Local Disclosure of Quantum Memory in Non-Markovian Dynamics

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Non-Markovian processes may arise in physics due to memory effects of environmental degrees of freedom. For quantum non-Markovianity, it is an ongoing debate to clarify whether such memory effects have a verifiable quantum origin, or whether they might equally be modeled by a classical memory. In this contribution, we propose a criterion to test locally for a truly quantum memory. The approach is agnostic with respect to the environment, as it solely depends on the local dynamics of the system of interest. Experimental realizations are particularly easy, as only single-time measurements on the system itself have to be performed. We study memory in a variety of physically motivated examples, both for a time-discrete case, and for time-continuous dynamics. For the latter, we are able to provide an interesting class of non-Markovian master equations with classical memory that allows for a physically measurable quantum trajectory representation.

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Introduction.—Applying quantum technologies to real-world problems requires a fundamental understanding of all underlying physical processes. Possible quantum advantages rely on our ability to cope with noise and dissipation, induced by the environment [1–7]. A detailed modeling of environmental impacts entails memory effects, showing non-Markovianity [8–15]. This requires advanced methods to describe quantum devices, yet non-Markovianity might also help to mitigate errors [16–18].

In recent years, it has become evident that non-Markovianity in quantum dynamics need not have a quantum origin [19–22]. The ability to distinguish memory effects arising from the coupling to an environmental quantum system from those of classical nature is of fundamental importance. On the one hand, it will help to improve the performance of quantum devices, as error-correction schemes differ in the two cases. On the other hand, such studies are inevitable when trying to prove the quantum nature of unfathomable degrees of freedom such as gravity [23,24].

Clearly, full operational access to the environment reveals its quantum nature, a situation hardly met in experiments. Indeed, standard open system theory aims at an effective dynamical description of the system of interest S, without any explicit reference to the environment E. Accordingly, we assume throughout that information is available from measurements on S only. The question arises whether such local information suffices to distinguish memory effects induced by an unknown quantum environment from those that may arise classically.

Recently, this question was addressed in the framework of process tensors [25–28]. A process tensor bears all information about the statistics of any possible sequence of measurements that could be performed locally on S. The classicality of the environmental memory can then be related to the separability of the process tensor [28]. While the process tensor is an elegant object from a theory point of view, its experimental determination is certainly challenging since it requires full multitime statistics of the process. By contrast, the results of this Letter are based on the system dynamics alone, and, thus, are both conceptually and experimentally more easily accessible.

We should note that there is an interesting angle to our approach, relating it to the existence of physically measurable quantum trajectories. We will explore these connections later, establishing non-Markovian master equations that allow for such a trajectory representation.

Formally, we define a *dynamics* $\mathcal{D}$ on S to be a family of completely positive trace-preserving (CPT) maps $\mathcal{D} = (\mathcal{E}_n)$ mapping the system state from the initial time t_0 to time t_n. This definition covers every physically valid evolution where the system and its environment are initially in a product state (uncorrelated). To determine the dynamics, channel tomography has to be performed for each $\mathcal{E}_n$, but no multitime statistics is needed. Besides this experimental advantage of the approach, it conforms very well with the traditional open quantum system frameworks based on dynamical maps and master equations.

In this Letter, we show how to disclose a truly quantum memory for non-Markovian dynamics, based on such local information. The proposed witness thus locally reveals a new, additional property of quantum non-Markovian dynamics, which is hidden for all known measures of non-Markovianity.

Classical and quantum memory.—Let us illustrate the idea with a simple toy model of a two-step dynamics $\mathcal{D}$,

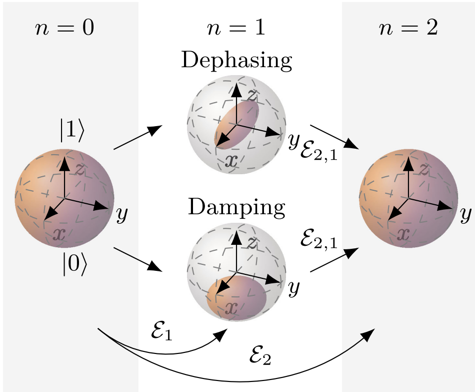

FIG. 1. Image of the Bloch sphere under the intermediate dephasing and damping dynamics, respectively. Both dynamics are non-Markovian as witnessed by the expansion during the second step. Dephasing is realizable with only classical memory, while amplitude damping is not. For this example, $f = 0.64$ and $g = 0.89$ in Eq. (2).

