Domain Wall Dynamics in Classical Spin Chains: Free Propagation, Subdiffusive Spreading, and Soliton Emission

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The nonequilibrium dynamics of domain wall initial states in a classical anisotropic Heisenberg chain exhibits a striking coexistence of apparently linear and nonlinear behaviors: the propagation and spreading of the domain wall can be captured quantitatively by *linear*, i.e., noninteracting, spin wave theory absent its usual justifications; while, simultaneously, for a wide range of easy-plane anisotropies, emission can take the place of stable solitons—a process and objects intrinsically associated with interactions and nonlinearities. The easy-axis domain wall only has transient dynamics, the isotropic one broadens diffusively, while the easy-plane one yields a pair of ballistically counterpropagating domain walls which, unusually, broaden *subdiffusively*, their width scaling as $t^{1/3}$.

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Introduction.—A prototypical setting for nonequilibrium dynamics is an initial state with two neighboring regions in different stationary states of the same Hamiltonian. A single sharp domain wall between distinct stationary states can move and spread, carrying energy—and, possibly, other conserved quantities. This type of dynamics has been studied in many contexts, including the spin-$\frac{1}{2}$ quantum XXZ chain [1–33], other quantum spin chains [30,34–38], quantum field theories [39–41], the continuum Landau-Lifshitz model of classical spin densities [30,42], two-dimensional quantum systems [43,44], and the simple exclusion process [45].

Here, we consider a *classical* one-dimensional anisotropic (XXZ) Heisenberg chain,

$$H = -J\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z), \qquad (1)$$

where the $\boldsymbol{S}_i \in S^2$ are classical $O(3)$ vectors at sites i of a chain, and we assume $J > 0$ is ferromagnetic. In addition to serving as one of the foundational models of magnetism and many-body spin physics, its dynamical properties have recently been subject to renewed investigation from various perspectives, including quantum-to-classical correspondence, nonlinearity, and integrability, and anomalous hydrodynamics [46–58].

We investigate domain wall dynamics in this, arguably, simplest incarnation of this problem. We find a rich phenomenology with a number of intriguing aspects, and a co-existence of linear and nonlinear behaviors: ballistically propagating domain walls which spread subdiffusively, showing the interplay of ballistic dynamics with subdiffusion, and which are well described by *linear* spin-wave theory; while, at the same time, we observe the emission of stable solitons, connecting to questions of dynamics and solitons in nearly integrable systems. It also opens a complementary perspective on the much studied related problem of quantum Heisenberg chains, where signatures of interesting phenomena such as KPZ scaling [25,59] have been experimentally observed [60,61].

We find qualitatively distinct behavior in the easy-plane, isotropic, and easy-axis cases ($0 \leq \Delta < 1$, $\Delta = 1$, and $\Delta > 1$, respectively). Our results and setup are summarized in Fig. 1. For easy-plane anisotropy, the domain wall splits into two ballistically counterpropagating ones [Fig. 1(c)]. Since the Hamiltonian is nonintegrable and intrinsically nonlinear, and since the propagating domain walls have high energy compared to the background, they can, in principle, emit or decay into other excitations—giving the nonequilibrium setup an inherent nonlinear flavor. It is therefore all the more surprising that, over the entire range of easy-plane anisotropy $\Delta \in [0, 1)$, domain walls propagate ballistically. This is reminiscent of the behavior of quasiparticles in integrable systems [14,30,62–72], or that of operator spreading [73–75]. For the latter, ballistic behavior is accompanied by diffusive broadening [73–75]. More generally, broadening in interacting many-body systems is

