Relativistic Perturbation Theory for Black-Hole Boson Clouds

Enrico Cannizzaro,[1,*] Laura Sberna,[2,†] Stephen R. Green,[3,‡] and Stefan Hollands[4,5,§]

[1]*Dipartimento di Fisica, "Sapienza" Università di Roma & Sezione INFN Roma1, Piazzale Aldo Moro 5, 00185, Roma, Italy*
[2]*Max Planck Institute for Gravitational Physics (Albert Einstein Institute) Am Muühlenberg 1, 14476 Potsdam, Germany*
[3]*School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*
[4]*Institut für Theoretische Physik, Universität Leipzig, Brüderstrasse 16, D-04103 Leipzig, Germany*
[5]*Max Planck Institute for Mathematics in the Sciences, Inselstrasse 16, D-04109 Leipzig, Germany*

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We develop a relativistic perturbation theory for scalar clouds around rotating black holes. We first introduce a relativistic product and corresponding orthogonality relation between modes, extending a recent result for gravitational perturbations. We then derive the analog of time-dependent perturbation theory in quantum mechanics, and apply it to calculate self-gravitational frequency shifts. This approach supersedes the nonrelativistic "gravitational atom" approximation, brings close agreement with numerical relativity, and has practical applications for gravitational-wave astronomy.

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Introduction.—Fundamental bosonic fields are ubiquitous in extensions of general relativity and the standard model. In black hole (BH) spacetimes, perturbations by massless bosonic fields are well known to be described by a series of damped sinusoids called quasinormal modes (QNMs) [1,2]. Unlike normal modes, which exist for conservative systems and have purely real spectrum, QNMs appear in dissipative systems and have complex frequencies $\omega = \omega_R + i\omega_I$, with the imaginary part setting their decay time. For BHs, dissipation arises due to radiation of the field through the horizon and away to infinity.

Massive fields around BHs admit an additional class of solutions known as quasibound states (QBSs). Whereas QNMs are radiative solutions, with frequency $|\omega| > \mu$, where μ is the field mass, QBSs are spatially confined by the Yukawa suppression and have $|\omega| < \mu$. Thus, QBSs do not radiate at infinity, although they still dissipate through the horizon. For spinning BHs, these modes can also undergo superradiant amplification, leading to the well-known superradiant instability (see, e.g., [3]). For astrophysical BHs, this process is efficient for $\mu \approx 10^{-20} - 10^{-10}$ eV, leading to the formation of a macroscopic boson cloud and the spin-down of the BH. This phenomenon translates into potentially observable signatures, such as gaps in the BH spin-mass (Regge) plane, gravitational-wave emission from the condensate (when the bosonic field is real), or signatures in binary systems [4–18]. Superradiant instabilities, therefore, represent a powerful probe of ultralight bosons beyond the standard model, such as axions or dark photons.

Given these (and other) prospects for deviations from linear mode evolution, there is considerable interest in calculating nonlinear perturbative effects involving QBSs or QNMs [5,19–22]. However, due to the non-Hermiticity of the system, the spectral theorem does not guarantee the orthogonality or completeness of these modes—which moreover often diverge at the BH horizon or infinity—so it is not clear *a priori* how to incorporate them into a perturbative framework.

For QBSs, the problem can be simplified using the "gravitational atom" or "hydrogenic" approximation. Indeed, at leading order in the gravitational coupling $\alpha = \mu M$, where M is the BH mass, and beyond the field's Compton length, $r \gg \mu^{-1}$, QBSs reduce to eigenfunctions of the hydrogen atom Hamiltonian. In this limit, the ingoing condition at the BH horizon is replaced by a regularity condition at the origin [12,23–25]. Thus, a "hydrogenic" inner product $(\cdot,\cdot)_{\mathrm{hyd}}$ can be defined, in analogy to quantum mechanics, and mode orthogonality is guaranteed by the spectral theorem in the absence of dissipative boundaries. (The same is not true for QNMs, which still radiate to infinity).

