories, Nucl. Phys. **28**, 529 (1961).

[3] C. Arzt, Reduced effective Lagrangians, Phys. Lett. B **342**, 189 (1995).

[4] L. J. Dixon, V. Kaplunovsky, and J. Louis, On effective field theories describing (2,2) vacua of the heterotic string, Nucl. Phys. **B329**, 27 (1990).

[5] R. Alonso, E. E. Jenkins, and A. V. Manohar, A geometric formulation of Higgs effective field theory: Measuring the curvature of scalar field space, Phys. Lett. B **754**, 335 (2016).

[6] R. Alonso, E. E. Jenkins, and A. V. Manohar, Geometry of the scalar sector, J. High Energy Phys. 08 (2016) 101.

[7] R. Nagai, M. Tanabashi, K. Tsumura, and Y. Uchida, Symmetry and geometry in a generalized Higgs effective field theory: Finiteness of oblique corrections versus perturbative unitarity, Phys. Rev. D **100**, 075020 (2019).

[8] A. Helset, A. Martin, and M. Trott, The geometric standard model effective field theory, J. High Energy Phys. 03 (2020) 163.

[9] T. Cohen, N. Craig, X. Lu, and D. Sutherland, Is SMEFT enough?, J. High Energy Phys. 03 (2021) 237.

[10] T. Cohen, N. Craig, X. Lu, and D. Sutherland, Unitarity violation and the geometry of Higgs EFTs, J. High Energy Phys. 12 (2021) 003.

[11] R. Alonso and M. West, Roads to the standard model, Phys. Rev. D **105**, 096028 (2022).

[12] Such terms can first be treated using integration by parts or derivative field redefinitions before covariantization, but these manipulations change the apparent field space geometry.

[13] For a broader overview of the geometry of jet bundles, see, e.g., [14–16]. For concurrent work applying jet bundles to the geometry of effective field theories, see Ref. [17].

[14] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, 1989).

[15] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics (Springer, New York, 1986).

[16] I. M. Anderson, Introduction to the Variational Bicomplex, in *Mathematical Aspects of Classical Field Theory*, Seattle, 1991, Contemp. Math. Vol. 132, edited by M. J. Gotay, J. E. Marsden, and V. Moncrief (American Mathematical Society, Providence, 1992), pp. 51–73.

[17] M. Alminawi, I. Brivio, and J. Davighi, Jet bundle geometry of scalar field theories, arXiv:2308.00017.

[18] T. Cohen, N. Craig, X. Lu, and D. Sutherland, On-shell covariance of quantum field theory amplitudes, Phys. Rev. Lett. **130**, 041603 (2023).

[19] C. Cheung, A. Helset, and J. Parra-Martinez, Geometry-kinematics duality, Phys. Rev. D **106**, 045016 (2022).

[20] N. Craig, Y.-T. Lee, X. Lu, and D. Sutherland, Effective field theories as Lagrange spaces, J. High Energ. Phys. 11 (**2023**) 69.

[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.061602, which includes [22–24], for the construction of a nonlinear connection at arbitrary order, the covariantization of scattering amplitudes for a general four-derivative scalar theory, and the relation between the covariant jet bundle geometry and Lagrange spaces.

[22] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Fundam. Theor. Phys. (Springer Netherlands, Dordrecht, 1994).

[23] M. Neagu, From Euler-Lagrange equations to canonical nonlinear connections, Arch. Math. **42**, 255 (2006).

[24] V. Balan and M. Neagu, *Jet Single-Time Lagrange Geometry and Its Applications* (Wiley, New York, 2011).

[25] The trivial bundle suffices since our ultimate interest lies in local rather than global geometry.

[26] C. Ehresmann, Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie, in *Geometrie Differentielle*, Colloq. Inter. du Centre Nat. de la Recherche Scientifique (Strasbourg, Paris, France, 1953), pp. 97–127.

[27] M. Neagu, C. Udriste, and A. Oana, Multi-time sprays and h-traceless maps on J1(T,M), Balkan J. Geom. Appl. **10**, 76 (2005).

[28] M. Neagu, Generalized metrical multi-time Lagrange geometry of physical fields, Forum Mathematicum **15**, 63 (2003).

[29] M. Neagu and C. Udriste, Torsion, curvature and deflection d-tensors on J1(T,M), Balkan J. Geom. Appl. **6**, 29 (2001).

[30] All other curvature components of ∇ vanish identically.

[31] A. Oana, Nonlinear connections on 2-jet bundle J2(T,M), Bull. Transilvania Univ. Brasov Ser. III **11**, 187 (2018).

[32] There are two summation conventions in effect here. There are $\binom{\dim\mathcal{T}+\mathsf{q}-1}{\mathsf{q}}$ multi-indices of order q but $(\dim\mathcal{T})^{\mathsf{q}}$ subscripts. If we repeat a subscript styled with the same Greek letter distinctly numbered, e.g., $\mu_1\mu_2$, we mean a sum over distinct multi-indices, not independent summations over each index μ_1 and μ_2.

[33] Any covariant derivative of the kinetic term in covariantized amplitudes like Eq. (22) will vanish.

[34] K. Finn, S. Karamitsos, and A. Pilaftsis, Frame covariant formalism for fermionic theories, Eur. Phys. J. C **81**, 572 (2021).

[35] R. Miron and G. Anastasiu, Compendium on the higher-order Lagrange spaces: The geometry of k–Osculator bundles. Prolongation of the Riemannian, Finslerian and Lagrangian structures. Lagrange spaces, Tensor N.S. **53**, 39 (1993).

[36] R. Miron and G. Anastasiu, Prolongation of the Riemannian, Finslerian and Lagrangian structures, Rev. Roumaine Math. Pures Appl. **41**, 237 (1996).

[37] R. Alonso and M. West, On the effective action for scalars in a general manifold to any loop order, Phys. Lett. B **841**, 137937 (2023).

[38] A. Helset, E. E. Jenkins, and A. V. Manohar, Renormalization of the standard model effective field theory from geometry, J. High Energy Phys. 02 (2023) 063.

[39] A. Helset, E. E. Jenkins, and A. V. Manohar, Geometry in scattering amplitudes, Phys. Rev. D **106**, 116018 (2022).

[40] K. Finn, S. Karamitsos, and A. Pilaftsis, Frame covariance in quantum gravity, Phys. Rev. D **102**, 045014 (2020).

[41] B. Assi, A. Helset, A. V. Manohar, J. Pagès, and C.-H. Shen, Fermion geometry and the renormalization of the standard model effective field theory, J. High Energy Phys. 11 (2023) 201.

[42] V. Gattus and A. Pilaftsis, Minimal supergeometric quantum field theories, Phys. Lett. B **846**, 138234 (2023).