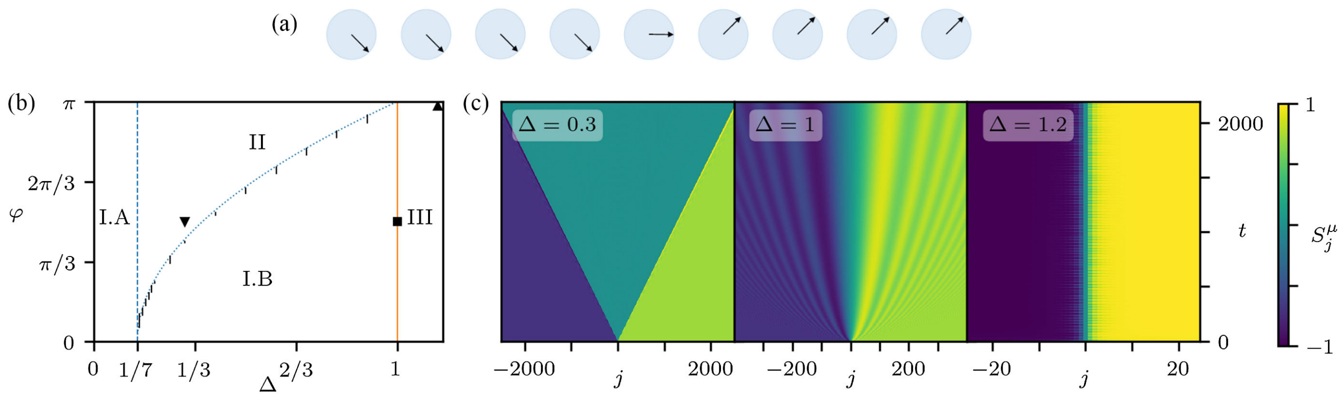

FIG. 1. (a) Schematic of the initial conditions (3), in the XY plane, shown for $\varphi = \pi/2$. (b) Boundaries between the different dynamical regimes as a function of the anisotropy Δ and amplitude φ. I.A and I.B are the two linear regimes (distinguished by whether the oscillations are behind or ahead of the counter-propagating domain walls, respectively), where all of the dynamical features are well described by linear spin wave theory; II is the easy-plane nonlinear regime, where solitons coexist with the spreading domain walls. III is the easy-axis regime, with a single, static domain wall. The isotropic point $\Delta = 1$ corresponds to the transition between I.B and III, and is well described by linear spin wave theory with a single, diffusively broadening domain wall. The vertical bars denote the uncertainty in determining the transition between I.B and II from the simulations. (c) Overview of the domain wall dynamics shown for the easy-plane $\Delta = 0.3$ (inverted shaded triangle), the isotropic point $\Delta = 1$ (shaded square) and the easy-axis $\Delta = 1.2$ (shaded triangle), respectively. Note the different ranges of the x axes. $\varphi = \pi/2$ for the easy-plane and the isotropic case, where S^y is plotted. S^z is plotted for the easy-axis case, with $\varphi = \pi$. Ballistic counterpropagation is observed in the easy-plane, diffusive melting of the original domain wall is seen at the isotropic point, while the easy-axis approaches a very narrow static soliton. Note that, since the subdiffusive domain wall spreading in the easy plane is parametrically slower than the ballistic propagation, and the emitted solitons move only very slightly slower than the domain walls and so are not well separated over the timescales shown, both of these are difficult to see on this overview plot—they are seen more readily in Fig. 2 instead.

typically diffusive, with exceptions usually associated with integrability or a lack of interactions.

In sharp contrast to this expectation, we show that the propagating domain walls broaden subdiffusively, as $\sim t^{1/3}$, in the entire easy-plane regime $\Delta \in [0, 1)$. We find that the propagation speed, profile, and $t^{1/3}$ scaling can be quantitatively obtained from *linear* spin wave theory. At the same time, above a critical angle $\varphi_c(\Delta)$ between the domains separated by the propagating domain wall [Fig. 1(b)], the linear behavior of the propagating domain walls coexists with the aforementioned, inherently *nonlinear* feature of the emission of solitons. We provide a heuristic picture for all of these processes.

At the isotropic Heisenberg point $\Delta = 1$, the domain walls can no longer propagate, and the subdiffusive spreading gives way to a diffusive melting of the original domain wall [Fig. 1(c)]. Nor can the domain walls propagate in the easy-axis case ($\Delta > 1$), where the melting is fully arrested and a static soliton is approached asymptotically [Fig. 1(c)].