given by the CPT maps

$$\mathcal{E}_1[\rho_S] = \mathrm{tr}_E\Big[U_1(\rho_S \otimes \rho_E)U_1^\dagger\Big],$$
$$\mathcal{E}_2[\rho_S] = \mathrm{tr}_E\Big[U_2U_1(\rho_S \otimes \rho_E)U_1^\dagger U_2^\dagger\Big], \tag{1}$$

where ρ_S and ρ_E are the initial states of system S and environment E, respectively. The global dynamics are mediated by unitaries $U_{1,2}$. We let both, S and E be qubits, and set $\rho_E = |0\rangle\langle 0|$. Crucially, for our toy model we fix the second unitary to be the inverse of the first, i.e., $U_2 = U_1^\dagger$. Accordingly, the second CPT map is trivial, $\mathcal{E}_2 = \mathbb{1}$. As for U_1, we consider two different choices:

$$U_1^{\mathrm{dephase}} = \exp[-if(\sigma_x \otimes \sigma_x)],$$
$$U_1^{\mathrm{damp}} = \exp[-ig(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)], \tag{2}$$

with real parameters f and g determining the strength of the map [29]. The first leads to a partial dephasing in the x basis, the second choice induces a partial amplitude damping (see Fig. 1). Almost any pure initial state of S gets entangled with the environment in this first step, and therefore mixed. The second interaction then rewinds these correlations and the system returns to its initial state. Thus, we witness non-Markovian dynamics according to all common criteria [29]. In this global picture it is fair to say that the repeated interaction with the same environmental quantum system leads to non-Markovianity. Clearly, E is that (quantum) memory.

However, once we look at the local dynamics $\mathcal{D}$ alone—meaning that we know the maps $(\mathcal{E}_1, \mathcal{E}_2)$ but we are ignorant about the global dynamics including E—the analysis is different: any single qubit dephasing dynamics, no matter what its true physical origin is, is indistinghuishable from a random unitary evolution [60,61]. Since classical memory suffices to keep track of the random choice of the unitary, no quantum environment E is needed [29].

In the case of the partial amplitude damping, the situation is less obvious. Remarkably, we will present a criterion below which verifies that the amplitude damping example indeed requires quantum memory, i.e., cannot be modeled by classical memory. To proceed, we need to define properly what we mean by *classical memory*:

Definition 1.—Given two CPT maps $\mathcal{E}_1$ and $\mathcal{E}_2$. The dynamics $\mathcal{D} = (\mathcal{E}_1, \mathcal{E}_2)$ can be realized with classical memory, if and only if there is at least one Kraus decomposition $\{M_i\}$ of $\mathcal{E}_1[\rho_S] = \sum_i M_i \rho_S M_i^\dagger$ and suitable CPT maps Φ_i such that

$$\mathcal{E}_2[\rho_S] = \sum_i \Phi_i\Big[M_i \rho_S M_i^\dagger\Big]. \tag{3}$$

Otherwise the dynamics is said to require *truly quantum memory.*

Let us elaborate why this definition embraces the idea of dynamics with classical memory. Equation (3) describes a sequential process. The Kraus decomposition $\{M_i\}$ can be seen as a local measurement on S which on average realizes the first map $\mathcal{E}_1$. The second step with CPT map Φ_i is *conditioned on that outcome i* of the first measurement. Crucially, the label i is classical data, storable in a classical memory. By contrast, for a dynamics that cannot be written in the form above, a persisting quantum environment has to be present throughout both dynamical steps, as suggested by our toy model.