The hydrogenic approximation (and its relativistic corrections [12,26]) has been widely used to compute various perturbative corrections to the linear problem [12,18,19,27–29]. For instance, to leading order, a potential term δV arising from, e.g., a binary companion, or a quartic self-interaction, gives rise to level mixing through the matrix element $(\Psi_{n\ell m}, \delta V \Psi_{n'\ell' m'})_{\mathrm{hyd}}$ [12]. The self-gravity of the state also gives rise to a shift in the mode frequency, proportional to the matrix element $(\Psi_{n\ell m}, \delta V \Psi_{n\ell m})_{\mathrm{hyd}}$ [4,27]. However, this approximation has two drawbacks: it breaks down for higher values of α, and it does not take into account the dissipative nature of the problem. To accurately model the phenomenology of massive fields around black holes, we require a *relativistic* perturbative framework, based on an appropriate notion of orthogonality between the modes.

In this Letter, we introduce a bilinear form for massive scalar fields in Kerr to take the place of the hydrogenic inner product in fully relativistic calculations. Under this bilinear form, which is a natural extension of the gravitational bilinear form of Ref. [30], Kerr QNMs and QBSs are truly orthogonal—for all values of α. The product reduces to the hydrogenic inner product in the limit $\alpha \to 0$, but it is also applicable in the relativistic regime, and forms the basis for a relativistic perturbation theory in terms of modes.

Using the relativistic product, we derive the analog of time-dependent perturbation theory in quantum mechanics for the scalar field. As an application, we calculate the leading relativistic frequency shift due to the self-gravity of a superradiant mode, and we find a significant improvement over the hydrogenic approximation when comparing to previously published numerical-relativity results [9], improving the agreement by a factor of 2.5 even at $\alpha = 0.4$. Our product therefore opens a new path to accurate nonlinear mode calculations.

We use $G_{\mathrm{N}} = c = \hbar = 1$ units throughout.

Bilinear form for massive scalars.—We first extend the bilinear form of [30] to scalar *massive* perturbations of Kerr and prove the orthogonality of scalar modes with both quasinormal and quasibound asymptotic conditions.

The Kerr line element for a black hole of mass M and spin parameter a is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{\Lambda}{\Sigma}\sin^2\theta\mathrm{d}\phi^2, \tag{1}$$

in Boyer-Lindquist coordinates, where $\Delta = r^2 + a^2 - 2Mr$, $\Sigma = r^2 + a^2\cos^2\theta$, $\Lambda = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$. We denote the event horizon (the greater root $r_\pm$ of Δ) by r_+ and define the tortoise coordinate, $\mathrm{d}r/\mathrm{d}r_* = \Delta/(r^2 + a^2)$.

The Lagrangian density for the complex Klein-Gordon equation on a Kerr background (which coincides with the Teukolsky equation $\mathcal{O}\Phi = 0$ for a spin $s = 0$ complex massive field [31]) reads

$$L = -\sqrt{-g}(g^{ab}\nabla_a\Phi^*\nabla_b\Phi + \mu^2\Phi^*\Phi), \tag{2}$$

where μ is the mass. A product between two solutions of the Klein-Gordon equation can be built as follows. Start from a "base" product (related to the symplectic form),

$$\Pi_\Sigma[\Phi_1, \Phi_2] = \int_\Sigma (\Phi_1\nabla_a\Phi_2 - \Phi_2\nabla_a\Phi_1)n^a\mathrm{d}V, \tag{3}$$

where Σ is a time slice with unit normal n^a. One can easily verify that, if Φ_1, Φ_2 are solutions, the product is conserved (i.e., independent of Σ) and that it is $\mathbb{C}$ linear in both entries, or bilinear.

Reference [30] showed that one can build, from this base product, an infinite number of conserved quantities by inserting symmetry operators of the equation of motion. In Kerr, one can make use of the symmetry operators associated with the time-translation and ϕ rotation isometries, $\mathcal{L}_t$ and $\mathcal{L}_\phi$, as well as with the Killing tensor of the spacetime. One can also use the symmetry operator associated with the t–ϕ spacetime symmetry, $\mathcal{J}$, whose action on a scalar field simply takes $t \to -t$ and $\phi \to -\phi$. Note that the Teukolsky operator and the t–ϕ reflection operator commute on $s = 0$ Weyl scalars, $\mathcal{O}\mathcal{J} = \mathcal{J}\mathcal{O}$.