The behavior for $\Delta \geq 1$ is analogous to that known for quantum spin-$\frac{1}{2}$ chains [3,42]—a classical-quantum analogy which is, in itself, remarkable. By contrast, the $t^{1/3}$ broadening of the domain wall that we find over the entire range $0 \leq \Delta < 1$ appears, in the quantum spin-$\frac{1}{2}$ case, only at the $\Delta = 0$ point [2,9,10,12] or at the light cone of fastest excitations [23,24,30,76], being associated with the non-interacting (free-fermion) nature of these cases. The existence and emission of the solitons have, to the best of our knowledge, not been previously observed—either in the quantum model or in a corresponding continuum Landau-Lifshitz model.

In the following, we provide details for these claims, and conclude with a discussion of the broader significance of this Letter.

Model.—We consider the classical XXZ spin chain, Eq. (1). The dynamics is given by the classical equations of motion,

$$\dot{S}_i^\mu = -\epsilon^{\mu\nu\lambda} J_\nu (S_{i+1}^\nu + S_{i-1}^\nu) S_i^\lambda, \tag{2}$$

which follow from the fundamental Poisson brackets $\{S_i^\mu, S_j^\nu\} = \delta_{ij}\epsilon^{\mu\nu\lambda} S_j^\lambda$, where $J_x = J_y = J = 1$ (which implicitly defines all units), and $0 \leq J_z = \Delta$. The XY point $\Delta = 0$ corresponds to the free-fermion limit of the quantum spin-$\frac{1}{2}$ chain, but is, in the classical case, an interacting model.

Easy-plane, $\Delta < 1$.—We consider a sharp domain wall in the in-plane components as the initial condition,

$$\begin{aligned} \boldsymbol{S}_{i<0} &= \cos(\varphi/2)\hat{\boldsymbol{x}} - \sin(\varphi/2)\hat{\boldsymbol{y}}, \\ \boldsymbol{S}_{i=0} &= \hat{\boldsymbol{x}}, \\ \boldsymbol{S}_{i>0} &= \cos(\varphi/2)\hat{\boldsymbol{x}} + \sin(\varphi/2)\hat{\boldsymbol{y}}, \end{aligned} \tag{3}$$

for some amplitude φ that sets the magnetization jump across the domain wall as illustrated in Fig. 1(a). The $O(2)$

isotropy implies that any choice of φ connects two easy-plane ground states. We set the spin at $i=0$ to lie halfway between the two domains, $\boldsymbol{S}_{i=0}=\hat{\boldsymbol{x}}$ (though the results do not depend on the choice of $\boldsymbol{S}_{i=0}$, so long as we do not select an unstable steady state [77]).

Numerically integrating the equations of motion (2) with initial conditions (3) and open boundaries reveals that two counterpropagating domain walls immediately emerge from $i=0$: a left-moving one connecting the $(-)$ domain to the expanding $\hat{\boldsymbol{x}}$ domain; and a right-moving one connecting the $\hat{\boldsymbol{x}}$ domain to the $(+)$ domain, as seen in Fig. 1(c).

The size of the $\hat{\boldsymbol{x}}$ domain grows linearly with time, implying ballistic domain wall motion. Moreover, the domain-wall velocity does not differ measurably from the long-wavelength group velocity of the spin wave expansion, $c=\sqrt{2(1-\Delta)}$ [cf. Eqs. (4)–(6), see also the Supplemental Material [78]], despite the nonlinearity of the equations of motion.

To investigate the long-time dynamics of the domain wall numerically, we switch to its co-moving frame [78]. We then find, numerically, that this easy-plane dynamics exhibits three qualitatively distinct regimes, cf. Fig. 1(b): two linear regimes, I.A and I.B, so-called because they are well described by *linear* spin wave theory in their entirety; and a nonlinear regime II characterized by an instability to the emission of solitons.

Within the linear regime we find, in addition to the ballistic motion of the domain walls, a subdiffusive spreading, with their width scaling as $t^{1/3}$. We demonstrate this scaling collapse of the full domain wall profiles in Figs. 2(a) and 2(b). In the nonlinear regime we observe the emission of a soliton during domain wall propagation shown in Fig. 2(c) which moves ballistically at a slower speed than the domain wall. We show with a purely ballistic scaling collapse in Fig. 2(d) that, indeed, this soliton does not disperse.

Spin-wave theory.—We next demonstrate that the spin-wave description of the easy-plane dynamics, remarkably, captures *all* of the relevant features in what we call the linear regimes, and correctly predicts velocity and width scaling of the domain walls even in the nonlinear regime.

