AdS_3 Pure Gravity and Stringy Unitarity

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We construct a unitary, modular-invariant torus partition function of a two-dimensional conformal field theory with a Virasoro primary spectral gap of $\Delta_* = [(c-1)/12]$ above the vacuum. The twist gap is identical, apart from two states $\mathcal{O}_*$ with spin scaling linearly in the central charge c. These states admit an AdS_3 interpretation as strongly coupled strings. All other states are black hole microstates.

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The quest for AdS_3 pure gravity still beckons. It is not fully known whether, or in what precise sense, such a consistent theory exists, either quantum mechanically or in the semiclassical limit. The latter is of particular physical interest, due to the existence of black holes and the emergence of spacetime.

Holographically speaking, the outstanding spectral problem is to find a torus partition function of a two-dimensional conformal field theory (CFT) that is mutually compatible with unitarity (a non-negative Virasoro primary spectral density) and modularity [exact $SL(2,\mathbb{Z})$-invariance of the partition function], while preserving the spectral gaps of a dual bulk theory with only black holes above a normalizable AdS_3 ground state. No known partition function satisfies these basic requirements.

There exists a diverse set of approaches to this problem which, famous as it is, we describe in condensed fashion. Summing over all smooth on-shell 3-manifolds $\mathcal{M}$ with $\partial\mathcal{M} = T^2$ [1], namely the $SL(2,\mathbb{Z})$ family of Bañados-Teitelboim-Zanelli (BTZ) black holes, generates a negative density of states in two regimes [1–3]: at large spin $j \to \infty$ near extremality,

$$\int_0^{t_0} dt\, \rho_{\mathrm{MWK},j}(t) \sim (-1)^j e^{\pi\sqrt{\xi j}}, \qquad t_0 \sim e^{-2\pi\sqrt{\xi j}}, \tag{1}$$

where

$$t := \min(h,\bar{h}) - \xi, \qquad j = h - \bar{h}, \qquad \xi := \frac{c-1}{24}, \tag{2}$$

and at the scalar black hole threshold,

$$\rho_{\mathrm{MWK},0}(t) = -6\delta(t) + (t > 0\,\text{continuum}). \tag{3}$$

The property (1) is especially severe: an exponentially large negative density despite an exponentially small window. From the bulk perspective, seeking a consistent pure gravity path integral requires reckoning with the sum over topologies; for related work, see [4–11]. (We note here some recent work in AdS_3/CFT_2 that studies fixed bulk topologies [12–17].)

Some valuable progress has been made. Explicit restoration of unitarity may be achieved by retreating from pure gravity [18,19], adding heavy point-particle matter which admits a geometric bulk interpretation. The construction of [20], which preserves the pure gravity spectrum, uses dimensional reduction to Jackiw-Teitelboim (JT) gravity to fix (1) with an infinite sum over off-shell Seifert manifolds, though it remains a mostly [21] implicit construction away from extremality and leaves (3) intact. Other approaches that forego a subset of the above conditions include [22–24].

Partition function.—The Virasoro primary partition function is defined as

$$Z_p(\tau) = \sqrt{y}|\eta(\tau)|^2 Z(\tau), \tag{4}$$

where $Z(\tau) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)})$ is the torus partition function (nonholomorphic) and $\tau := x + iy$. The following modular-invariant $Z_p(\tau)$ is unitary at sufficiently large ξ:

$$\mathcal{Z}(\tau) = Z_{\mathrm{MWK}}(\tau) + Z_{\mathrm{string}}(\tau), \tag{5}$$

where

$$Z_{\mathrm{MWK}}(\tau) := \sum_{\gamma\in SL(2,\mathbb{Z})/\Gamma_\infty} \sqrt{\mathrm{Im}(\gamma\tau)}\,|q_\gamma^{-\xi}(1-q_\gamma)|^2,$$

$$Z_{\mathrm{string}}(\tau) := \sum_{\gamma\in SL(2,\mathbb{Z})/\Gamma_\infty} \sqrt{\mathrm{Im}(\gamma\tau)}\left(2q_\gamma^{\xi/4}\bar{q}_\gamma^{-\xi/4} + \mathrm{c.c.}\right), \tag{6}$$

with $q_\gamma := e^{2\pi i\gamma\tau}$. These are Poincaré sums over $SL(2,\mathbb{Z})$ modulo Γ_∞, the set of modular T transformations [25]. As we substantiate below, the unitary range of ξ includes $\xi \gg 1$, and provisionally appears to hold for all $\xi \in 2\mathbb{Z}_+$. The reason for the "string" moniker will be explained momentarily.