The product relevant for the orthogonality relation can be built from the t–ϕ reflection operator [30]. For scalar massive (or massless) perturbations *with compact support* it is given by

$$\langle\!\langle \Phi_1, \Phi_2 \rangle\!\rangle = \Pi_\Sigma[\mathcal{J}\Phi_1, \Phi_2]. \tag{4}$$

In Boyer-Lindquist coordinates, the bilinear form reads

$$\begin{aligned}\langle\!\langle \Phi_1, \Phi_2 \rangle\!\rangle = \int_{r_+}^{\infty} \mathrm{d}r \int_{S^2} \mathrm{d}^2\Omega \Bigg[&\frac{2Mra}{\Delta}(\mathcal{J}\Phi_1\partial_\phi\Phi_2 - \Phi_2\partial_\phi\mathcal{J}\Phi_1) \\ &+ \frac{\Sigma}{\Delta}\left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right) \\ &\times (\mathcal{J}\Phi_1\partial_t\Phi_2 - \Phi_2\partial_t\mathcal{J}\Phi_1)\Bigg],\end{aligned} \tag{5}$$

where $\mathrm{d}^2\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$. In addition to being bilinear and conserved, one can easily prove, in analogy to Ref. [30], that (1) the bilinear form is symmetric, $\langle\!\langle \Phi_1, \Phi_2 \rangle\!\rangle = \langle\!\langle \Phi_2, \Phi_1 \rangle\!\rangle$; and (2) the time-translation symmetry operator is symmetric with respect to the bilinear form, $\langle\!\langle \mathcal{L}_t\Phi_1, \Phi_2 \rangle\!\rangle = \langle\!\langle \Phi_1, \mathcal{L}_t\Phi_2 \rangle\!\rangle$.

Extension to mode solutions.—Quasinormal and quasibound states are mode solutions of the Teukolsky equation, $\Phi_{\ell m\omega} = e^{-i\omega t + im\phi}R_{\ell m\omega}(r)S_{\ell m\omega}(\theta)$, where $S_{\ell m\omega}$ are the $s = 0$ spin-weighted spheroidal harmonics with angular numbers ℓ, m [31] and the radial solution can be defined in terms of an asymptotic series involving a three-term recursion relation [32,33]. The modes are required to be regular at the horizon, $\Phi \sim e^{-ik_H r_*}$ as $r_* \to -\infty$, where $k_H = \omega - m\Omega_H$ and Ω_H is the angular frequency of the outer horizon $\Omega_H = a/(2Mr_+)$. At infinity, the two families satisfy

$$\Phi \sim r^{-1}e^{ikr_*}, \qquad r_* \to \infty \text{ (QNMs)}, \tag{6}$$

$$\Phi \sim r^{-1}e^{-ikr_*}, \qquad r_* \to \infty \text{ (QBSs)}, \tag{7}$$

where $k = \sqrt{\omega^2 - \mu^2}$.

Because the radial solutions have noncompact support, and for $\omega_{\mathrm{I}} < 0$ actually diverge as $r_* \to -\infty$ (QNMs and QBSs) and as $r_* \to +\infty$ (QNMs), we must find a suitable, finite extension of the bilinear form (4). In analogy with Ref. [30], we extend the definition of the bilinear form to a

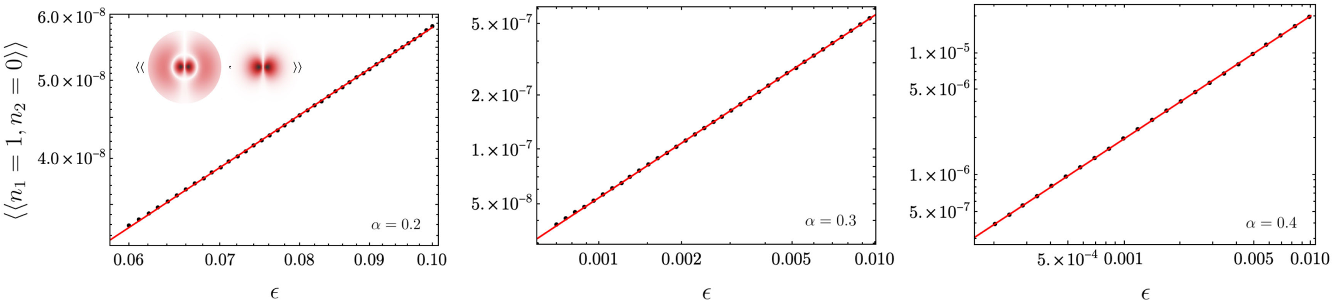

FIG. 1. The relativistic product between two $\ell = m = 1$ QBSs in Schwarzschild, as a function of the counterterm regularization point $\epsilon = \bar{r}/r_+ - 1$, for different scalar field masses. The red curve is a power-law fit, showing convergence to zero. In the top-left corner, we show the absolute value of the radial mode functions around the BH. Modes are normalized to have $\langle\langle n, n\rangle\rangle = 1$ in the regularization limit.