We expand each spin about the $\hat{\boldsymbol{x}}$ domain,

$$\boldsymbol{S}_i=\hat{\boldsymbol{x}}\sqrt{1-l_i^2}+\boldsymbol{l}_i, \tag{4}$$

and retain only terms linear in $\boldsymbol{l}_i$ in the equations of motion (which is, *a priori*, not controlled, as φ is large).

The analytical solution of the resulting problem is presented in the Supplemental Material [78], but the central asymptotic result is readily stated: the spin-wave dispersion is given by

$$\omega_q\sim c|q|-\alpha|q|^3+\ldots, q\sim 0, \tag{5}$$

where

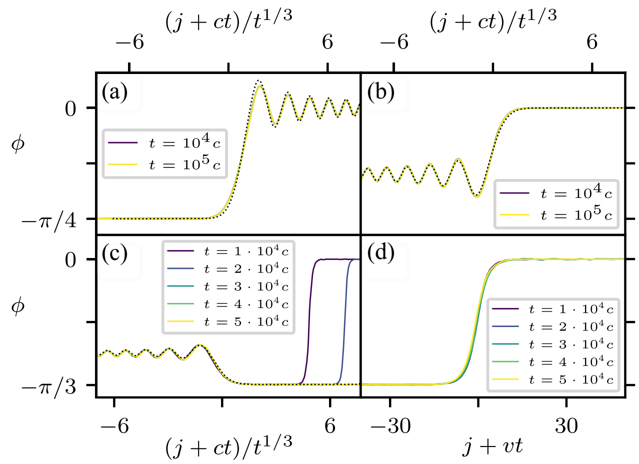

FIG. 2. Dynamics in the easy-plane case. Dotted lines show spin wave predictions, where relevant. Only the left-moving domain walls are shown, and lighter colors indicate later times. (a) Linear regime I.A ($\Delta=0$, $\varphi=\pi/2$), showing the ballistic propagation and subdiffusive spreading of the domain wall, with the oscillations trailing. (b) Linear regime I.B ($\Delta=0.3$, $\varphi=3\pi/10$), where the oscillations are now ahead of the domain wall. (c) Nonlinear regime II ($\Delta=0.25$, $\varphi=\pi/2$), showing that the domain wall decays by emitting a soliton, though its speed and width scaling are unaffected; the soliton moves at a slower velocity $v<c$, and the separation increases with time. (d) Same parameters and times as (c), but in the comoving frame of the soliton.

$$c=2\sqrt{1-\Delta},\qquad \alpha=\frac{1-7\Delta}{12c}, \tag{6}$$

and, at long times, the left-moving domain wall is a function $\mathcal{D}$ of the variable $(j+ct)/(3\alpha t)^{1/3}$:

$$S_j^y(t)\sim\sin\left(\frac{\varphi}{2}\right)\mathcal{D}\left(\frac{j+ct}{(3\alpha t)^{1/3}}\right). \tag{7}$$

The linear spin wave calculation thus correctly predicts, asymptotically, two ballistically counterpropagating domain walls, each with a width scaling as $w(t)\propto t^{1/3}$. We also observe good quantitative agreement of the spin-wave prediction (dotted) with the profiles obtained in the full numerical simulation (solid curves) in Figs. 2(a) and 2(b). The integral form is different from that appearing in the quantum free-fermion case [1,12] but is similar to those appearing in recent studies of caustics and catastrophes at light cones [79–81].

Soliton emission in the nonlinear regime.—We next discuss the emission of solitons in the easy-plane regime. As observed in Figs. 2(c) and 2(d) the moving domain wall can emit a stable (nondispersing) ballistically propagating soliton connecting two ground states. We find that this emission only takes place above a critical, anisotropy-dependent amplitude $\varphi>\varphi_c(\Delta)$, and, in particular, only in the regime $1/7<\Delta<1$, as shown in Fig. 1(b). The energy carried by these solitons is seen to depend both on the