Further remarks: The definition of a dynamics $\mathcal{D}$ requires that $\mathcal{E}_2$ is a CPT map from the initial time t_0 to time t_2. By contrast, the average map from the intermediate time t_1 to t_2, given by $\mathcal{E}_{2,1} = \mathcal{E}_2 \circ \mathcal{E}_1^{-1}$, is in general *not* CPT (see also Fig. 1). Moreover, for the actual implementation of the measurement $\{M_i\}$ and the channels Φ_i, independent ancillary quantum systems might be necessary. However, these can always be discarded after use, so they do not serve as a memory.

Markovian quantum dynamics satisfy Eq. (3) trivially with $\Phi_i = \Phi = \mathcal{E}_{2,1}$, there is no memory at all. Any random unitary process (e.g., the dephasing in Fig. 1) can be written in the form of Eq. (3) of classical memory, as explained earlier. By contrast, the amplitude damping toy model cannot be realized in this way, as will follow from our theorem below.

As the main result of this Letter, we next provide a sufficient criterion for a locally known dynamics $\mathcal{D} = (\mathcal{E}_1, \mathcal{E}_2)$ to *not* be realizable by means of classical memory according to Definition 1. Its relevance is twofold. First, if the criterion holds, we have proof of a persistent quantum environment E. Second, note that Definition 1 is the most general physically measurable pure-state quantum trajectory representation of the given dynamics $\mathcal{D}$. Disclosing quantum memory, therefore, rules out the existence of such quantum trajectories. We will elaborate on these issues in a time-continuous limit in more detail below.

Criterion.—For the criterion, we need the concept of entanglement of assistance. Consider a bipartite quantum state χ_{SA} of system S and ancilla A (not to be confused with the environment E). Let $E[\chi_{\mathrm{SA}}]$ be an entanglement monotone (e.g., entanglement of formation or concurrence) [62]. The entanglement of assistance $E^\sharp$ is then [63,64]

$$E^\sharp[\chi_{\mathrm{SA}}] \coloneqq \max_{\{p_k,|\psi_k\rangle\}} \sum_k p_k E[|\psi_k\rangle], \tag{4}$$

i.e., the average entanglement *maximized* over any pure-state decomposition of χ_{SA}. Now assume that χ_{SA} describes the Choi state of the map $\mathcal{E}$ acting on the system S [65], i.e.,

$$\chi_{\mathrm{SA}}[\mathcal{E}] = \chi[\mathcal{E}] = (\mathcal{E} \otimes \mathbb{1})|\phi^+\rangle\langle\phi^+|, \quad \text{with}$$
$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j_{\mathrm{S}}\rangle|j_{\mathrm{A}}\rangle, \tag{5}$$

where d is the dimension of the system and the $|j_{\mathrm{S,A}}\rangle$ form an orthonormal basis in S and A, respectively. We find the following theorem:

Theorem 1.—Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two CPT maps. If for the Choi states χ_1 and χ_2 of $\mathcal{E}_1$ and $\mathcal{E}_2$ we observe

$$E^\sharp[\chi_1] < E[\chi_2], \tag{6}$$

the dynamics $\mathcal{D} = (\mathcal{E}_1, \mathcal{E}_2)$ requires quantum memory.

Proof.—Suppose the dynamics $\mathcal{D} = (\mathcal{E}_1, \mathcal{E}_2)$ only requires classical memory as defined in Definition 1. Then the local measurement $\{M_i\}$ implementing the channel $\mathcal{E}_1$ on S decomposes the corresponding Choi state χ_1 into the pure-state decomposition $\{p_i, |\psi_i\rangle\}$ with $|\psi_i\rangle = (M_i \otimes \mathbb{1})|\phi^+\rangle/\sqrt{p_i}$, $|\phi^+\rangle$ as in Eq. (5), and p_i being the probability for outcome i. The average entanglement in this decomposition $\{p_i, |\psi_i\rangle\}$ is upper bounded by the entanglement of assistance:

$$E^\sharp[\chi_1] = \max_{\{p_k,|\psi_k\rangle\}} \sum_k p_k E[|\psi_k\rangle] \geq \sum_i p_i E[|\psi_i\rangle]. \tag{7}$$