complex radial integration contour $\mathcal{C}$, such that the radial integral is absolutely convergent. We define the bilinear form over a pair of QNMs or QBSs with complex frequencies ω_1, ω_2 by integrating over a complex r_* contour such that

$$\arg r_* + \arg(\omega_1 + \omega_2) = -\pi/2, \quad r_* \to -\infty, \tag{8}$$

and running along the real axis elsewhere. If the product is over one or two QNMs, we also take

$$\arg r_* + \arg(\pm k_1 \pm k_2) = \pi/2, \quad r_* \to \infty, \tag{9}$$

where the plus (minus) sign holds for QNMs (QBSs).

Explicitly, the bilinear form on modes reads

$$\langle\langle \Phi_1, \Phi_2\rangle\rangle_{\text{modes}} = i\delta_{m_1 m_2} e^{-i(\omega_1-\omega_2)t} \int_{\mathcal{C}} \mathrm{d}r \frac{K}{\Delta} R_1 R_2, \tag{10}$$

where

$$K(r) = \alpha_{12}(r^2 + a^2)^2(\omega_2 + \omega_1) - 2Mra\alpha_{12}(m_1 + m_2) - \gamma_{12}(\omega_2 + \omega_1)a^2\Delta(r), \tag{11}$$

$$\alpha_{12} = 2\pi \int_0^\pi \mathrm{d}\theta \, \sin\theta S_1(\theta) S_2(\theta), \tag{12}$$

$$\gamma_{12} = 2\pi \int_0^\pi \mathrm{d}\theta \, \sin^3\theta S_1(\theta) S_2(\theta). \tag{13}$$

Note that, as demonstrated for Kerr QNMs in Ref. [30], this product can be used to project initial data onto modes, resulting in the known mode excitation coefficients [34–36]. In the hydrogenic limit, this reduces to the familiar inner product on the (real) hydrogenic mode functions,

$$\langle\langle \Phi_1, \Phi_2\rangle\rangle \to \delta_{m_1 m_2} \int_0^\infty \mathrm{d}r \, r^2 R_1 R_2(r) \int_0^\pi \mathrm{d}\theta \, \sin\theta S_1 S_2(\theta) \equiv (\Phi_1, \Phi_2)_{\text{hyd}}, \tag{14}$$

up to an overall factor. In this limit, no regularization is required.

For QBSs in Schwarzschild, it is convenient to adopt an alternative regularization involving counterterm subtraction [37]. This is particularly useful when mode solutions are only known numerically and thus cannot be easily continued into the complex r_* plane. For Schwarzschild, the integrals over r and θ factorize, and the latter gives rise to the usual orthogonality relation for spherical harmonics. The radial integration can then be regularized by subtracting suitable counterterms,

$$\langle\langle \Phi_1, \Phi_2\rangle\rangle_{\text{Schwarzschild QBS}} = i\delta_{m_1 m_2}\delta_{l_1 l_2}(\omega_1 + \omega_2) \lim_{\bar{r}_* \to -\infty} \left[\int_{\bar{r}_*}^\infty dr_* \, X_1(r'_*) X_2(r'_*) + \frac{i}{\omega_1 + \omega_2} X_1(\bar{r}_*) X_2(\bar{r}_*) + \mathcal{O}(r_*^{-1}) \right], \tag{15}$$

where $X(r) = rR(r)$. For long-lived states ($M|\omega_I| \ll 1$), only the leading counterterm is needed to make the product finite. Further details of the counterterm subtraction method, including a discussion of higher-order counterterms, are provided in the Supplemental Material [38]. Note that this method works for QBSs, since regularizing the QNM divergence at infinity would require an infinite series of subtractions.

Mode orthogonality.—With the finite bilinear form in hand, from 2 we obtain

$$(\omega_1 - \omega_2)\langle\langle \Phi_1, \Phi_2\rangle\rangle = 0 \tag{16}$$

for a pair of QNMs or QBSs with frequencies ω_1, ω_2. Then, either $\langle\langle \Phi_1, \Phi_2\rangle\rangle = 0$ or $\omega_1 = \omega_2$, proving that QNMs and

QBSs are orthogonal. In particular, modes of the two families are also mutually orthogonal.