From a CFT point of view, $Z_{\text{string}}(\tau)$ is a Poincaré sum over two copies of a Virasoro primary seed state $\mathcal{O}_*$ with quantum numbers

$$(\Delta_*, j_*) = \left(2\xi, \frac{\xi}{2}\right) \Leftrightarrow (t_*, \bar{t}_*) = \left(-\frac{\xi}{4}, \frac{\xi}{4}\right), \tag{7}$$

and its parity image with $h_* \leftrightarrow \bar{h}_*$. We have employed the "reduced twist" variable t along with its partner $\bar{t} \coloneqq \max(h, \bar{h}) - \xi$. We have chosen the state in (6) to be doubly degenerate, a natural choice that preserves integrality, but $\mathcal{Z}(\tau)$ is unitary for a finite range of degeneracies $d_* > 1$ (see Supplemental Material [26], Appendix B).

Let us state the spectral properties of the partition function $\mathcal{Z}(\tau)$, deferring its unitarity to the next subsection. The spectrum is shown in Fig. 1. The gap in conformal dimension above the vacuum is exactly

$$\Delta_* = \frac{c-1}{12}, \tag{8}$$

with no corrections. This is the value anticipated by the Virasoro modular bootstrap program (e.g., [29–33]) as the optimal gap at *large* c, on the basis of black hole universality: the conformal dimension (8) corresponds to the massless limit of the semiclassical BTZ black hole. The state-of-the-art bootstrap upper bound on the spectral gap at large c is the numerical result [32]

$$\Delta_* \lesssim \frac{c}{9.08} \qquad (c \gg 1), \tag{9}$$

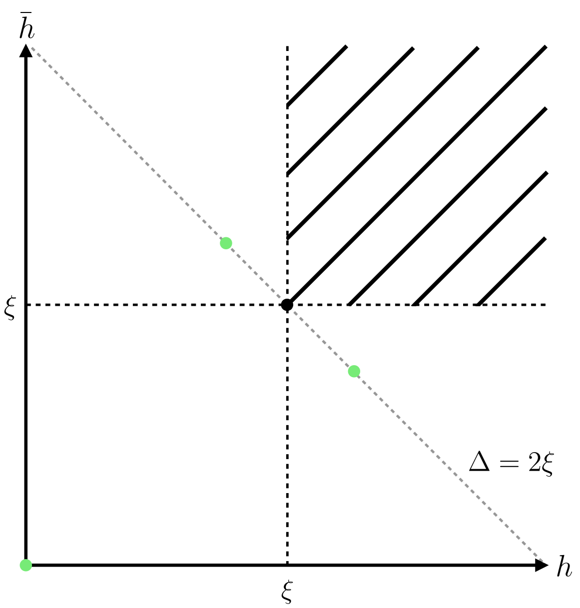

FIG. 1. The Virasoro primary spectrum of $\mathcal{Z}(\tau)$. Green dots denote the vacuum state and two (parity-invariant) states with $(h_*, \bar{h}_*) = (\frac{5}{4}\xi, \frac{3}{4}\xi)$, interpretable in AdS_3 as strongly coupled strings. All other states exceed the semiclassical black hole threshold: $\min(h, \bar{h}) \geq \xi$. The density of states is positive.

with a slightly weaker analytical bound [33]. (See [34–54] for further bootstrap work on Virasoro spectra at large c.) The explicit realization by $\mathcal{Z}(\tau)$ of the gap (8) while preserving unitarity at $\xi \gg 1$ (the first such example, to our knowledge) is also noteworthy because of the paucity of pure CFT arguments that a gap this large is possible. Conversely, $\mathcal{Z}(\tau)$ shows constructively that without incorporating discreteness into the modular bootstrap [52], the optimal bound on the gap cannot be lower than $\Delta_* = 2\xi$. This statement applies for all values of ξ for which $\mathcal{Z}(\tau)$ is unitary.

As for the twist spectrum, all Virasoro primaries besides $\mathbb{1}$ and $\mathcal{O}_*$ have $t \geq 0$. There is a positive integer number of scalar states at $t = 0$ [see (13)]. The spectrum of $t > 0$ states is continuous. This can be understood rather generally in terms of coarse graining. At large ξ, this can be thought of as a consequence of ignorance of exponentially small effects in c—for example, smearing over the mean level spacing $\sim e^{-S_{\text{Cardy},j}(t)}$. We explain these points of interpretation in the section below on "Summary and Random (matrix) comments".