Local quantum channels can only reduce the entanglement [62]. Therefore, defining $\rho_i \coloneqq (\Phi_i \otimes \mathbb{1})|\psi_i\rangle\langle\psi_i|$, where Φ_i is a CPT map that can depend on the previous outcome, we have

$$\sum_i p_i E[|\psi_i\rangle] \geq \sum_i p_i E[\rho_i]. \tag{8}$$

The decomposition $\{p_i, \rho_i\}$ represents the Choi state χ_2 of the second map $\mathcal{E}_2$, i.e., $\sum_i p_i \rho_i = \chi_2$. However, the average entanglement in this decomposition is lower bounded by the entanglement of the state χ_2 itself,

$$\sum_i p_i E[\rho_i] \geq \min_{\{p_k,|\varphi_k\rangle\}} \sum_k p_k E[|\varphi_k\rangle] = E[\chi_2], \tag{9}$$

where the minimization runs over all pure-state decompositions of χ_2. ■

Discrete example.—First, we show a two-step dynamics that, upon changing a parameter, can be tuned from the case of verifiable quantum memory according to Theorem 1 to the case of classical memory, obeying a representation as in Definition 1. We consider a map $\mathcal{A}_p$ representing a thermal amplitude damping of a single qubit given by Kraus operators

$$M_1 = z_-\sqrt{p}\sigma_-, \quad M_2 = z_-(\sqrt{1-p}\sigma_+\sigma_- + \sigma_-\sigma_+),$$
$$M_3 = z_+\sqrt{p}\sigma_+, \quad M_4 = z_+(\sigma_+\sigma_- + \sqrt{1-p}\sigma_-\sigma_+), \tag{10}$$

where the strength of the channel is given by $p \in [0,1]$ and $z_\pm = 1/\sqrt{1+e^{\pm\beta}}$, with β a dimensionless inverse temperature. The zero-temperature amplitude damping channel with ground state $|0\rangle$ as its fixed point emerges as $\beta \to \infty$. At finite temperature, M_3 and M_4 model absorption from a thermal bath.

We consider a sequence of two maps of this class, i.e., a dynamics $\mathcal{D} = (\mathcal{E}_1, \mathcal{E}_2) = (\mathcal{A}_{p_1}, \mathcal{A}_{p_2})$, with p_n the damping strength at time t_n. For the sake of this example, we fix the inverse temperature $\beta = 0.51$, the first damping strength $p_1 = 0.9$, and investigate the nature of the required memory as a function of the second strength p_2. We choose the concurrence $\mathcal{C}$ as the entanglement monotone E in Eq. (4) and write $\mathcal{C}^\sharp$ for the concurrence of assistance. In Fig. 2 we plot $\mathcal{C}^\sharp[\chi_1] - \mathcal{C}[\chi_2]$ and satisfy the criterion for $p_2 < 0.11$ (orange region). Thus, the corresponding non-Markovian dynamics requires quantum memory. For $p_2 > 0.86$, the dynamics can be modeled by classical memory (blue region). We provide an explicit representation as in Definition 1 (see caption of Fig. 2 for details).

Time-continuous example.—Let us apply the criterion of Theorem 1 to the zero-temperature non-Markovian amplitude damping master equation,

$$\dot{\rho} = \mathcal{L}_t[\rho] = \frac{\gamma_-(t)}{2}([\sigma_-\rho, \sigma_+] + [\sigma_-, \rho\sigma_+]). \tag{11}$$

Here $\gamma_-(t)$ is the instantaneous damping rate which in the non-Markovian case changes sign over time [29].