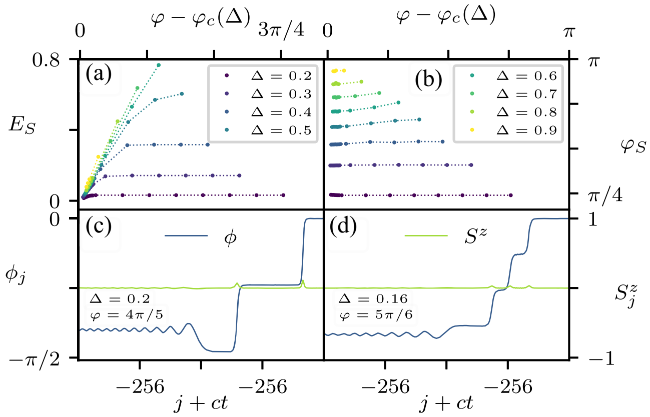

FIG. 3. Soliton emission in the easy-plane dynamics. (a),(b) Dependence of the soliton energy E_S and soliton amplitude φ_S, respectively, on the initial amplitude φ. We observe that the amplitude of the emitted solitons is almost constant. (c),(d) Two- and three-soliton emission, respectively, when $\varphi \gg \varphi_S$.

anisotropy Δ and the initial amplitude of the domain wall φ. Importantly, however, we observe that, at fixed Δ, the angular amplitude φ_S of the emitted soliton (the in-plane angle between the two ground states the soliton connects) does not depend on φ [see Fig. 3(b)], and, in fact, is equal to the critical value $\varphi_c = \varphi_S$. Finally, we observe n-soliton emission if $\varphi > (2n-1)\varphi_S$, as shown in Figs. 3(c) and 3(d).

To explain this phenomenology we begin with an observation on the kinematics of magnons. When $\Delta < 1/7$, the spin-wave dispersion has *negative* curvature at small q; this ensures that two-magnon scattering is elastic. In contrast, inelastic scattering is possible for $\Delta > 1/7$, allowing the dynamic instability towards soliton emission [78].

To explain why the emitted soliton's amplitude φ_S depends only on Δ (i.e., is unique for a given Hamiltonian), we propose the following heuristic model of soliton production. We assume that the model supports a two-parameter family of soliton solutions, which we may take to be their energy E_S and velocity v_S. These two parameters, then, uniquely determine the other physical properties, such as the width and amplitude. Now, as the interactions are local, and the soliton is observed to be created at the ballistically-moving centre of the domain wall over an extended time, the speed of the soliton must initially be matched to the Δ-dependent domain wall speed so that energy can be efficiently transferred—that is, $v_S = c(\Delta)$. Further, since the soliton is seeded by the domain wall, it must begin with zero energy $E_S \to 0$. This fixes the two parameters, and so picks out a unique initial soliton with some amplitude $\varphi_S(\Delta)$. As the dynamics proceeds, energy is transferred from the domain wall to the soliton, slowing down the latter and leading to its separation from the domain wall; but the amplitude φ_S is a nonlocal property [82], and so cannot be changed by local dynamical processes after the soliton and domain wall begin to separate.

Finally, given the fixed soliton amplitude φ_S, we can provide an energetic argument for the stability regions. The domain-wall energy depends monotonically on its amplitude, which must, therefore, decrease if soliton emission is to occur. The initial amplitude of the domain wall is $\varphi/2$, and after the emission the new amplitude is $|\varphi_S - \varphi/2|$. Thus, emission is possible only if $\varphi > \varphi_c = \varphi_S$. This also implies that n-soliton emission is possible if $\varphi > (2n-1)\varphi_S$, as observed in Figs. 3(c) and 3(d).

Easy-axis and isotropic dynamics.—We briefly remark on the domain wall dynamics at the isotropic point ($\Delta = 1$) and in the easy-axis case ($\Delta > 1$).