Density of states: The corresponding Virasoro primary density of states, related to our partition function as

$$\frac{\mathcal{Z}(\tau)}{\sqrt{y}} = \sum_{j=0}^{\infty} (2 - \delta_{j,0}) \cos(2\pi j x) \int_{\mathbb{R}} d\Delta\, e^{-2\pi y(\Delta - 2\xi)} \rho_j(\Delta) \tag{10}$$

can be derived straightforwardly using existing methods for Poincaré sums. We have, in terms of reduced twist t,

$$\rho_j(t) = \rho_{\text{MWK},j}(t) + \rho_{\text{string},j}(t) \tag{11}$$

for every spin j. The Maloney-Witten-Keller (MWK) density $\rho_{\text{MWK},j}(t)$ is recalled in Supplemental Material [26], Appendix A. The new term is, for $j \neq 0$,

$$\rho_{\text{string},j}(t) = \frac{4}{\sqrt{t\bar{t}}} \sum_{s=1}^{\infty} f_{j,j_*;s} \cos\left(\frac{2\pi}{s}\sqrt{\xi\bar{t}}\right) \cosh\left(\frac{2\pi}{s}\sqrt{\xi t}\right) + (j_* \to -j_*, t \leftrightarrow \bar{t}), \tag{12}$$

where $f_{j,j_*;s} \coloneqq S(j, j_*; s)/s$ with $S(j, j_*; s)$ a Kloosterman sum. For $j = 0$, such sums must be regularized; using standard methods nicely summarized in [19], the result is the $j = 0$ specialization of the $j \neq 0$ densities, augmented by a constant subtraction; see Supplemental Material [26], Appendix A.

There are two hurdles to establishing positivity: one must cancel the negativity of the MWK partition function in the $j \to \infty$ regime, and at the scalar threshold $t = 0$, both without introducing new negativity.

At $j \to \infty$, the negativity (1) is resolved by construction: we have added states with reduced twist $t_* = -\xi/4$, designed precisely to avoid the large-spin negativity in accordance with the arguments of [3] and the subsequent approach of [18,19]. (We added two such states, but any number $d_* > 1$ would do; we review this in Supplemental Material [26], Appendix B.) The states $\mathcal{O}_*$ have asymptotically large spin as $\xi \to \infty$. It is exactly this property which admits the novelty of a spectral gap $\Delta_* = 2\xi$ without introducing further negativity elsewhere in the spectrum—and indeed, as we now show, curing the scalar negativity (3) in the process.

The scalar density of states is

$$\rho_0(t) = \delta(t+\xi) + [-6 + 8\sigma_0(j_*)]\delta(t) + \tilde{\rho}_0(t). \quad (13)$$

The first term is the vacuum state. The second, formerly problematic, term has been rendered strictly positive, for any $j_* = \xi/2$. Happily, it is also an integer, a welcome surprise. Unlike previous approaches to this negativity, its resolution does not require the addition of an "extra" *ad hoc* $+6\delta(t)$ [2], instead coming for free in $\rho_{\mathrm{string},0}(t)$. We note a number-theoretic feature of this degeneracy: if j_* is prime, then $\sigma_0(j_*) = 2$.

The last term, $\tilde{\rho}_0(t)$, is the continuum with support on $t > 0$, given explicitly in Supplemental Material [26], Appendix A. Its positivity requires a more careful analysis because various large-ξ suppression factors are absent when $j = 0$, i.e., $t = \bar{t}$, as can be seen in (12); however, $\tilde{\rho}_0(t)$ is indeed positive for all $t \geq 0$. We provide details in Supplemental Material [26], Appendix B, but can sketch the essential point here. In the regime $\xi t \gg 1$, the scalar MWK density is positive and exponentially larger in magnitude than the string density. As $\xi t \sim \mathcal{O}(1)$, positivity is nontrivial as both densities are of the same order and the string density is term-wise oscillatory in t. With an eye toward semiclassical gravity, we focus on $\xi \gg 1$, taking the regime of $x := 2\pi\sqrt{\xi t}$ fixed. The proof of positivity proceeds in two steps: first, we show that the sum of the $s = 1$ and $s = 2$ terms in the regularized density is positive; and second, we show that the $s \geq 2$ terms are individually positive (see Supplemental Material [26], Appendix B)

Numerical evaluation of the scalar density at large but finite ξ confirms these analytic results, as shown in Fig. 2. Indeed, we see that positivity appears to hold all the way down to $j_* = 1$, i.e., $\xi = 2$, formally the smallest central charge in our construction [55].