For the dynamics resulting from this master equation, we find that the concurrence of assistance of the Choi state is equal to the concurrence (of formation) for all times, $\mathcal{C}^\sharp[\chi(t)] = \mathcal{C}[\chi(t)]$, $\forall\ t$. In the non-Markovian case, $\mathcal{C}$ is a nonmonotonous function. Thus, there are times $t_2 > t_1$ such that $\mathcal{C}^\sharp[\chi(t_1)] < \mathcal{C}[\chi(t_2)]$, which shows by virtue of

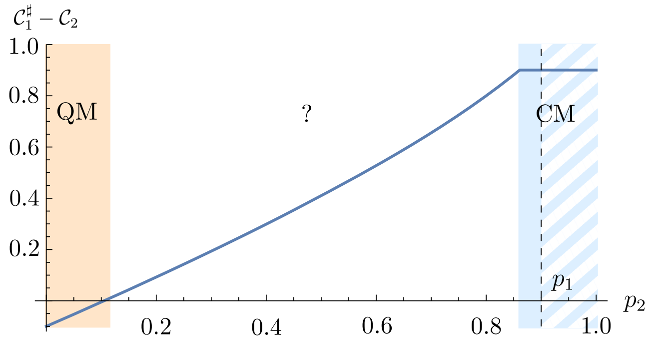

FIG. 2. Entanglement difference as a function of the strength parameter p_2 of the second dynamical step of a thermal amplitude damping channel (see text, other parameters $p_1 = 0.9$ and $\beta = 0.51$). For $p_2 < 0.11$, the criterion in Theorem 1 is satisfied (orange) and the dynamics requires quantum memory (QM). For $p_2 < p_1$, the damping gets partially rewound and the dynamics is non-Markovian. Yet for $0.86 \leq p_2 < p_1$ (solid blue) we can explicitly construct a representation as in Definition 1, and therefore only classical memory (CM) is needed—see Supplemental Material [29]. For $p_2 \geq p_1$ the dynamics is Markovian and thus does not require memory at all (blue hatched region). For $0.11 \leq p_2 < 0.86$ (white) we cannot decide whether truly quantum memory is required.

Theorem 1 that zero-temperature non-Markovian amplitude damping cannot be realized by means of classical memory.

However, heuristically extending the scenario to a thermal bath, one finds that for sufficiently high temperatures the criterion is no longer violated [29]. This does not necessarily mean that the dynamics can be explained by classical memory, but it shows that at higher temperatures it becomes harder to locally verify the quantum nature of the memory.

Dynamics with classical memory.—Disclosing quantum memory for time-continuous dynamics requires the consideration of the dynamical map at two distinct times, as seen in the previous example. However, to ensure that classical memory is sufficient, one has to explicitly provide a representation in terms of the time-continuous generalization of Definition 1.

For a dynamics $\mathcal{D} = (\mathcal{E}_n)_{n=1}^N$, with N discrete time steps, a representation with classical memory takes the form

$$\mathcal{E}_n[\rho] = \sum_{i_1,\ldots,i_n} M_{i_n}^{(i_1,\ldots,i_{n-1})} \ldots M_{i_2}^{(i_1)} M_{i_1} \rho M_{i_1}^\dagger M_{i_2}^{(i_1)\dagger} \ldots \ldots M_{i_n}^{(i_1,\ldots,i_{n-1})\dagger}, \qquad 1 \leq n \leq N, \tag{12}$$

where the superscripts indicate that the measurement operators at a certain step can depend on *all* previous outcomes. For suitably chosen measurements $\{M_{i_n}\}$, this construction allows for a time-continuous limit.

Equation (12) describes the most general form of a physically measurable pure-state trajectory representation of a dynamics. Hence, for a dynamics which requires truly quantum memory according to Theorem 1, a pure-state unraveling is immediately ruled out. On the other hand, a non-Markovian dynamics which can be written in this way, i.e., which only requires classical memory, admits a pure-state trajectory representation by construction. This clarifies that the often debated existence of physically measurable non-Markovian quantum trajectories depends on the classicality of the memory needed to implement the dynamics [29]. In the following, we provide some time-continuous examples.