We now numerically compute the product (15) between two QBSs in Schwarzschild with different radial numbers n [39]. We do so in the hydrogenic ($\alpha = M\mu \ll 1$) and relativistic ($\alpha \simeq 1$) regimes. To compute the quasibound frequencies and radial solutions, we use the Leaver continued fraction method [32]. We perform product integrals (15) numerically using *Mathematica*.

Figure 1 shows the product between the $\ell = m = 1$ fundamental mode and the first overtone as a function of the integral regulator $\epsilon = \bar{r}/r_+ - 1$. Different panels span the hydrogenic regime and the relativistic regime. The product between the two modes goes to zero as a power law as $\epsilon \to 0$ in all cases, confirming numerically the orthogonality to a precision of order 10^{-7}. For higher values of α, we are able to probe the integral for smaller r due to better convergence resulting from milder divergences at the horizon. We obtain similar results also for higher radial overtones.

Relativistic perturbation theory.—We now describe our relativistic approach to compute transitions between modes due to a perturbation. To emphasize the similarity to ordinary Schrödinger perturbation theory in quantum mechanics, we work in the Hamiltonian formulation, writing the metric in Arnowitt-Deser-Misner form $g^{ab} = -N^{-2}(t^a - N^a)(t^b - N^b) + h^{ab}$ (see Appendix E of [40]), assumed to be some perturbation of Kerr. Starting from the Lagrangian (2) we introduce the momentum $\Pi = N^{-1}\sqrt{h}(t^a - N^a)\nabla_a\Phi$ and the Hamiltonian, leading to equations in first order form,

$$\begin{pmatrix} \dot{\Phi} \\ \dot{\Pi} \end{pmatrix} \equiv \mathcal{L}_t \begin{pmatrix} \Phi \\ \Pi \end{pmatrix} = H \begin{pmatrix} \Phi \\ \Pi \end{pmatrix}, \tag{17}$$

where

$$H \equiv \begin{pmatrix} N^a D_a & N/\sqrt{h} \\ \sqrt{h}(D^a N D_a - N\mu^2) & D_a N^a \end{pmatrix}. \tag{18}$$

In phase-space notation, the relativistic product takes the form $\langle\!\langle (\Phi_1, \Pi_1)^{\mathrm{T}}, (\Phi_2, \Pi_2)^{\mathrm{T}} \rangle\!\rangle = \int_C (\Phi_1 \circ \mathcal{J}\Pi_2 + \Pi_1 \circ \mathcal{J}\Phi_2) \mathrm{d}^3x$.

For a general perturbation, H is time-dependent. We make an ansatz for the column vector $F = (\Phi, \Pi)^{\mathrm{T}}$ associated with a solution in terms of a superposition of modes with time-dependent amplitudes [41],

$$F(t) = \sum_q c_q(t) F_{0q}(t), \tag{19}$$

where F_{0q} are the quasibound and quasinormal modes of the unperturbed problem with Hamiltonian H_0, i.e., $H_0 F_{0q} \equiv \mathcal{L}_t F_{0q} = -i\omega_q F_{0q}$, so that $F_{0q}(t)$ has harmonic $e^{-i\omega_q t}$ time dependence.

We decompose the Hamiltonian as $H = H_0 + \delta H$, where the subscript 0 denotes quantities associated with the Klein-Gordon equation in the Kerr metric. The scheme rests on the facts that $\langle\!\langle F_{0q}, F_{0q'} \rangle\!\rangle = \delta_{qq'}$ and that H_0 is symmetric [42] relative to our relativistic product on two-vector states F_i. A standard calculation mirroring quantum mechanics then gives the perturbation series for the time-dependent excitation coefficients,

$$\dot{c}_n \langle\!\langle \Phi_n, \Phi_n \rangle\!\rangle = \sum_q c_q \langle\!\langle F_{0n}, \delta H(t) F_{0q} \rangle\!\rangle. \tag{20}$$