A bulk string interpretation.—The above construction is purely on the CFT side. Is there an AdS_3 gravity interpretation of the highly spinning operator $\mathcal{O}_*$ and its modular images?

One appealing answer is that $\mathcal{O}_*$ is a strongly coupled string, and its modular images, stringy contributions to the black hole spectrum. While an operator like $\mathcal{O}_*$ with $t < 0$ and $\bar{t} > 0$ cannot be dual to a smooth BTZ black hole nor to a conical defect (such solutions with real mass and angular momenta do not exist in semiclassical AdS_3 gravity coupled to point particles), spinning strings in AdS_3 can, and indeed do, satisfy this condition.

The spectrum of folded, spinning Nambu-Goto strings coupled to gravity in AdS_3 was studied in [57] in the classical limit. The Virasoro primary string spectrum is parametrized by a string tension λ and an angular velocity ω. The string tension is given in terms of AdS, string and Planck scales as

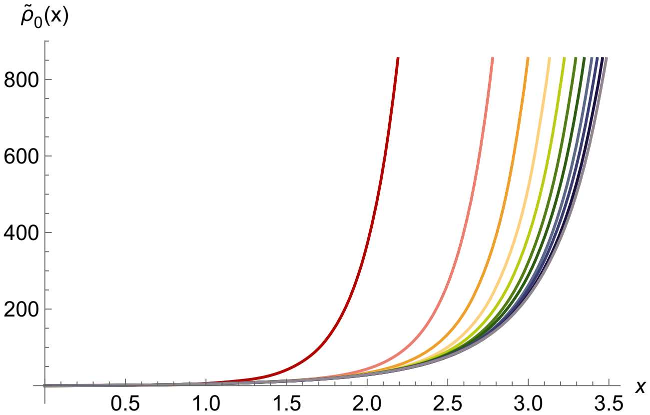

FIG. 2. Plot of the regularized scalar primary density of states $\tilde{\rho}_0(x)$ of the partition function $\mathcal{Z}(\tau)$, as a function of $x = 2\pi\sqrt{\xi t}$, with ξ ranging from $\xi = 2$ (red) to $\xi = 102$ (blue) in steps of 10. The curves are positive for all $x \geq 0$. (Obtained by summing over $s \leq 200$ in the regularized density, see Supplemental Material [26], Appendix A.).

$$\lambda = \frac{1}{2\pi}\frac{L_{\mathrm{AdS}}}{\ell_s}\frac{\ell_p}{\ell_s}, \quad (14)$$

where $\ell_p = 8\pi G_N$.

For a given λ, the string spectrum starts at the origin $t = \bar{t} = -\xi$ and ends at the extremality bound $t = 0$ or $\bar{t} = 0$. Matching the string spectrum to the quantum numbers (7) of the operator $\mathcal{O}_*$ yields the unique result

$$\lambda_* = 1, \qquad \omega_* = 2. \quad (15)$$

This string is strongly coupled: from (14), an AdS-sized string with $\lambda_* = 1$ requires $\ell_p/\ell_s \sim \mathcal{O}(1)$, which is the ratio that defines an effective string coupling $g_s = (\ell_p/\ell_s)^{>0}$ (where the exponent depends on the details of the putative string background [58]).

The specific value $\lambda_* = 1$ happens to enjoy a certain synergy with the equations of [57]. For generic λ and ω, the solutions of [57] are given in terms of elliptic integrals. However, at $\lambda = 1$—and only at $\lambda = 1$—the solution simplifies dramatically, as the string embedding equation becomes algebraic. It is simple enough to recall explicitly in a few lines. The Lorentzian spacetime metric outside the string is locally AdS_3 with the corresponding mass and angular momentum,

$$ds^2 = \frac{1}{16}(-du^2 + dv^2) - \left(z - \frac{1}{256z}\right)du\,dv + \frac{dz^2}{4z^2}, \quad (16)$$

where, in the conventions of [57], the conformal boundary is at $z \to \infty$. The string embedding is determined by functions $u(\sigma,\tau)$, $v(\sigma,\tau)$, and $z(\sigma)$, where (σ,τ) are world sheet coordinates with induced metric

$$h = \Omega^2(\sigma)(-d\tau^2 + d\sigma^2),$$
$$\Omega^2(\sigma) = 3\,\frac{[z_L - z(\sigma)][z(\sigma) - z_R]}{z(\sigma)}, \tag{17}$$

where (15) implies $z_L = (3/16)$, $z_R = -(1/48)$, and

$$z(\sigma) = \frac{(32 - 25\cos^2\sigma + 5\sqrt{25\cos^4\sigma - 64\cos^2\sigma + 64})}{384}. \tag{18}$$