As mentioned earlier, any dynamics with random unitary representation can be realized with classical memory (see also Refs. [66,67]). Another simple case is a probabilistic mixture of multiple Markovian dynamics. The prime example is the master equation of eternal non-Markovianity requiring two bits of classical memory [20]. There, the outcome of an initial random choice with probabilities p_i determines which of three different Markovian dynamics with generators $\mathcal{L}_i$ is implemented for all times, $\mathcal{E}_t = p_1 e^{t\mathcal{L}_1} + p_2 e^{t\mathcal{L}_2} + p_3 e^{t\mathcal{L}_3}$. This dynamical map is in general non-Markovian with respect to the CP-divisibility criterion [20,21,68,69]. Nevertheless, it has an obvious pure-state trajectory representation. Further dynamics with classical memory are given by quantum semi-Markov processes, where the application of the next step depends on a (classical) waiting time distribution [22,70–73].

The richness of dynamics with classical memory is, however, far greater. Equation (12) can serve as a starting point to derive new non-Markovian master equations with classical memory based on a quantum-jump-inspired trajectory representation, as we show next.

We use a qubit and start from a standard quantum jump trajectory which describes amplitude damping (jump operator σ_-). The classical memory keeps track of whether the jump has already occurred. If so, the jump operator is replaced by σ_+. One bit of classical memory is sufficient for the implementation of this scheme. Integrating the succession of maps over all possible jump times yields the non-Markovian time-local master equation

$$\mathcal{L}_t[\rho] = \frac{1}{2} \sum_{k=1,2} \gamma_k(t) \left(\left[L_k, \rho L_k^\dagger \right] + \left[L_k \rho, L_k^\dagger \right] \right), \tag{13}$$

with

$$\gamma_1(t) = \frac{\kappa(\kappa t - 1)}{2(\kappa t - e^{\kappa t})}, \qquad \gamma_2(t) = \frac{\kappa(e^{\kappa t} - 1)}{8(e^{\kappa t} - \kappa t)}, \qquad L_1 = \sigma_-, \qquad L_2 = \sigma_z.$$

A detailed derivation is presented in the Supplemental Material [29]. Let us stress that the non-Markovian master equation (13) has a physically realizable, measurable quantum jump representation by construction.

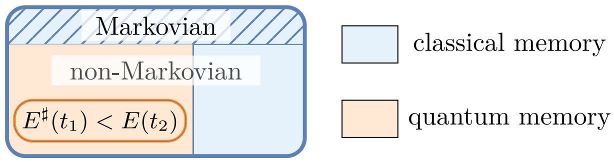

FIG. 3. Non-Markovian quantum dynamics may emerge from classical or truly quantum memory. Markovian dynamics trivially falls into the first class. Some dynamics requiring a truly quantum memory can be detected by the criterion proposed in Theorem 1 (encircled region within the orange area). An example is the non-Markovian amplitude damping [see Eq. (11)]. Examples of non-Markovian dynamics with classical memory (solid blue region) encompass Eq. (13) and the master equation of eternal non-Markovianity [20]. We conjecture that there exist dynamics which require quantum memory (orange region) but which cannot be detected by our criterion.

It is interesting to note that the dynamics given by Eq. (13) is P-indivisible and thus non-Markovian in a stricter sense than the P-divisible master equation of eternal non-Markovianity discussed earlier.

Conclusion.—Non-Markovian quantum dynamics is associated with memory effects. However, this memory is not necessarily provided by environmental quantum degrees of freedom but may be classical. In this Letter, we investigate the nature of that memory from a local viewpoint. Focusing on the dynamics in the open system alone, we make no assumption about the physics of the environment.

We start from a definition for a dynamics requiring classical memory only. As the main result, we then present a criterion in terms of an inequality whose satisfaction rules out any such realization of the given dynamics. Crucially, this criterion depends solely on information about the single-time local dynamics of the open system, no multi-time statistics is required. Its tomography and thus the disclosure of environmental quantum memory is in experimental sight.

We illustrate the concept with several discrete and time-continuous examples with and without truly quantum memory, including cases which can be tuned between the two regimes. In particular, we show how to construct a class of non-Markovian time-local master equations admitting a pure-state quantum jump trajectory representation based on classical memory. No such unraveling can exist for a dynamics which requires truly quantum memory.