If δH is approximately t-independent, we have an (approximate) perturbed quasinormal or quasibound mode $F = F_0 + \delta F$, defined by the "eigenvalue equation" $HF = -i(\omega_0 + \delta\omega)F$ and appropriate boundary conditions, with a frequency shift $\delta\omega$. Taking an inner product $\langle\!\langle F_0, \cdot \rangle\!\rangle$ with the unperturbed QNM or QBS and going through exactly the same steps as in ordinary time-independent quantum mechanics perturbation theory immediately yields the usual formula,

$$-i\delta\omega = \frac{\langle\!\langle F_0, \delta H F_0 \rangle\!\rangle}{\langle\!\langle F_0, F_0 \rangle\!\rangle}, \tag{21}$$

at first perturbation order. Substituting this back into the eigenvalue equation and taking an inner product $\langle\!\langle F_{0q}, \cdot \rangle\!\rangle$ with all unperturbed QNM or QBS F_{0q} orthogonal to F_0 then gives

$$\delta F_0 = \sum_q \frac{\langle\!\langle F_{0q}, \delta H F_0 \rangle\!\rangle}{-i\langle\!\langle F_{0q}, F_{0q} \rangle\!\rangle(\omega_0 - \omega_{0q})} F_{0q}, \tag{22}$$

using (20) at first order. In the Supplemental Material [38], we also derive the perturbation equations for the excitation coefficients in the second-order formalism.

Frequency shift due to self-gravity.—We apply our relativistic perturbative framework to calculate the frequency shift $\delta\omega_n$ of an (unstable) mode Φ_n close to the superradiant bound in Kerr due to its self-gravity. We assume that the squared amplitude A^2 of the mode and the rotation parameter a/M are both relatively small and neglect effects that are not linear in these quantities. We show in the Supplemental Material [38] that, under these assumptions, the perturbed metric $\delta g^{ab} = g^{ab} - g_0^{ab}$ can be written in the form $\delta g^{ab} \approx -\delta[N^{-2}(t^a - N^a)(t^b - N^b)]$ where $\delta N^a \approx N_0^a \delta N/N_0$. Following [19], we therefore take a semi-Newtonian, approximation for the gravitational potential sourced by a mode. With this, the perturbed Hamiltonian of the scalar field is $\delta H \approx \delta V H_0$, where $\delta V = \delta N/N_0$ is given approximately by

$$\delta V(r) \approx -\mu^2 \left[\frac{1}{r} \int_{r_+}^{r} \mathrm{d}^3 r' |\Phi_n|^2 + \int_r^\infty \mathrm{d}^3 r' \frac{|\Phi_n|^2}{r'} \right], \tag{23}$$

where the integration is carried out over flat space and we have taken the leading order (spherical) multipole.

To estimate the correction $\delta\omega_n$ to the mode frequency ω_n in Kerr we use (21), with $\delta H \approx \delta V H_0$ and $H_0(\Phi_n, \Pi_n)^{\mathrm{T}} = -i\omega_n(\Phi_n, \Pi_n)^{\mathrm{T}}$. Reverting to 1-component form, we get

$$\frac{\delta\omega_n}{\omega_n} \approx \frac{\langle\!\langle \Phi_n, \delta V \Phi_n \rangle\!\rangle}{\langle\!\langle \Phi_n, \Phi_n \rangle\!\rangle}. \tag{24}$$

This approach is similar in spirit to that outlined in Refs. [22,43–45]. In the nonrelativistic limit, this formula reduces to that found in Refs. [19,27]. Note that superradiantly unstable QBSs, which have $\omega_I > 0$ and decay at infinity, have no divergence at the horizon and therefore require no regularization of the product.

For completeness, we also write the equation for the time evolution of the excitation coefficients, better suited to when the perturbation $\delta H \approx \delta V H_0$ is time-dependent,

$$\dot{c}_n \langle\!\langle \Phi_n, \Phi_n \rangle\!\rangle = -i \sum_q \omega_q \langle\!\langle \Phi_n, \delta V \Phi_q \rangle\!\rangle. \tag{25}$$

We now calculate numerically the frequency shift (24) for superradiant modes with $\ell = m = 1$. For a given coupling α, we set the BH spin to be close to the superradiant bound $m\Omega_H \gtrsim \omega_R$, the same setup as [19]. For this application, we use the Black Hole Perturbation Toolkit to compute the modes' spin-weighted spheroidal harmonics [46].