Opposite points on the string are identified, "sewing up" the spacetime [60]. The outermost radius of the string (where it folds) is at $z = z_L$, while the center of the string is at $z = z(0) = 1/12$. The spacetime ends at the string, avoiding a naked singularity.

So we see that the state $\mathcal{O}_*$ admits an interpretation as a highly spinning string coupled to gravity in AdS_3. That it is strongly coupled dovetails nicely with how AdS_3 pure gravity could possibly arise in string theory: strong coupling is necessary to gap the light string modes to the Planck scale.

Black hole microstates: Our construction adds not only the states $\mathcal{O}_*$, but their $SL(2,\mathbb{Z})$ images too. These states are heavy, but are *not* BTZ black holes (fully captured by the MWK sum over smooth Euclidean saddles) nor their orbifolds (which are modular images of conical defect geometries).

Instead, these are new black hole microstates made of strongly coupled strings. The Euclideanized, modular-transformed solutions of [57] are small black strings, in the following specific sense: whereas a BTZ/conical defect solution with the same quantum numbers would be nakedly singular, the strings shroud this region by terminating the spacetime. These geometries may be thought of as quantum AdS_3 versions of the stringy cloak of [61] and other small black strings (e.g., [62,63]). That they are "small"—the modular transforms of a single string, rather than a parametrically large number of them—is also visible thermodynamically in the different functional forms of the stringy and BTZ microcanonical entropies: $\rho_{\text{string},j}(t)$ is oscillatory as a function of t, unlike the BTZ density $\rho_{\text{MWK},j}(t)$, and is exponentially subleading to $\rho_{\text{MWK},j}(t)$, term-by-term in the modular sum, away from the near-extremal regime $\xi t \lesssim \mathcal{O}(1)$ where the BTZ black hole becomes highly quantum [64]. This fluctuating behavior signals that the stringy degrees of freedom give genuinely new contributions to the black hole Hilbert space, distinct from the semiclassical BTZ geometries or quotients thereof.

$SL(2,\mathbb{Z})$ spectral representation.—As a slight detour, it is enlightening to give an alternative representation of $Z_{\text{string}}(\tau)$. The spectral gap condition $\Delta_* = 2\xi$ implies that $Z_{\text{string}}(\tau) \in L^2(\mathcal{F})$, and hence admits a harmonic decomposition in the $SL(2,\mathbb{Z})$ spectral eigenbasis, comprising the completed Eisenstein series $E^*_{\frac{1}{2}+i\omega}(\tau)$ with $\omega \in \mathbb{R}$ and Maass cusp forms $\phi_n(\tau)$ (e.g., [65–67]). Denoting their spin-j Fourier coefficients as $\mathsf{a}_j^{(s)}$ and $\mathsf{b}_j^{(n)}$, respectively, and using the conventions of [21], we have

$$Z_{\text{string}}(\tau) = \int_{\mathcal{C}_{\text{crit}}} \mathsf{a}_{j_*}^{(s)} \frac{\Gamma\left(\frac{\frac{1}{2}-s}{2}\right)\Gamma\left(\frac{s-\frac{1}{2}}{2}\right)}{\Lambda(s)\Lambda(1-s)} E_s^*(\tau) + \sum_{n=1}^{\infty} \mathsf{b}_{j_*}^{(n)} \Gamma\left(-\frac{i\omega_n}{2}\right)\Gamma\left(\frac{i\omega_n}{2}\right)\phi_n(\tau), \tag{19}$$

where $\mathcal{C}_{\text{crit}}$ denotes $[(4\pi i)^{-1}$ times] contour integration along $s = \frac{1}{2} + i\omega$, and $\Lambda(s) \coloneqq \pi^{-s}\Gamma(s)\zeta(2s)$ is the completed Riemann zeta function. For details see Supplemental Material [26], Appendix C.