Our criterion is sufficient but not necessary—refinements are thus desirable (see Fig. 3). The presented concepts serve as an immediate starting point for further investigations, which include characterizing the size of the quantum or classical memory, criteria for unital dynamics, and the construction of physically realizable non-Markovian trajectories of the diffusive type. More generally, our Letter shows that an environment-agnostic perspective can be a valuable tool for characterizing environmental properties without making prior assumptions about the underlying physics.

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[1] E. Knill, R. Laflamme, and W. H. Zurek, Proc. R. Soc. A **454**, 365 (1998).
[2] A. W. Harrow and A. Montanaro, Nature (London) **549**, 203 (2017).
[3] J. Preskill, Quantum **2**, 79 (2018).
[4] J. Roffe, Contemp. Phys. **60**, 226 (2019).
[5] C. D. Bruzewicz, J. Chiaverini, R. McConnell, and J. M. Sage, Appl. Phys. Rev. **6**, 021314 (2019).
[6] *Quantum Computing: Progress and Prospects*, edited by E. Grumbling and M. Horowitz (National Academies Press, Washington, DC, 2019).
[7] S. Wang, E. Fontana, M. Cerezo, K. Sharma, A. Sone, L. Cincio, and P. J. Coles, Nat. Commun. **12**, 6961 (2021).
[8] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. **13**, 093004 (2011).
[9] H.-P. Breuer, J. Phys. B **45**, 154001 (2012).
[10] Á. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
[11] M. J. W. Hall, Phys. Rev. A **89**, 042120 (2014).
[12] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. **88**, 021002 (2016).
[13] L. Li, M. J. Hall, and H. M. Wiseman, Phys. Rep. **759**, 1 (2018).
[14] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
[15] Á. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
[16] O. Oreshkov and T. A. Brun, Phys. Rev. A **76**, 022318 (2007).
[17] Z.-X. Man, Y.-J. Xia, and R. Lo Franco, Phys. Rev. A **92**, 012315 (2015).
[18] K. X. Wei, E. Pritchett, D. M. Zajac, D. C. McKay, and S. Merkel, arXiv:2302.10881.
[19] B. Vacchini, J. Phys. B **45**, 154007 (2012).
[20] N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, Sci. Rep. **7**, 6379 (2017).
[21] S. N. Filippov, J. Piilo, S. Maniscalco, and M. Ziman, Phys. Rev. A **96**, 032111 (2017).
[22] N. Megier, M. Ponzi, A. Smirne, and B. Vacchini, Entropy **23**, 905 (2021).
[23] D. Carney, H. Müller, and J. M. Taylor, PRX Quantum **2**, 030330 (2021).
[24] Y. Ma, T. Guff, G. W. Morley, I. Pikovski, and M. S. Kim, Phys. Rev. Res. **4**, 013024 (2022).
[25] S. Milz, D. Egloff, P. Taranto, T. Theurer, M. B. Plenio, A. Smirne, and S. F. Huelga, Phys. Rev. X **10**, 041049 (2020).
[26] C. Giarmatzi and F. Costa, Quantum **5**, 440 (2021).
[27] M. Nery, M. T. Quintino, P. A. Guérin, T. O. Maciel, and R. O. Vianna, Quantum **5**, 538 (2021).