At the isotropic point, there can be no propagating ferromagnetic domain walls, because all components of the magnetization are conserved; instead, we observe that the initially sharp domain wall spreads diffusively [Fig. 4(a)]. This can be understood within the linear spin-wave picture. At $\Delta = 1$, the dispersion switches from an odd-power expansion to the even expansion [78]

$$\omega_q = 2[1 - \cos(q)] \sim q^2 + \cdots, \qquad q \sim 0. \tag{8}$$

There are no linear (dispersionless) terms—so the center of the domain wall does not move—and the width is now controlled by the quadratic, not cubic, term. Details of the calculation are presented in [78]. The final, asymptotic answer can be conveniently expressed in terms of the normalized Fresnel integrals,

$$S_j^y(t) \sim \sin\left(\frac{\varphi}{2}\right)\left[\mathcal{C}\left(\frac{j}{\sqrt{2\pi t}}\right) + \mathcal{S}\left(\frac{j}{\sqrt{2\pi t}}\right)\right], \tag{9}$$

which shows good quantitative agreement with the full solution as seen in Fig. 4(a).

For the easy axis, we change the initial conditions so that the domain wall occurs in the z components, ensuring that the state has finite energy. Specifically,

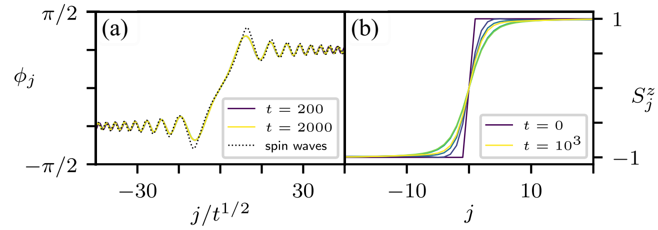

FIG. 4. Dynamics (a) at the isotropic point, and (b) in the easy-axis regime ($\Delta = 1.2$). There are no propagating domain walls in either case: the initial state spreads diffusively at the isotropic point, whilst it approaches the static soliton in the easy-axis case.

$$S_{i<0} = -\hat{z}, \qquad S_{i=0} = \hat{x}, \qquad S_{i>0} = +\hat{z}. \tag{10}$$

Now, since the z magnetization is conserved, there can be no propagating domain wall solutions; some dissipative spin-wave radiation escapes, before the state settles down, in an oscillatory manner [Fig. 4(b)], to the static soliton,

$$S_j^z = \tanh[j\cosh^{-1}(\Delta)] \tag{11}$$

(see Ref. [78] for the derivation of this soliton solution).

Conclusions and outlook.—Our Letter of domain wall dynamics in the classical anisotropic Heisenberg spin chain reveals a remarkably rich and unexpected phenomenology, including ballistic propagation and subdiffusive spreading of domain walls in the easy-plane regime, alongside the existence and emission of stable solitons—a highly nonlinear phenomenon coincident with a description of many aspects of the dynamics in the framework of *linear* spin-wave theory.

The fact that all of the essential features of the regimes I.A and I.B (where no solitons are emitted, cf. Fig. 1) are captured by a linearized description is, itself, remarkable—and connects this Letter to the broader question of under what conditions nonlinear settings—e.g., *a priori* beyond the linear response regime—may still be described by simplified linear theories. This issue has appeared prominently, for example, in the study of KPZ dynamics [60,61,72] expected for small jumps in the initial condition, but in fact observed for larger ones. Further, how a description of the "doubly nonlinear" phenomenon of the *emission* of (single or even multiple) *stable solitons* can coexist with a linear description of the propagation of the emitting domain wall is a tantalizing open question for future theoretical work.

This Letter also sheds some light on the question of when, and to what extent, classical treatments can account for *a priori* complex quantum dynamics, by providing closely related instances of where this appears to be (im) possible: while the $\Delta \geq 1$ regimes and the XY point ($\Delta = 0$) appear to be entirely analogous both classically and quantum-mechanically [15,22,32,33,42], the $0 < \Delta < 1$ regime is qualitatively distinct in the classical case. While reflecting some properties of the quantum $\Delta = 0$ case [9,10], the phenomenon of soliton emission has not been observed in previous studies of either the $S = \frac{1}{2}$ quantum case or the continuum Landau-Lifshitz model.

Overall, it has become clear that spin chains, not just quantum but also classical, host many unexplored features. The classical Heisenberg spin chain in particular has proven to be a fruitful platform to uncover and understand complex phenomena, and is presumably good for many surprises in future studies.

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[78] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.057202, which also contains Refs. [64–66], for additional details on the frame-shifting procedure, the spin-wave calculations, and the kinematics of two-magnon scattering.

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