Presenting $Z_{\text{string}}(\tau)$ in spectral form reveals some interesting features and curiosities.

First, the modular average of $Z_{\text{string}}(\tau)$ vanishes:

$$\langle Z_{\text{string}} \rangle \coloneqq \int_{\mathcal{F}} \frac{dxdy}{y^2} Z_{\text{string}}(\tau) = 0. \tag{20}$$

This follows from the vanishing of the Eisenstein spectral overlap in (19) at $s = 0$, which defines the modular average in general. We note that this property is shared by Narain CFTs [66].

Next, $Z_{\text{string}}(\tau)$ may be written as the action of an $SL(2,\mathbb{Z})$ Hecke operator $T_{\xi/2}$ [68] on a "primitive" partition function, $\mathcal{Z}_{\text{string}}(\tau)$, defined as $Z_{\text{string}}(\tau)$ but with the Fourier coefficients evaluated at $j_* = 1$:

$$\mathcal{Z}(\tau) = Z_{\text{MWK}}(\tau) + T_{\xi/2}\mathcal{Z}_{\text{string}}(\tau). \tag{21}$$

In this way, the entire family of unitary partition functions indexed by ξ may be generated by a Hecke action, implementing shifts in central charge. This shares a superficial likeness with Witten's construction of holomorphic extremal CFT partition functions [69], with obvious differences.

Finally, there is a profound conjecture in number theory, the "horizontal" Sato-Tate conjecture for Maass cusp forms of $SL(2,\mathbb{Z})$, which has interesting consequences for the spectral decomposition [70–73]. The conjecture states that for prime $j \to \infty$ and any fixed n, the normalized Fourier coefficients $\mathsf{b}_j^{(n)}/\mathsf{b}_1^{(n)}$ are equidistributed with respect to Wigner's semicircle distribution. This (and $\mathsf{b}_1^{(n)} \neq 0$, which follows from Hecke relations applied to Hecke-Maass cusp forms) implies that

$$\lim_{j\to\infty} \langle\langle \mathsf{b}_j^{(n)} \rangle\rangle = 0 \qquad (\text{fixed}\, n), \tag{22}$$

where $\langle\langle \cdot \rangle\rangle$ indicates a statistical average. Therefore, even though $(Z_{\text{string}}, \phi_n) \propto \mathsf{b}_{j_*}^{(n)} \neq 0$, they vanish on average in the large central charge limit $j_* \to \infty$ (for prime j_*) [74]. In this sense, the Eisenstein term seems to more directly underlie the unitarity of $\mathcal{Z}(\tau)$. It would be nice to understand this from a physical, quantum chaos point of view.

Summary and random (matrix) comments.—Our main result is the construction of the unitary partition function $\mathcal{Z}(\tau)$ given in (5), with the spectral gaps depicted in Fig. 1.

From the AdS_3 gravity point of view, despite the dimension gap above the vacuum state to the black hole threshold $\Delta_* = [(c-1)/12]$, this is not a semiclassical pure gravity path integral in the strict sense, due to the spinning states $\mathcal{O}_*$ with subthreshold twist. At any finite spin, these states are not visible, and the theory contains only black hole states. The degeneracies of all discrete states are integers.

We have advanced a bulk interpretation of $\mathcal{O}_*$ as a strongly coupled spinning string, though other interpretations may well be possible (or preferred). We view this as an indicative toy model for a genuine string theory compactification to AdS_3 pure gravity. A complete approach would include higher Regge trajectories; corrections to the spectrum from excitations around the spin-j ground states of [57]; and the other ingredients, such as fluxes and their brane sources, required to solve the strongly coupled string field equations (whatever they may be). Nevertheless, the problem of finding a direct, physically sensible quantization of AdS_3 gravity which would lead to our partition function (or another one with the desired properties) remains outstanding.

Randomness: Our construction cures the negativity from the sum over smooth bulk saddles semiclassically, rather than quantum mechanically. Quantum effects are not just present in a consistent theory, but are expected to be crucial in the engineering of a *bona fide* theory of AdS_3 pure gravity: there are strong indications that if such a theory exists, off-shell geometries encoding random matrix behavior of the chaotic spectrum play a central role in unitarizing the spectrum [8,20,76]. An explicit determination of the leading-order random matrix contribution to the semiclassical path integral of pure gravity with torus boundary, denoted as $Z_{\rm RMT}(\tau)$, was made in [21].