[28] P. Taranto, M. T. Quintino, M. Murao, and S. Milz, arXiv:2307.11905.
[29] See Supplemental Material, which includes Refs. [11–15,30–59], at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.060402 for details.
[30] S. Lorenzo, F. Plastina, and M. Paternostro, Phys. Rev. A **88**, 020102(R) (2013).
[31] A. Smirne, N. Megier, and B. Vacchini, Quantum **5**, 439 (2021).
[32] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2007).
[33] S. Kretschmer, K. Luoma, and W. T. Strunz, Phys. Rev. A **94**, 012106 (2016).
[34] B. M. Garraway, Phys. Rev. A **55**, 2290 (1997).
[35] L. Diósi, N. Gisin, and W. T. Strunz, Phys. Rev. A **58**, 1699 (1998).
[36] D. Heineken, K. Beyer, K. Luoma, and W. T. Strunz, Phys. Rev. A **104**, 052426 (2021).
[37] M. Beth Ruskai, S. Szarek, and E. Werner, Linear Algebra Appl. **347**, 159 (2002).
[38] M. Ziman, P. Štelmachovič, and V. Bužek, Open Syst. Inf. Dyn. **12**, 81 (2005).
[39] M. Ziman and V. Bužek, in *Quantum Dynamics and Information* (World Scientific, Singapore, 2010), pp. 199–227.
[40] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2009).
[41] T. A. Brun, Am. J. Phys. **70**, 719 (2002).
[42] L. Diósi, Phys. Rev. Lett. **100**, 080401 (2008).
[43] H. M. Wiseman and J. M. Gambetta, Phys. Rev. Lett. **101**, 140401 (2008).
[44] H.-P. Breuer, Phys. Rev. A **70**, 012106 (2004).
[45] C. Pellegrini and F. Petruccione, J. Phys. A **42**, 425304 (2009).
[46] S. Krönke and W. T. Strunz, J. Phys. A **45**, 055305 (2012).
[47] A. Smirne, M. Caiaffa, and J. Piilo, Phys. Rev. Lett. **124**, 190402 (2020).
[48] N. Megier, W. T. Strunz, and K. Luoma, Phys. Rev. Res. **2**, 043376 (2020).
[49] L. Diósi and W. T. Strunz, Phys. Lett. A **235**, 569 (1997).
[50] H.-P. Breuer, B. Kappler, and F. Petruccione, Phys. Rev. A **59**, 1633 (1999).
[51] M. W. Jack, M. J. Collett, and D. F. Walls, J. Opt. B **1**, 452 (1999).
[52] J. Piilo, S. Maniscalco, K. Härkönen, and K.-A. Suominen, Phys. Rev. Lett. **100**, 180402 (2008).
[53] R. Hartmann and W. T. Strunz, J. Chem. Theory Comput. **13**, 5834 (2017).
[54] G. Gasbarri and L. Ferialdi, Phys. Rev. A **98**, 042111 (2018).
[55] V. Link, W. T. Strunz, and K. Luoma, Entropy **24**, 352 (2022).
[56] T. Becker, C. Netzer, and A. Eckardt, Phys. Rev. Lett. **131**, 160401 (2023).
[57] R. I. Karasik and H. M. Wiseman, Phys. Rev. Lett. **106**, 020406 (2011).
[58] S. Daryanoosh and H. M. Wiseman, New J. Phys. **16**, 063028 (2014).
[59] K. Beyer, K. Luoma, and W. T. Strunz, Phys. Rev. A **97**, 032113 (2018).
[60] M. Gregoratti and R. F. Werner, J. Mod. Opt. **50**, 915 (2003).
[61] J. Helm and W. T. Strunz, Phys. Rev. A **81**, 042314 (2010).
[62] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).
[63] D. P. DiVincenzo, C. A. Fuchs, H. Mabuchi, J. A. Smolin, A. Thapliyal, and A. Uhlmann, in *Lecture Notes in Computer Science*, edited by C. P. Williams (Springer, Berlin, Heidelberg, 1999), pp. 247–257.
[64] J. A. Smolin, F. Verstraete, and A. Winter, Phys. Rev. A **72**, 052317 (2005).
[65] M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).
[66] T. Grotz, L. Heaney, and W. T. Strunz, Phys. Rev. A **74**, 022102 (2006).
[67] D. Chruściński and F. A. Wudarski, Phys. Lett. A **377**, 1425 (2013).
[68] H.-P. Breuer, G. Amato, and B. Vacchini, New J. Phys. **20**, 043007 (2018).
[69] V. Jagadish, R. Srikanth, and F. Petruccione, Phys. Rev. A **101**, 062304 (2020).
[70] H.-P. Breuer and B. Vacchini, Phys. Rev. Lett. **101**, 140402 (2008).
[71] B. Vacchini, Phys. Rev. Lett. **117**, 230401 (2016).
[72] D. Chruściński and A. Kossakowski, Phys. Rev. A **94**, 020103(R) (2016).
[73] N. Megier, A. Smirne, and B. Vacchini, Entropy **22**, 796 (2020).