In Fig. 2, we compare for several α our perturbative calculation of $\delta\omega/M_{\mathrm{cloud}}$ against the numerical-relativity estimate of $\partial\omega/\partial M_{\mathrm{cloud}}$ from [19]. We find excellent agreement, including significant improvement over the hydrogenic approximation, which begins to fail around $\alpha \simeq 0.3$. For $\alpha = 0.4$, the error is reduced from 28% to 11%. The remaining disagreement is likely due to the approximation

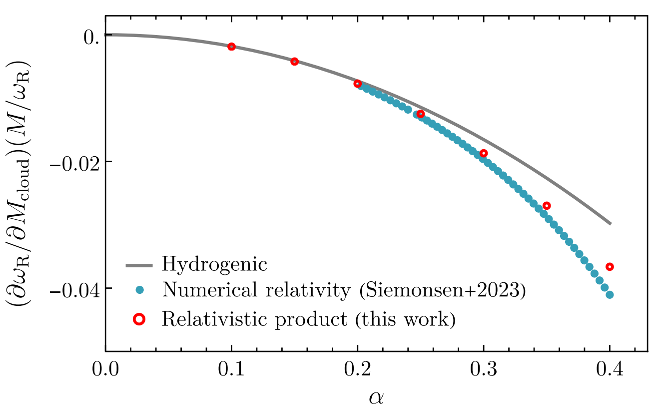

FIG. 2. Frequency shift due to the self-gravity of a superradiant mode in Kerr ($\ell = m = 1$, $n = 0$). We compare our result based on the relativistic product with the hydrogenic approximation, and with the fully relativistic (numerical) frequency shift from Ref. [19]. For the analytic results, we plot $\delta\omega/M_{\mathrm{cloud}}$, which should be a good approximation of the derivative for small cloud masses.

that $\delta\omega$ is linear in the cloud mass ($\partial\omega/\partial M_{\mathrm{cloud}} \simeq \delta\omega/M_{\mathrm{cloud}}$), to our semi-Newtonian approximation for the perturbed equation, and to the monopolar approximation of the Newtonian potential.

In the Supplemental Material [38], we include another example, calculating relativistic matrix elements of tidal perturbers [12], and again find $O(10\%)$ corrections to the hydrogenic approximation. This example is relevant for gravitational-wave signals from extreme or intermediate mass-ratio binaries (see also [22]).

Conclusions.—In this Letter, we introduced a bilinear form for massive scalar-field perturbations of Kerr and showed that modes are orthogonal with respect to this product. Our bilinear form replaces the standard quantum mechanics inner product—often employed in a hydrogenic approximation—making no assumption on the strength of the gravitational coupling α. We also introduced an approach to compute perturbative corrections to mode evolution due to a perturbation, and applied this to recover frequency shifts due to the self-gravity of a superradiant state. For large values of α, accurate results were previously only obtainable using numerical relativity.

Our bilinear form and perturbative framework have both conceptual and practical importance. Other applications could be to compute corrections due to self-interaction terms such as quartic potentials [27], or in the sine-Klein-Gordon equation for the QCD axion [47]. In future work, we also hope to explore transitions between quasinormal and quasibound modes, and to rigorously derive angular selection rules for massive perturbations in Kerr using the bilinear form.

Another natural extension would be to generalize our product to massive spin-1 fields. This scenario presents a number of difficulties as the Proca equation is not separable using the standard Teukolsky formalism. Nevertheless, an ansatz yielding separability of the Proca equation in Kerr spacetime was recently discovered [48,49], and could allow for a generalization of the bilinear form.

Finally, in the context of BH binaries, the gravitational product [30] could be used with the second-order Teukolsky equation [50,51] to estimate nonlinear corrections to the BH ringdown. This could be used to inform waveform development and address recent questions on nonlinear effects during the ringdown [20,21,52–54]. We hope to report on these interesting problems in the future.

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*enrico.cannizzaro@uniroma1.it
†laura.sberna@aei.mpg.de
‡stephen.green2@nottingham.ac.uk
§stefan.hollands@uni-leipzig.de

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[39] The product between states with different ℓ, m numbers is trivial, as their orthogonality in Schwarzschild follows from the orthogonality of the spherical harmonics. The orthogonality between different ℓ modes becomes non-trivial in Kerr [30] due to the spin-weighted spheroidal harmonics.

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[42] The symmetry $\langle\langle F_1, H_0 F_2\rangle\rangle = \langle\langle H_0 F_1, F_2\rangle\rangle$ can be shown by integration by parts, and in contrast to 2 does not require F_i to be solutions.

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