In any theory of semiclassical AdS_3 gravity (pure or otherwise), the black hole spectrum is chaotic, and its path integral should encode random fluctuations for quantum consistency. Such random matrix contributions are absent in $\mathcal{Z}(\tau)$. We may explain this fact, as well as the continuous spectrum in the chaotic regime $t > 0$, by interpreting $\mathcal{Z}(\tau)$ as the partition function of a microscopic compact CFT that has been subject to coarse graining.

As shown in [21] using a formalism built on a CFT trace formula, the random matrix contribution to the density of states, properly understood, vanishes upon coarse graining the spectrum over a suitable microcanonical window in twist, δt [77]. Because this window is necessarily larger than the exponentially small mean level spacing of the chaotic spectrum, the coarse graining simultaneously explains both the absence of random matrix contributions to (5) and its continuous spectrum while remaining compatible with a microscopic CFT interpretation. Given our explicit construction, we can determine δt: it is the characteristic wavelength of the oscillations of $\rho_{{\rm string},j}(t)$ in (12), namely, $\delta t \sim 1/\xi$. We emphasize that this coarse-graining interpretation does not rely on a $\xi \gg 1$ limit, and is compatible with compactness of a putative underlying CFT; there could, of course, be as-yet-unknown bootstrap constraints that rule this possibility out.

Note that in a $\xi \gg 1$ limit, $\mathcal{Z}(\tau)$ is also compatible with other interpretations, in particular with a hypothetical ensemble average over (possibly near) CFTs, or with other, perhaps independent, constructions of "approximate CFT" [78]. While we have presented a microscopic CFT interpretation in part to emphasize that a departure from standard AdS/CFT physics is not required at this level, semiclassical gravity seems unable to distinguish among these [78–80], at least perturbatively in G_N.

A complementary view on this coarse-grained interpretation comes from the formalism of [21]. Since $Z_{\rm string}(\tau)$ is the modular completion of a non–black hole state, we do not expect it to encode random matrix behavior *per se* [21,78,81,82]. Applying the results of [21] to $Z_{\rm string}(\tau)$ helps to ratify this perspective. In (19) we provided the $SL(2,\mathbb{Z})$ spectral decomposition of $Z_{\rm string}(\tau)$. A canonical diagnostic of random matrix universality is the presence of a linear ramp in the coarse-grained spectral form factor, with a specific coefficient prescribed by the random matrix ensemble. We can ask whether $Z_{\rm string}(\tau)$ generates this ramp after squaring and taking the diagonal approximation. A necessary and sufficient condition for the ramp was derived in [21], as an exponential decay condition on the spectral overlaps at $\omega \to \infty$. One readily checks that $Z_{\rm string}(\tau)$ does not satisfy this criterion, instead decaying as a power law [83].

Stringiness: On the other hand, $Z_{\rm string}(\tau)$ exhibits some behavior that lies somewhere "in between" chaotic and nonchaotic. Define a microcanonical coarse graining over mean twist,

$$\overline{f(t_1)f(t_2)} := \int_0^\infty dt'\, f(t'+\epsilon)f(t'-\epsilon)W(t-t'), \tag{23}$$

where $t = [(t_1+t_2)/2]$ and $\epsilon = [(t_1-t_2)/2]$. Applying this to $f(t) = \rho_{{\rm string},j}(t)$ at fixed j using (12) produces a nonzero variance upon coarse graining over windows $\delta t \gtrsim (1/\xi)$. However, its oscillatory behavior leads to suppression relative to the disconnected average. In particular, at $\xi \gg 1$,

$$\frac{{\rm Var}[\rho_j(t)]}{\bar{\rho}_j(t)^2} \approx e^{-4\pi\sqrt{\xi(t+j)}} \qquad (\xi \gg 1), \tag{24}$$

where $\bar{\rho}_j(t) = \rho_{{\rm MWK},j}(t)$. In the extremal limit $t \to 0$, the suppression factor is $e^{-S_{0,j}}$, where $S_{0,j} = 4\pi\sqrt{\xi j}$ is the extremal spin-j BTZ black hole entropy. In contrast, wormholes encoding chaotic behavior are suppressed as $e^{-2S_{0,j}}$ in the extremal limit [8,21,76,84,85]. It would be worthwhile to understand this intermediate behavior as a nonperturbative effect, possibly associated to strongly coupled strings, in a UV complete AdS_3 gravity path integral.

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